

MATH431: 2D Graphics Basics

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1 Overview of Geometric Transformations in 2D

Suppose we have an object in $\mathbb{R}^2$ that we wish to manipulate. For the sake of simplicity this object will be composed of points. Some standard manipulation might be things like:

- (a) Translations: Shifting the object horizontally and vertically.
- (b) Rotations: Rotating the object about an arbitrary point.
- (c) Reflections: Reflecting the object in an arbitrary line.
- (d) Perspective: Although this makes more sense in $\mathbb{R}^3$ it's worth taking a look at this in $\mathbb{R}^2$ because it makes the problems clear in a simpler context.

Moreover ideally we would like to use linear algebra to do these things because it is fast and cheap, computationally speaking.

More specifically given a point represented by a vector $\mathbf{v}$ we would like to represent each of the transformations above by a matrix M so that applying the transformation can be done by doing the product $M\mathbf{v}$.

Note: This is an ideal situation. In reality much of the gruntwork calculation is baked into the software or hardware and transparent to the user. However it's still certainly true that simpler calculations are faster, all other things being equal. Consequently our ongoing goal is simplicity wherever possible.

2 Translations

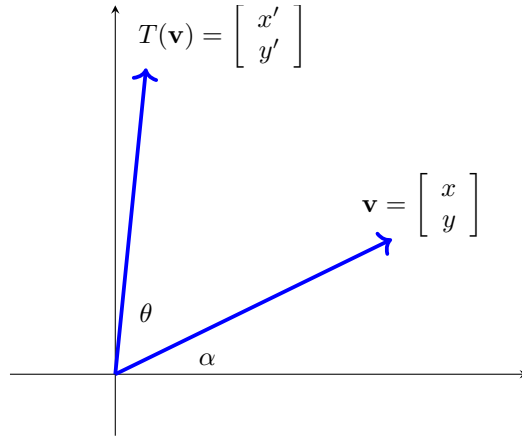
Immediately we have a problem. Given a point represented by a vector $\mathbf{v} = [x, y]^T$ a translation by a units horizontally and b units vertically would have to be a transformation $T([x, y]^T) = [x + a, y + b]^T$. But this cannot be represented by a matrix because it is not linear and it is not linear because $T([0, 0]^T) = [a, b]^T \neq [0, 0]^T$ unless $a = b = 0$ which is no translation at all.

As we've mentioned above this is not a huge issue in the sense that addition is a fairly easy operation but it is a separate operation from matrix multiplication, so it's worth keeping this in mind.

3 Rotations

Rotations about points other than the origin are not linear for the same reason argued above. If you rotate the origin about a point other than the origin it will end up off the origin, unless the rotation is by 2π , which is no rotation at all.

So how about rotation about the origin? Let's construct it, so consider the following picture:



First note that:

$$\begin{aligned}x &= \sqrt{x^2 + y^2} \cos \alpha \\y &= \sqrt{x^2 + y^2} \sin \alpha\end{aligned}$$

It then follows that from the addition formula for cosine:

$$\begin{aligned}x' &= \sqrt{x^2 + y^2} \cos(\alpha + \theta) \\&= \sqrt{x^2 + y^2} (\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\&= \sqrt{x^2 + y^2} \cos \alpha \cos \theta - \sqrt{x^2 + y^2} \sin \alpha \sin \theta \\&= x \cos \theta - y \sin \theta\end{aligned}$$

And that from the addition formula for sine:

$$\begin{aligned}y' &= \sqrt{x^2 + y^2} \sin(\alpha + \theta) \\&= \sqrt{x^2 + y^2} (\sin \alpha \cos \theta + \sin \theta \cos \alpha) \\&= \sqrt{x^2 + y^2} \sin \alpha \cos \theta + \sqrt{x^2 + y^2} \sin \theta \cos \alpha \\&= y \cos \theta + x \sin \theta \\&= x \sin \theta + y \cos \theta\end{aligned}$$

This can be written as a matrix product:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} \\ = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

It follows that rotation about the origin counterclockwise by θ radians is linear with the matrix representation:

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Because we've explicitly constructed a matrix which does exactly what we wanted we know for a fact that rotation about the origin counterclockwise by θ radians is linear and that we have our matrix.

