MATH431: Geometric Algebra

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1 Introduction and Computation Note

Some of the calculations in this chapter are fairly intense. I did some of the really bad ones in Python with the `clifford` package. In those cases I've noted this. Some I did by hand for practice.

2 Scalars, Vectors, Bivectors, Trivectors

2.1 Scalars

Definition 2.1.1. A scalar in $\mathbb{R}^n$ can be thought of as a zero-dimensional entity projecting in zero directions out from the origin. Scalars are said to have *grade* 0. The set of scalars is denoted and defined by:

$$\wedge^0\mathbb{R}^n = \text{span}\,\{c|c \in \mathbb{R}\}$$

□

Note 2.1.1. It may seem weird to write "span" here when the result is just $\mathbb{R}$ itself but go with it for now.

□

Note 2.1.2. We could also just write:

$$\wedge^0\mathbb{R}^n = \text{span}\,\{1\}$$

However for now leave the definition as above.

□

2.2 Vectors

Definition 2.2.1. A vector in $\mathbb{R}^n$ can be thought of as a one-dimensional entity projecting out in one direction out from the origin. Vectors are said to have *grade* 1. The set of vectors is denoted and defined by:

$$\wedge^1\mathbb{R}^n = \text{span}\,\{\mathbf{a}|\mathbf{a} \in \mathbb{R}^n\}$$

□

Note 2.2.1. It may seem weird to write "span" here when the result is just $\mathbb{R}^n$ itself but go with it for now.

□

Note that in $\mathbb{R}^n$ we have the standard basis:

$$\{\mathbf{e_1}, ..., \mathbf{e_n}\}$$

and all vectors may be written as linear combinations of those.

Note 2.2.2. Given this, we could also just write:

$$\wedge^1 \mathbb{R}^n = \text{span}\{\mathbf{e_1}, ...\mathbf{e_n}\}$$

However for now leave the definition as above.

□

2.3 Bivectors and the Wedge Product of Two Vectors

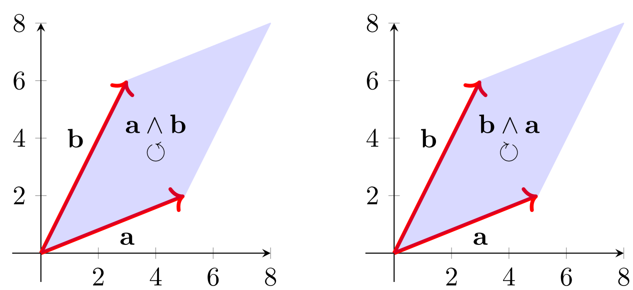
To create our newest addition we introduce a new way of combining vectors.

Definition 2.3.1. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ we define the *outer product* (*exterior product*, *wedge product*) $\mathbf{a} \wedge \mathbf{b}$ as the oriented parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.

□

By "oriented" the implication is that if we follow $\mathbf{a}$ first and create a loop around the parallelogram we get an orientation. Thus $\mathbf{b} \wedge \mathbf{a}$ would have the opposite orientation.

Example 2.1. If $\mathbf{a} = 5\mathbf{e_1} + 2\mathbf{e_2}$ and $\mathbf{b} = 3\mathbf{e_1} + 6\mathbf{e_2}$ then the following two pictures illustrate $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$. Note the orientations given by the circulating arrows.



□

We adopt the convention that reversing the orientation negates the parallelogram and consequently $\mathbf{b} \wedge \mathbf{a} = -(\mathbf{a} \wedge \mathbf{b})$. It follows from this that $\mathbf{a} \wedge \mathbf{a} = 0$, which makes sense because no parallelogram (or a degenerate parallelogram) is created.

Definition 2.3.2. We define a *bivector* to be any linear combination of such outer products (oriented parallelograms). More formally a *bivector* in $\mathbb{R}^n$ for $n \geq 2$ is any element in the set:

$$\wedge^2 \mathbb{R}^n = \text{span}\left\{\mathbf{a} \wedge \mathbf{b} \,\middle|\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n\right\}$$

Bivectors are said to have *grade* 2.