We can now use it to rotate.

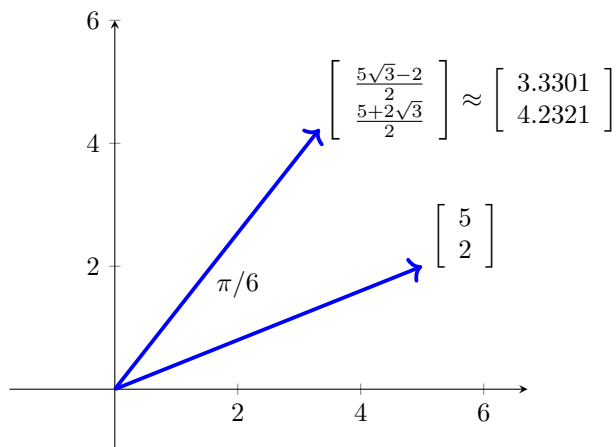
Example 3.1. The result of rotating the point $(5, 2)$ counterclockwise by $\pi/6$ radians about the origin is:

$$R_{\pi/6} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5(\sqrt{3}/2) - 2(1/2) \\ 5(1/2) + 2(\sqrt{3}/2) \end{bmatrix}$$

Or the point

$$(5(\sqrt{3}/2) - 2(1/2), 5(1/2) + 2(\sqrt{3}/2)) = \left(\frac{5\sqrt{3} - 2}{2}, \frac{5 + 2\sqrt{3}}{2} \right)$$

□



Exercise 3.1. Find the result of rotating the point $(-3, 7)$ counterclockwise by $\pi/4$ radians.

□

Exercise 3.2. Find the result of rotating the point $(-2, -7)$ clockwise by $7\pi/6$ radians.

□

Exercise 3.3. Show computationally that a rotation by θ_1 radians followed by a rotation by θ_2 radians results in a rotation by $\theta_1 + \theta_2$ radians.

□

Exercise 3.4. A non-vertical line in space can be given by the slope-intercept form $y = mx + b$ for $m, b \in \mathbb{R}$.

- Find the image of the point $(0, b)$ under rotation by θ .
- Find the image of the point $(1, m + b)$ under rotation by θ .
- Find the slope of the image of the line. This should not contain b . Why does this make sense? For which values of θ will this be undefined? Explain geometrically what is happening for such values.
- Write the equation of the new line in the form $y = m'x + b'$ where m' and b' may depend on m, b, θ .
- Where does the line $y = 2x + 6$ get mapped to when $\theta = \pi/6$?
- In the case where the slope is undefined what will the equation of the new line be?

□

Exercise 3.5. Any line in 2D space can be given by an equation of the form $ax + by = c$ for $a, b, c \in \mathbb{R}$.

- If a rotation by θ radians is applied to this line, find the equation of the resulting line. Your final answer should be in the form

$$a'x + b'y = c'$$

where a', b' , and c' may depend on a, b, c, θ . You may need more than one case.

- Where does the line $y = 5$ get mapped to when $\theta = \pi/3$?
- Where does the line $x = 5$ get mapped to when $\theta = \pi/4$?
- Where does the line $2x + 4y = 7$ get mapped to when $\theta = \pi/6$?

□

Exercise 3.6. It's fairly clear intuitively that rotation preserves distances (formally it's an example of an *isometry*) but show computationally that this is true. In other words show that if P and Q are points then we have:

$$\text{dist}(R_\theta(P), R_\theta(Q)) = \text{dist}(P, Q)$$

□

4 Rotation Addendum in 3D

It's possible to rewrite our rotation formula in a way that extends its use. Consider that the mapping works as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} = \cos \theta \begin{bmatrix} x \\ y \end{bmatrix} + \sin \theta \begin{bmatrix} -y \\ x \end{bmatrix}$$

The vector $[-y, x]^T$ is the result of rotating $[x, y]^T$ counterclockwise by $\pi/2$ about the origin. Consequently what this reformulation is saying is that if $\mathbf{v}$ is a vector and if $\mathbf{w}$ is the result of rotating $\mathbf{v}$ counterclockwise by $\pi/2$ about the origin then the rotation map is:

$$\mathbf{v} \mapsto (\cos \theta)\mathbf{v} + (\sin \theta)\mathbf{w}$$

Since this mapping is a simple linear combination it is actually valid even in 3D if we understand the context. Imagine a plane through the origin and a perpendicular axis $\hat{\mathbf{u}}$ (in this case $\hat{\mathbf{u}}$ is a unit normal vector).

Suppose $\mathbf{v}$ is a vector in this plane and $\mathbf{w}$ is the result of rotating $\mathbf{v}$ by $\pi/2$ about the $\hat{\mathbf{u}}$ axis obeying the right-hand rule (thumb follows $\hat{\mathbf{u}}$ and fingers curl from $\mathbf{v}$ to $\mathbf{w}$). Then the result of rotating $\mathbf{v}$ by θ radians about $\hat{\mathbf{u}}$ (again following the right-hand rule) will obey the same rotation map above.

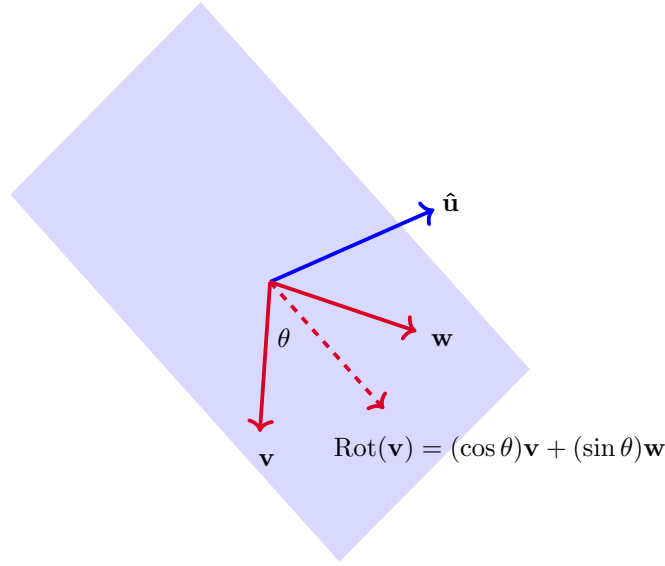
We'll use the following notation when it's clear what the axis and angle are or when the specifics are irrelevant:

$$\text{Rot}(\mathbf{v}) = (\cos \theta)\mathbf{v} + (\sin \theta)\mathbf{w}$$

Note that the perpendicular axis $\hat{\mathbf{u}}$ is useful only for understanding the direction of rotation (how $\mathbf{v}$ and $\mathbf{w}$ need to relate), it doesn't initially show in the formula. Really the formula just indicates that $\mathbf{v}$ rotates toward $\mathbf{w}$ within the plane they span and $\hat{\mathbf{u}}$ doesn't come into play using that interpretation.

It is of course critical that $|\mathbf{v}| = |\mathbf{w}|$ in order to use the formula.

This picture might clarify.