□

Note 2.3.1. Now the use of the word "span" seems to be relevant since we can take linear combinations of oriented parallelograms. These linear combinations should for now just be taken to be symbolic, don't think about what it might mean to "add" two parallelograms.

□

Note 2.3.2. We currently have no analogy in order to write:

$$\wedge^2\mathbb{R}^n = \text{span}\{???\}$$

We will, though.

□

Note 2.3.3. Just to emphasize, it's tempting to think that it's only expressions of the form $\mathbf{a} \wedge \mathbf{b}$ which are bivectors but it's not just those, it's all linear combinations of those!

□

Example 2.2. An example of a bivector in $\mathbb{R}^3$ (with helpful parentheses) is:

$$10\left[(2\mathbf{e_1}) \wedge (\mathbf{e_1} + 2\mathbf{e_2})\right] + 7\left[(2\mathbf{e_2} + 2\mathbf{e_3}) \wedge (4\mathbf{e_1} - \mathbf{e_3})\right]$$

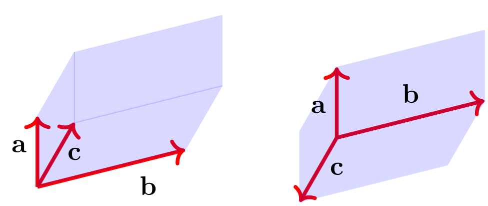
□

2.4 Trivectors and the Wedge Product of Three Vectors

Definition 2.4.1. Given three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we define the *outer product* (*exterior product*, *wedge product*) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the oriented parallelepipid with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

□

By "oriented" the implication is that if we take the orientation of $\mathbf{a} \wedge \mathbf{b}$ and apply the right hand rule, if $\mathbf{c}$ agrees we get an orientation. If it disagrees we get the opposite orientation.



Definition 2.4.2. We define a *trivector* to be any linear combination of such triple outer products (oriented parallelepipids). More formally a *trivector* in $\mathbb{R}^n$ for $n \geq 3$ is any element in the set:

$$\wedge^3\mathbb{R}^n = \text{span}\left\{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \middle| \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n\right\}$$

Trivectors are said to have *grade* 3.

$\square$

Note 2.4.1. Again the use of the word "span" is relevant here since a trivecor can be any linear combination of oriented parallelepipid.

$\square$

Note 2.4.2. Again we currently have no analogy in order to write:

$$\wedge^3\mathbb{R}^n = \text{span}\,\{???\}$$

We will, though.

$\square$

Example 2.3. An example of a trivector in $\mathbb{R}^3$ (with helpful parentheses) is:

$$5\left[(-1\mathbf{e_1}+\mathbf{e_2})\wedge(\mathbf{e_2})\wedge(3\mathbf{e_1}+2\mathbf{e_3})\right]-7\left[(\mathbf{e_2}-2\mathbf{e_3})\wedge(5\mathbf{e_2})\wedge(2\mathbf{e_1}+\mathbf{e_3})\right]$$

$\square$

2.5 Highervectors?

We can go further, but we won't. We'll stop at trivectors.

2.6 Multivectors

We abstract even further to allow sums of various of the former.

Definition 2.6.1. A *multivector* is defined as an abstract linear combination of scalars, vectors, bivectors and (in $\mathbb{R}^3$) trivectors. In other words:

$$\begin{aligned}\wedge\mathbb{R}^2 &= \text{LC of scalars, vectors, and bivectors}\\ \wedge\mathbb{R}^3 &= \text{LC of scalars, vectors, bivectors, and trivectors}\end{aligned}$$

$\square$

Example 2.4. Here are some examples:

(a) $A = 2 + \mathbf{a}$ is a multivector in $\wedge\mathbb{R}^2$ and in $\wedge\mathbb{R}^3$.

(b) $B = 3 + 3\mathbf{e_1} + (\mathbf{e_2}\wedge 3\mathbf{e_3})$ is a multivector in $\wedge\mathbb{R}^2$ and in $\wedge\mathbb{R}^3$.