Example 4.1. Observe that the vectors

$$\mathbf{v} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \text{ and } \mathbf{w} = 0\hat{\mathbf{i}} + \frac{3}{\sqrt{5}}\hat{\mathbf{j}} - \frac{6}{\sqrt{5}}\hat{\mathbf{k}}$$

are perpendicular and have the same magnitude. Consequently if we wish to rotate $\mathbf{v}$ toward $\mathbf{w}$ by $3\pi/4$ radians we calculate:

$$\begin{aligned} \text{Rot}(\mathbf{v}) &= (\cos(3\pi/4))\mathbf{v} + (\sin(3\pi/4))\mathbf{w} \\ &= -\frac{\sqrt{2}}{2}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) + \frac{\sqrt{2}}{2}\left(0\hat{\mathbf{i}} + \frac{3}{\sqrt{5}}\hat{\mathbf{j}} - \frac{6}{\sqrt{5}}\hat{\mathbf{k}}\right) \end{aligned}$$

It's worth noting that since $\frac{3\pi}{4} > \frac{\pi}{2}$ this actually rotates $\mathbf{v}$ past $\mathbf{w}$ in the plane they span.

□

Exercise 4.1. Observe that the vectors $\mathbf{v} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{w} = 0\hat{\mathbf{i}} + \sqrt{6}\hat{\mathbf{j}} + \sqrt{6}\hat{\mathbf{k}}$ are perpendicular and have the same magnitude. Find the vector which results when $\mathbf{v}$ is rotated $\frac{\pi}{3}$ radians towards $\mathbf{w}$ in the plane they span and the vector which results when $\mathbf{w}$ is rotated $\frac{\pi}{3}$ radians towards $\mathbf{v}$ in the plane they span.

□

Exercise 4.2. The vectors $\mathbf{v} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{w} = 1\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 2\hat{\mathbf{j}}$ are perpendicular but do not have the same magnitude. Find the vector which results when $\mathbf{v}$ is rotated $\frac{5\pi}{6}$ radians towards $\mathbf{w}$ in the plane they span.

□

Exercise 4.3. The vectors $\mathbf{v} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}$ and $\mathbf{w} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ are neither perpendicular nor have the same magnitude. They do of course span a plane. Find the vector which results when $\mathbf{v}$ is rotated $\frac{5\pi}{6}$ radians towards $\mathbf{w}$ in the plane they span.

□

In a more likely scenario it's $\hat{\mathbf{u}}$ and $\mathbf{v}$ which are given. In this context $\hat{\mathbf{u}}$ defines both the plane and a sense of counterclockwise within that plane again with the right-hand rule. The thumb follows $\hat{\mathbf{u}}$ and the curl of the fingers are designated as counterclockwise.

In this scenario $\hat{\mathbf{u}}$ becomes essential in finding $\mathbf{w}$ because $\mathbf{w} = \hat{\mathbf{u}} \times \mathbf{v}$.

Why is this? Well the right-hand rule shows that $\mathbf{w}$ points in the direction we need it to and it's the right length because we know that $\hat{\mathbf{u}}$ and $\mathbf{v}$ are perpendicular and $\hat{\mathbf{u}}$ is a unit vector (important!) and so we know that

$$|\mathbf{w}| = |\hat{\mathbf{u}} \times \mathbf{v}| = |\hat{\mathbf{u}}||\mathbf{v}|\sin\theta = (1)|\mathbf{v}|\sin(\pi/2) = |\mathbf{v}|$$

Thus in reality we can write:

$$\text{Rot}(\mathbf{v}) = (\cos\theta)\mathbf{v} + (\sin\theta)(\hat{\mathbf{u}} \times \mathbf{v})$$

Example 4.2. Observe that the vectors $\hat{\mathbf{u}} = \frac{1}{\sqrt{14}}(1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ and $\mathbf{v} = 3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 1\hat{\mathbf{k}}$ are perpendicular. Consequently to rotate $\mathbf{v}$ about the axis $\hat{\mathbf{u}}$ by $\pi/4$ radians we calculate:

$$\text{Rot}(\mathbf{v}) = (\cos(\pi/4))\mathbf{v} + (\sin(\pi/4))(\hat{\mathbf{u}} \times \mathbf{v})$$