(c) $C = 2 + \mathbf{a} + 3\mathbf{b} + (\mathbf{a}\wedge\mathbf{b})$ is a multivector in $\wedge\mathbb{R}^2$ and in $\wedge\mathbb{R}^3$ where $\mathbf{a}$ and $\mathbf{b}$ are any vectors.

(d) $D = 4\mathbf{v_1} + 3\mathbf{v_2}\wedge\mathbf{v_3} + \mathbf{v_4}\wedge\mathbf{v_5} + 2\mathbf{v_6}\wedge\mathbf{v_7}\wedge\mathbf{v_8}$ is a multivector in $\wedge\mathbb{R}^3$ but not in $\wedge\mathbb{R}^2$ because it contains a trivector.

$\square$

2.7 The Inner Product

Definition 2.7.1. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ we define the *inner product* (*scalar product*, *dot product*):

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The inner product produces a scalar thereby dropping us from vectors down to scalars.

□

Exercise 2.1. Suppose $\mathbf{a}$ and $\mathbf{b}$ have lengths 7 and 10 respectively and meet at angle of $\pi/3$. Find their inner product.

□

Theorem 2.7.1. If $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}$ then:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Proof. From the Law of Cosines we have:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$
$$(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\mathbf{a} \cdot \mathbf{b}$$

The result follows after cancellation. □

Practical Calculation 0.1. If you are given $\mathbf{a}$ and $\mathbf{b}$, to calculate $\mathbf{a} \cdot \mathbf{b}$ write $\mathbf{a}$ and $\mathbf{b}$ as linear combinations of $\mathbf{e_i}$ and apply the above theorem. The result will be a scalar.

□

Example 2.5. If $\mathbf{a} = 2\mathbf{e_1} + 3\mathbf{e_2}$ and $\mathbf{b} = 5\mathbf{e_1} - 1\mathbf{e_2} + 17\mathbf{e_3}$ then:

$$\mathbf{a} \cdot \mathbf{b} = (2)(5) + (3)(-1) + (0)(17) = 7$$

□

Definition 2.7.2. We say that two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are *perpendicular* if the angle between them is $\pi/2$, which is equivalent to saying that $\mathbf{a} \cdot \mathbf{b} = 0$.

□

2.8 The Outer Product $\mathbf{a} \wedge \mathbf{b}$ Revisited

For vectors $\mathbf{a}$ and $\mathbf{b}$ we defined the outer product $\mathbf{a} \wedge \mathbf{b}$ as the oriented parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. What we will do next is introduce some axioms which make geometric sense but have possibly unexpected consequences.

Axiom 2.8.1. We insist on the following axioms. For each we have noted what these mean geometrically. Consider that they make geometric sense.

(a) $\mathbf{a} \wedge (\mathbf{b} \pm \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \pm (\mathbf{a} \wedge \mathbf{c})$
The meaning of the $+$ version is that the sum of the two parallelograms which meet along the edge $\mathbf{a}$ equals the resulting large parallelogram.

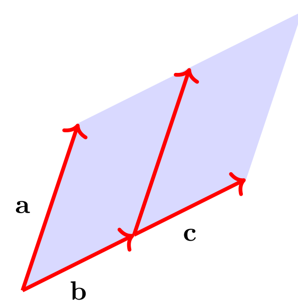
(b) $(\mathbf{b} \pm \mathbf{c}) \wedge \mathbf{a} = (\mathbf{b} \wedge \mathbf{a}) \pm (\mathbf{c} \wedge \mathbf{a})$
The meaning is the same as the above.

(c) $\alpha(\mathbf{a} \wedge \mathbf{b}) = (\alpha\mathbf{a}) \wedge \mathbf{b} = \mathbf{a} \wedge (\alpha\mathbf{b})$
The meaning is that scaling an entire parallelogram by α is equivalent to scaling just one side by α.

(d) $\mathbf{b} \wedge \mathbf{a} = -(\mathbf{a} \wedge \mathbf{b})$
The meaning is that switching the order of the sides of the parallelogram reverses the orientation.
Note that from (d) we get the special case $\mathbf{a} \wedge \mathbf{a} = 0$ for any $\mathbf{a}$.