Since we have:

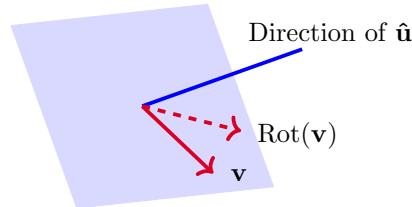
$$\hat{\mathbf{u}} \times \mathbf{v} = \frac{1}{\sqrt{14}}(-2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

we get the result:

$$\text{Rot}(\mathbf{v}) = \frac{\sqrt{2}}{2}(3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 1\hat{\mathbf{k}}) + \frac{\sqrt{2}}{2} \frac{1}{\sqrt{14}}(-2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

□

The following picture is the result. We've put the direction of $\hat{\mathbf{u}}$ rather than $\hat{\mathbf{u}}$ itself since $\hat{\mathbf{u}}$ is a unit vector and consequently is so short as to be tricky to have in the picture.



Exercise 4.4. Let $\mathbf{u} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ (not a unit vector) and let $\mathbf{v} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Find $\hat{\mathbf{u}}$ (normalize $\mathbf{u}$), prove that $\hat{\mathbf{u}}$ and $\mathbf{v}$ are perpendicular and then find the result when $\mathbf{v}$ is rotated by $\pi/6$ around $\hat{\mathbf{u}}$.

□

Exercise 4.5. Consider the plane defined by $x + 2y + 4z = 0$ and assumed normal vector arising from the normalization of $\mathbf{N} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. Find a generic formula for rotation of this plane about this vector and give this formula as a mapping of a point $(x, y, z) \mapsto (?, ?, ?)$.

□

Exercise 4.6. Given a plane $\mathcal{P}$, a point $\mathbf{p} \in \mathcal{P}$, and a unit vector $\hat{\mathbf{u}}$ anchored at $\mathbf{p}$, we can still rotate $\mathcal{P}$ about $\mathbf{p}$ (and about the axis $\hat{\mathbf{u}}$) in a counterclockwise direction as defined by $\hat{\mathbf{u}}$ even when $\mathbf{p} \neq \mathbf{0}$, although this transformation is not linear. We do so by translating $\mathbb{R}^3$ so that $\mathbf{p}$ is at the origin, then rotate, then translate back.

Let $\mathcal{P}$ be the plane $x + 2y + 3z = 6$. Let the center and axis be defined by $\mathbf{p} = [2, 2, 0]^T$ and $\hat{\mathbf{u}}$ be the normalization of $\mathbf{N} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Find the resulting point when $(0, 3, 0)$ is rotated by $\pi/6$ radians in this way.

□

5 Reflections

As with rotations it's clear that a transformation which reflects in a line not through the origin will not be linear since it will take the origin off itself. But how about reflection in a line through the origin?

There are various approaches to this but the most elementary one is to approach the problem as follows. Suppose a line through the origin makes an angle of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ radians with the x -axis. Denote by F_θ the reflection (F for Flip) in that line.

What we can do is rotate about the origin to move this line to the x -axis, then reflect in the x -axis using the transformation:

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and then rotate back. Notice that the above transformation simply negates the y -coordinate.

So then in other words if a line makes an angle of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ with the x -axis then we would simply reflect by:

$$F_\theta = R_\theta F_0 R_{-\theta}$$

Note 5.0.1. Note that is a matrix product and so its action is right-to-left. To apply it to a point (vector) $\mathbf{v}$ we would do

$$\mathbf{v} \mapsto R_\theta F_0 R_{-\theta} \mathbf{v}$$

Or if you prefer:

$$\text{Flip}(\mathbf{v}) = R_\theta F_0 R_{-\theta} \mathbf{v}$$

□

Since the product is a matrix the resulting transformation is linear.

Interestingly if we work the details of this out we find:

$$\begin{aligned} F_\theta &= R_\theta F_0 R_{-\theta} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \dots \\ &= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= R_{2\theta} F_0 \end{aligned}$$

So reflecting in the line with angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is equivalent to first reflecting in the x -axis and then rotating counterclockwise by 2θ :

$$F_\theta = R_\theta F_0 R_{-\theta} = R_{2\theta} F_0$$

Definition 5.0.1. The resulting matrix is the *reflection matrix*.

□

Exercise 5.1. Fill in the ... part of the calculation above.

□

Exercise 5.2. Calculate the reflection matrix for the line which makes an angle of $\frac{\pi}{3}$ with the x -axis. Then use it to calculate the reflection of the point $(5, -3)$.

□

Exercise 5.3. The line $y = mx$ is represented by the vector $\mathbf{v} = [1, m]^T$ which by basic trigonometry satisfies $1 = \sqrt{1 + m^2} \cos \theta$ and $m = \sqrt{1 + m^2} \sin \theta$ (a picture can help you see this). Use this fact and the above formula for $F_\mathbf{v}$ to show that reflection in the line $y = mx$ is represented by the matrix:

$$\frac{1}{m^2 + 1} \begin{bmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{bmatrix}$$

□

Exercise 5.4. Calculate the matrix which reflects in the line $y = 3x$ and use it to reflect the points $(1, 2)$ and $(3, -4)$.

□

Exercise 5.5. Another approach to reflection in $y = mx$ is to start with a point (x_0, y_0) , construct the line perpendicular to $y = mx$ and through (x_0, y_0) , and then locate the opposite point on this new line which is the same distance from $y = mx$ as (x_0, y_0) . Do so and show that the result is the same as the matrix two exercises above. This is pretty straightforward when done carefully.

□

Exercise 5.6. Work through some of the above calculations by using reflection in the y -axis instead of reflection in the x -axis and show that the results are all equivalent.

□

Exercise 5.7. Show that reflecting in the line (vector) with angle θ is equivalent to first reflecting in the y -axis and then rotating counterclockwise by some amount.

□

6 Two Reflections Equals One Rotation

From the previous calculation:

$$F_\theta = R_{2\theta}F_0$$

We can solve for $R_{2\theta}$ as follows, noting that $F_0^{-1} = F_0$:

$$\begin{aligned} F_\theta &= R_{2\theta}F_0 \\ F_\theta F_0^{-1} &= R_{2\theta} \\ F_\theta F_0 &= R_{2\theta} \end{aligned}$$

Thus we can see that:

$$R_\theta = F_{\theta/2}F_0$$

It follows that any rotation at all can be constructed using reflection in the x -axis as well as one other reflection.

Note 6.0.1. Given that we defined reflections in terms of rotations this obviously becomes self-referential. We are not suggesting that this is necessarily the best way to calculate a rotation but rather we are just pointing out a connection. In addition having this formula at our disposal can help us with further calculations and observations.

□

Example 6.1. To rotate by $\pi/6$ radians we simply do the following:

$$R_{\pi/6} = F_{\pi/12}F_0$$

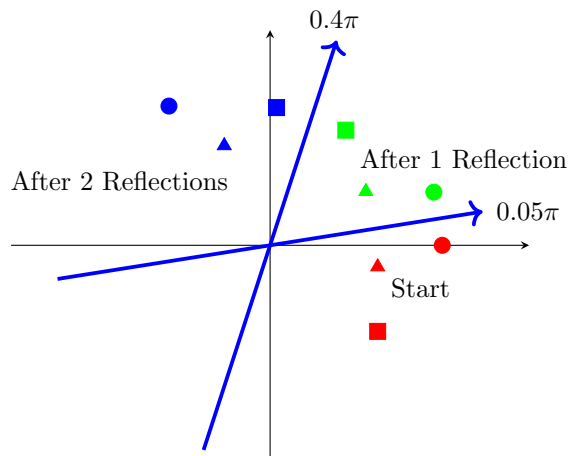
□

Exercise 6.1. Write $R_{\pi/3}$ as a product of reflections as in the previous example.