□

Here is a picture which illustrates the geometric meaning of the addition version of (a) in that the large parallelogram with sides $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$ equals the sum of the small parallelograms with sides $\mathbf{a}$ and $\mathbf{b}$ for the left one and sides $\mathbf{a}$ and $\mathbf{c}$ for the right one.



As a consequence of these axioms it turns out that all outer products of vectors may be rewritten as linear combinations of various outer products of basis vectors, specifically of outer products of the form $\mathbf{e_i} \wedge \mathbf{e_j}$ for $i < j$.

Practical Calculation 0.2. To calculate the outer product of two vectors use these axioms and organize the result as a linear combation of various outer products of basis vectors.

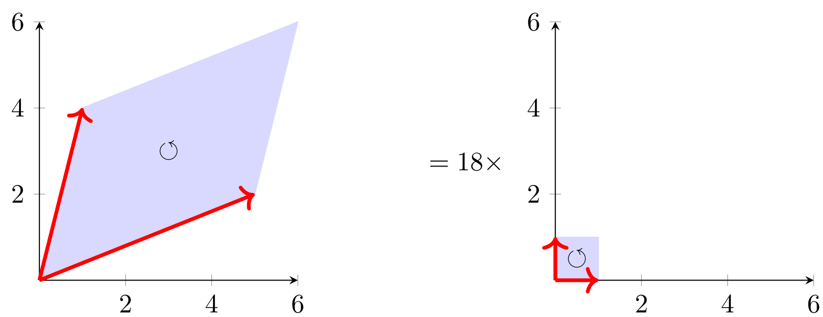
□

Example 2.6. If $\mathbf{a} = 5\mathbf{e_1} + 2\mathbf{e_2}$ and $\mathbf{b} = 1\mathbf{e_1} + 4\mathbf{e_2}$ then

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (5\mathbf{e_1} + 2\mathbf{e_2}) \wedge (1\mathbf{e_1} + 4\mathbf{e_2}) \\
&= ((5\mathbf{e_1} + 2\mathbf{e_2}) \wedge (1\mathbf{e_1})) + ((5\mathbf{e_1} + 2\mathbf{e_2}) \wedge (4\mathbf{e_2})) \\
&= 5(\mathbf{e_1} \wedge \mathbf{e_1}) + 2(\mathbf{e_2} \wedge \mathbf{e_1}) + 20(\mathbf{e_1} \wedge \mathbf{e_2}) + 8(\mathbf{e_2} \wedge \mathbf{e_2}) \\
&= 5(0) - 2(\mathbf{e_1} \wedge \mathbf{e_2}) + 20(\mathbf{e_1} \wedge \mathbf{e_2}) + 8(0) \\
&= 18(\mathbf{e_1} \wedge \mathbf{e_2})
\end{aligned}$$

□

Consider that this example states that the oriented parallelogram formed by **a** and **b** equals 18 oriented squares.



Example 2.7. If $\mathbf{a} = 2\mathbf{e_1} - 3\mathbf{e_2} + 4\mathbf{e_3}$ and $\mathbf{b} = \mathbf{e_1} - 7\mathbf{e_2}$ then

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (2\mathbf{e_1} - 3\mathbf{e_2} + 4\mathbf{e_3}) \wedge (\mathbf{e_1} - 7\mathbf{e_2}) \\
&= 2\mathbf{e_1} \wedge \mathbf{e_1} - 14\mathbf{e_1} \wedge \mathbf{e_2} - 3\mathbf{e_2} \wedge \mathbf{e_1} + 12\mathbf{e_2} \wedge \mathbf{e_2} + 4\mathbf{e_3} \wedge \mathbf{e_1} - 28\mathbf{e_2} \wedge \mathbf{e_2} \\
&= 2(0) - 13\mathbf{e_1} \wedge \mathbf{e_2} + 3\mathbf{e_1} \wedge \mathbf{e_2} + 12(0) - 4\mathbf{e_1} \wedge \mathbf{e_3} - 28(0) \\
&= -10\mathbf{e_1} \wedge \mathbf{e_2} - 4\mathbf{e_1} \wedge \mathbf{e_3}
\end{aligned}$$

□

Exercise 2.2. Let $\mathbf{a} = 2\mathbf{e_1} + 3\mathbf{e_2}$ and $\mathbf{b} = 6\mathbf{e_1} - 7\mathbf{e_2}$. Calculate $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$.

□

Exercise 2.3. Let $\mathbf{a} = 2\mathbf{e_1} + 3\mathbf{e_2} + 8\mathbf{e_3}$ and $\mathbf{b} = -7\mathbf{e_1} - 3\mathbf{e_2} - 2\mathbf{e_3}$. Calculate $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$.

□

It's evident as mentioned earlier that every outer product of two vectors generates a linear combination of various $\mathbf{e_i} \wedge \mathbf{e_j}$.

However we can be more formal and note:

Theorem 2.8.1. In $\mathbb{R}^2$ the outer product of any two vectors equals a multiple of $\mathbf{e_1} \wedge \mathbf{e_2}$. More specifically if $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$ and if θ is the angle from $\mathbf{a}$ to $\mathbf{b}$ then

$$\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta (\mathbf{e_1} \wedge \mathbf{e_2})$$

Since $|\mathbf{a}||\mathbf{b}| \sin\theta$ is the area of the parallelogram that $\mathbf{a}$ and $\mathbf{b}$ create, this theorem basically states that in $\mathbb{R}^2$ each parallelogram equals the number of unit squares equal to its area.

Proof. We have:

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (a_1\mathbf{e_1} + a_2\mathbf{e_2}) \wedge (b_1\mathbf{e_1} + b_2\mathbf{e_2}) \\
&= a_1b_1(\mathbf{e_1} \wedge \mathbf{e_1}) + a_1b_2(\mathbf{e_1} \wedge \mathbf{e_2}) + a_2b_1(\mathbf{e_2} \wedge \mathbf{e_1}) + a_2b_2(\mathbf{e_2} \wedge \mathbf{e_2}) \\
&= a_1b_1(0) + +a_1b_2(\mathbf{e_1} \wedge \mathbf{e_2}) + a_2b_1(-(\mathbf{e_1} \wedge \mathbf{e_2}) + a_2b_2(0) \\
&= (a_1b_2 - a_2b_1)(\mathbf{e_1} \wedge \mathbf{e_2}) \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_2}) \\
&= |\mathbf{a}||\mathbf{b}| \sin\theta (\mathbf{e_1} \wedge \mathbf{e_2})
\end{aligned}$$

□

From this we get:

Corollary 2.8.1. In $\mathbb{R}^2$ every bivector is simply a multiple of $\mathbf{e_1} \wedge \mathbf{e_2}$.

Proof. Apply the above theorem and then combine. □

And:

Theorem 2.8.2. In $\mathbb{R}^3$ the outer product of any two vectors equals a linear combination of $\mathbf{e_1} \wedge \mathbf{e_2}$, $\mathbf{e_2} \wedge \mathbf{e_3}$, and $\mathbf{e_1} \wedge \mathbf{e_3}$. More specifically if $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}$ then

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (a_1b_2 - a_2b_1)(\mathbf{e_1} \wedge \mathbf{e_2}) + (a_2b_3 - a_3b_2)(\mathbf{e_2} \wedge \mathbf{e_3}) + (a_1b_3 - a_3b_1)(\mathbf{e_1} \wedge \mathbf{e_3}) \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_2}) + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} (\mathbf{e_2} \wedge \mathbf{e_3}) + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_3})
\end{aligned}$$

Warning: This looks a bit like a cross product, and is related to a cross product, but isn't a cross product because it's not producing a vector!

From this we get:

Corollary 2.8.2. In $\mathbb{R}^3$ every bivector is simply a linear combination of $\mathbf{e_1} \wedge \mathbf{e_2}$, $\mathbf{e_2} \wedge \mathbf{e_3}$, and $\mathbf{e_1} \wedge \mathbf{e_3}$.