□

It turns out that the previous statement can be generalized to say that the product of any two reflections equals a rotation.

A picture with a few points can help convince us of this. Here we see a circle, a square and a triangle reflected first in a line with angle of 0.05π and then in a line with angle of 0.4π . We can see that the successive reflections formed a rotation, the question is how far.



This is not hard to figure out. Suppose the first reflection makes an angle of θ_1 with the positive x -axis while the second makes an angle of θ_2 with the positive x -axis. Let's calculate the result of reflecting in the first and then the second. The lengthy matrix calculation here is omitted, just the result is given. Here the notation F_θ is used to denote a reflection in the line which makes an angle of θ with the positive x -axis.

$$\begin{aligned}
F_{\theta_2} F_{\theta_1} &= R_{\theta_2} F_0 R_{-\theta_2} R_{\theta_1} F_0 R_{-\theta_1} \\
&= \dots \\
&= \begin{bmatrix} \cos(2(\theta_2 - \theta_1)) & -\sin(2(\theta_2 - \theta_1)) \\ \sin(2(\theta_2 - \theta_1)) & \cos(2(\theta_2 - \theta_1)) \end{bmatrix}
\end{aligned}$$

This shows that the result rotates counterclockwise about the origin by $2(\theta_2 - \theta_1)$. An alternate way of thinking about this is that it rotates the plane from the first reflection toward the second reflection by twice the angle between them.

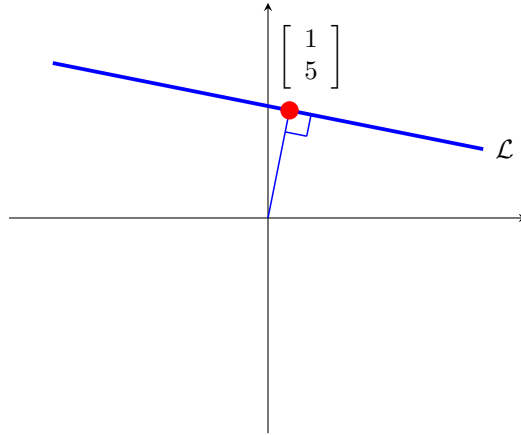
7 Closest Point Parametrization of a Line

7.1 Method

Classic ways of representing a line in $\mathbb{R}^2$ are $y = mx + b$, $ax + by = c$ and via a parametrization $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$.

Another interesting way to represent a line $\mathcal{L}$ which does not pass through the origin is via the *closest point parametrization of a line*. Let $\mathbf{p} \in \mathbb{R}^2$ with $\mathbf{p} \neq \mathbf{0}$. This defines a line as follows - first draw the line from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\mathbf{p}$ and then draw the line $\mathcal{L}$ through $\mathbf{p}$ perpendicular to this first line.

Example 7.1. Here is the line parametrized by $\mathbf{p} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$:



□

Exercise 7.1. Consider now the line $\mathcal{L}$ parametrized by $\mathbf{p} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

- (a) Draw a rough sketch of $\mathcal{L}$.

- (b) Find an equation which is satisfied iff $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{L}$ and rewrite it slope-intercept form.

□

Exercise 7.2. Which point parametrizes the line $y = 3x + 2$?

□

Exercise 7.3. Suppose $\mathbf{p} \neq \mathbf{0}$ represents $\mathcal{L}$. Show that $R_\theta(\mathbf{p})$ represents $R_\theta(\mathcal{L})$ and $F_{\mathbf{v}}(\mathbf{p})$ represents $F_{\mathbf{v}}(\mathcal{L})$.

□

Exercise 7.4. Suppose two lines are represented by:

$$\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Find the point at which they meet.