Proof. Apply the above theorem and then combine. □

Example 2.8. For the vectors:

$$\mathbf{a} = 2\mathbf{e_1} + 3\mathbf{e_2} + 4\mathbf{e_3}$$
$$\mathbf{b} = 7\mathbf{e_1} + 1\mathbf{e_2} + 5\mathbf{e_3}$$

We have:

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} &= \begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_2}) + \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} (\mathbf{e_2} \wedge \mathbf{e_3}) + \begin{vmatrix} 2 & 4 \\ 7 & 5 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_3}) \\ &= -19(\mathbf{e_1} \wedge \mathbf{e_2}) + 11(\mathbf{e_2} \wedge \mathbf{e_3}) - 18(\mathbf{e_1} \wedge \mathbf{e_3})\end{aligned}$$

□

Theorem 2.8.3. As a result of the above we have:

$$\wedge^2\mathbb{R}^2 = \text{span}\{\mathbf{e_1} \wedge \mathbf{e_2}\}$$
$$\wedge^2\mathbb{R}^3 = \text{span}\{\mathbf{e_1} \wedge \mathbf{e_2}, \mathbf{e_2} \wedge \mathbf{e_3}, \mathbf{e_1} \wedge \mathbf{e_3}\}$$

□

2.9 The Outer Product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ Revisited

For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we defined the outer product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the oriented parallelepipid with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. What we will do next is introduce some axioms which make geometric sense but have unexpected consequences. These axioms are similar to those for the outer product of two vectors so we've been less verbose.

Axiom 2.9.1. We insist on the following axioms.

(a) The outer product distributes over addition and subtraction.

(b) Multiplication by a scalar is associative.

(c) Interchanging two adjacent terms negates the outer product of three vectors, for example $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -\mathbf{a} \wedge \mathbf{c} \wedge \mathbf{b}$.

□

Invoking the same axioms above we extend the triple outer product to arbitrary vectors and we get the following in $\mathbb{R}^3$.

Theorem 2.9.1. In $\mathbb{R}^3$ the outer product of any three vectors equals a multiple of $\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$. More formally if $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}$ and $\mathbf{c} = c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3}$ then

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3})$$

Proof. Omitted. $\square$

Practical Calculation 0.3. Given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, to calculate $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ write $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as linear combination of $\mathbf{e_i}$ and apply the above axioms. The result will be a linear combination of various $\mathbf{e_i} \wedge \mathbf{e_j} \wedge \mathbf{e_k}$.

$\square$

Exercise 2.4. If $\mathbf{a} = 3\mathbf{e_2} - 2\mathbf{e_2} + 0\mathbf{e_3}$, $\mathbf{b} = 1\mathbf{e_1} + 2\mathbf{e_2} + 4\mathbf{e_3}$, and $\mathbf{c} = 6\mathbf{e_1} - 1\mathbf{e_2} + 7\mathbf{e_3}$ calculate all possible triple outer products.
Hint: Up to interchange/sign how many are there?

$\square$

Theorem 2.9.2. As a result of the above we have:

$$\wedge^3\mathbb{R}^3 = \operatorname{span}\{\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}\}$$

$\square$

3 The Geometric Product

Definition 3.0.1. We define the geometric product of multivectors as follows:

(a) For any set of distinct basis vectors the geometric product equals the outer product.

(b) $\mathbf{e_i}\mathbf{e_i} = 1$

(c) $\mathbf{e_j}\mathbf{e_i} = -\mathbf{e_i}\mathbf{e_j}$

(d) The geometric product is associative.

(e) The geometric product distributes over addition and subtraction.

$\square$

Example 3.1. Here are some simple examples of how these play out in application:

(a) We have $\mathbf{e_1}\mathbf{e_2} = \mathbf{e_1} \wedge \mathbf{e_2}$ and $\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} = \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$.

(b) We have $\mathbf{e_1}\mathbf{e_1} = 1$.

(c) We have $\mathbf{e_2}\mathbf{e_1} = -\mathbf{e_1}\mathbf{e_2}$.

(d) We have $\mathbf{e_1}\mathbf{e_2}\mathbf{e_1}\mathbf{e_3} = \mathbf{e_1}(\mathbf{e_2}\mathbf{e_1})\mathbf{e_3} = -\mathbf{e_1}(\mathbf{e_1}\mathbf{e_2})\mathbf{e_3} = -\mathbf{e_2}\mathbf{e_3}$.

(e) We have $(3\mathbf{e_1} + 2\mathbf{e_2})(4\mathbf{e_1} - 5\mathbf{e_3}) = 12\mathbf{e_1}\mathbf{e_1} - 15\mathbf{e_1}\mathbf{e_2} + 8\mathbf{e_2}\mathbf{e_1} - 10\mathbf{e_2}\mathbf{e_3} = 12 - 12\mathbf{e_1}\mathbf{e_2} - 10\mathbf{e_1}\mathbf{e_3}$.

$\square$

Before doing a few more examples, note that the rules of the geometric product allow us to rewrite all multivectors and products of multivectors as linear combinations of geometric products of basis vectors.

In other words, comprehensively we have:

$$\begin{aligned}\wedge^0\mathbb{R}^2 &= \text{span}\{1\}\\ \wedge^1\mathbb{R}^2 &= \text{span}\{\mathbf{e_1},\mathbf{e_2}\}\\ \wedge^2\mathbb{R}^2 &= \text{span}\{\mathbf{e_1}\mathbf{e_2}\}\end{aligned}$$

And we have:

$$\begin{aligned}\wedge^0\mathbb{R}^3 &= \text{span}\{1\}\\ \wedge^1\mathbb{R}^3 &= \text{span}\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}\\ \wedge^2\mathbb{R}^3 &= \text{span}\{\mathbf{e_1}\mathbf{e_2},\mathbf{e_2}\mathbf{e_3},\mathbf{e_1}\mathbf{e_3}\}\\ \wedge^3\mathbb{R}^3 &= \text{span}\{\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}\}\end{aligned}$$

Here is an example with two vectors:

Example 3.2. If $\mathbf{a} = 2\mathbf{e_1} - 3\mathbf{e_2} + 4\mathbf{e_3}$ and $\mathbf{b} = \mathbf{e_1} - 7\mathbf{e_2}$ then

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= (2\mathbf{e_1} - 3\mathbf{e_2} + 4\mathbf{e_3})\wedge(\mathbf{e_1} - 7\mathbf{e_2})\\ &= 2\mathbf{e_1}\wedge\mathbf{e_1} - 14\mathbf{e_1}\wedge\mathbf{e_2} - 3\mathbf{e_2}\wedge\mathbf{e_1} + 21\mathbf{e_2}\wedge\mathbf{e_2} + 4\mathbf{e_3}\wedge\mathbf{e_1} - 28\mathbf{e_3}\wedge\mathbf{e_2}\\ &= 2(0) - 14\mathbf{e_1}\wedge\mathbf{e_2} + 3\mathbf{e_1}\wedge\mathbf{e_2} + 21(0) - 4\mathbf{e_1}\wedge\mathbf{e_3} + 28\mathbf{e_2}\wedge\mathbf{e_3}\\ &= -11\mathbf{e_1}\mathbf{e_2} + 28\mathbf{e_2}\mathbf{e_3} - 4\mathbf{e_1}\mathbf{e_3}\end{aligned}$$

Here is an example with a vector and a bivector:

Example 3.3. Suppose $A = 2\mathbf{e_1} + \mathbf{e_3}$ and $B = 3\mathbf{e_1}\mathbf{e_2} - 5\mathbf{e_2}\mathbf{e_3}$ then:

$$\begin{aligned}AB &= (2\mathbf{e_1} + \mathbf{e_3})(3\mathbf{e_1}\mathbf{e_2} - 5\mathbf{e_2}\mathbf{e_3})\\ &= 6\mathbf{e_1}\mathbf{e_1}\mathbf{e_2} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 3\mathbf{e_3}\mathbf{e_1}\mathbf{e_2} - 5\mathbf{e_3}\mathbf{e_2}\mathbf{e_3}\\ &= 6\mathbf{e_2} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 3\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 5\mathbf{e_2}\\ &= 11\mathbf{e_2} - 7\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}\end{aligned}$$