□

7.2 Advantages and Disadvantages

The major advantages to using the closest point parametrization of a line is that to rotate the line around the origin, or to reflect the line in another line through the origin, we simply operate on that point.

This is counterbalanced by the major disadvantages:

- Determining if another point is on this line is not as obvious as with something like $y = mx + b$.
- Translation of the line is not easy.
- Rotation of the line around a point other than the origin is not easy.
- Reflection of the line through another line not through the origin is not easy.

Of course formulas may be developed for the three above transformations but they take some work.

Exercise 7.5. If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is the closest point parametrization of a line and we translate it by $\begin{bmatrix} a \\ b \end{bmatrix}$, what is the closest point parametrization of the resulting line?

□

Exercise 7.6. Suppose $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is the closest point parametrization of a line.

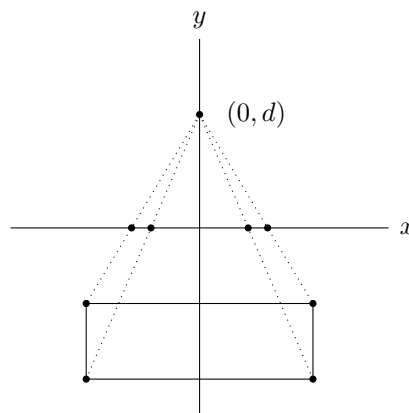
How could you determine whether the point $\begin{bmatrix} x \\ y \end{bmatrix}$ is on this line? Think of at least two different ways.

□

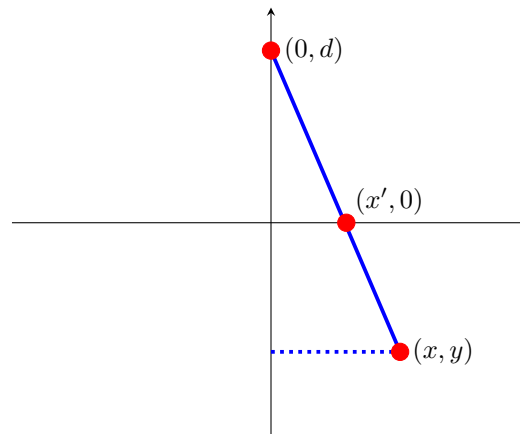
8 Perspective Projection

Perspective will make more sense when we move to 3D but we can work up a 2D version that nails down the basic problem.

Imagine an object in the xy -plane below the x -axis. If you position your eye at the point $(0, d)$ and look downwards and imagine that the viewing plane is the x -axis then you will see the object as if it is projected with perspective to the x -axis:



Let's look at just one point:



A simple calculation with similar triangles tells us that the point (x, y) maps to the point $(x', 0)$ via:

$$\frac{x'}{d} = \frac{x}{d-y}$$

$$x' = \frac{dx}{d-y}$$

This is not a linear transformation, meaning there is no 2×2 matrix P such that:

$$P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

This is easy to see - if such a mapping were a linear transformation then it would take the basis vectors $[1, 0]^T$ and $[0, 1]^T$ to $[d(1)/(d-0), 0]^T = [1, 0]^T$ and $[d(0)/(d-1), 0]^T = [0, 0]^T$ respectively, meaning it would be the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

However clearly this matrix does not do what we're asking of it:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \neq \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

Exercise 8.1. We showed that this perspective projection was not linear by assuming that it was, constructing the associated matrix, then showing that the matrix failed to do what we want. Instead we could take the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

and show that it failed the definition of linearity. Do so.

□

Exercise 8.2. Explain the difference (in terms of the geometric result) between the object moving away from the viewing plane and the eye moving away from the viewing plane.

□

9 Conclusion

In 2D none of translation, rotation, reflection or projection work as we'd like. We get rotation only if it's about the origin and reflection only if it's in a line through the origin. However we've learned a few useful things, such as how two successive reflections yield a rotation as well as how perspective projection ought to work. This knowledge will help us as we move forward.