$\square$

Here is an example with two multivectors:

Example 3.4. If $A = 2 + 3\mathbf{e_1} + \mathbf{e_1e_2}$ and $B = 4 + \mathbf{e_1} - 2\mathbf{e_2}$ then:

$$\begin{aligned}
AB &= (2 + 3\mathbf{e_1} + \mathbf{e_1e_2})(4 + \mathbf{e_1} - 2\mathbf{e_2}) \\
&= 2(4 + \mathbf{e_1} - 2\mathbf{e_2}) + 3\mathbf{e_1}(4 + \mathbf{e_1} - 2\mathbf{e_2}) + \mathbf{e_1e_2}(4 + \mathbf{e_1} - 2\mathbf{e_2}) \\
&= 8 + 2\mathbf{e_1} - 4\mathbf{e_2} + 12\mathbf{e_1} + 3\mathbf{e_1e_1} - 6\mathbf{e_1}\mathrm{e}_2 + 4\mathbf{e_1e_2} + \mathbf{e_1e_2e_1} - 2\mathbf{e_1e_2e_2} \\
&= 8 + 2\mathbf{e_1} - 4\mathbf{e_2} + 12\mathbf{e_1} + 3 - 6\mathbf{e_1}\mathrm{e}_2 + 4\mathbf{e_1e_2} - \mathbf{e_2} - 2\mathbf{e_1} \\
&= 8 + 12\mathbf{e_1} - 5\mathbf{e_2} - 2\mathbf{e_1e_2}
\end{aligned}$$

□

Here is an example where we have to do a lot more initial work. We are given bivectors A and B and we have to rewrite them first:

Example 3.5. Suppose $A = (2\mathbf{e_1} + 3\mathbf{e_2}) \wedge (\mathbf{e_2} - \mathbf{e_3})$ and $B = 4\mathbf{e_1} \wedge (\mathbf{e_2} + 2\mathbf{e_3})$. Before proceeding with any calculations we write:

$$\begin{aligned}
A &= 2\mathbf{e_1} \wedge \mathbf{e_2} - 2\mathbf{e_1} \wedge \mathbf{e_3} + 3\mathbf{e_2} \wedge \mathbf{e_2} - 3\mathbf{e_2} \wedge \mathbf{e_3} \\
&= 2\mathbf{e_1e_2} - 2\mathbf{e_1e_3} + 3(0) - 3\mathbf{e_2e_3} \\
&= 2\mathbf{e_1e_2} - 2\mathbf{e_1e_3} - 3\mathbf{e_2e_3} \\
&= 2\mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 2\mathbf{e_1e_3}
\end{aligned}$$

And we write:

$$\begin{aligned}
B &= 4\mathbf{e_1} \wedge \mathbf{e_2} + 8\mathbf{e_1} \wedge \mathbf{e_3} \\
&= 4\mathbf{e_1e_2} + 8\mathbf{e_1e_3}
\end{aligned}$$

Then for example we can do AB easily:

$$\begin{aligned}
AB &= (2\mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 2\mathbf{e_1e_3})(4\mathbf{e_1e_2} + 8\mathbf{e_1e_3}) \\
&= 2\mathbf{e_1e_2}(4\mathbf{e_1e_2} + 8\mathbf{e_1e_3}) - 3\mathbf{e_2e_3}(4\mathbf{e_1e_2} + 8\mathbf{e_1e_3}) - 2\mathbf{e_1e_3}(4\mathbf{e_1e_2} + 8\mathbf{e_1e_3}) \\
&= 8\mathbf{e_1e_2e_1e_2} + 16\mathbf{e_1e_2e_1e_3} - 12\mathbf{e_2e_3e_1e_2} - 24\mathbf{e_2e_3e_1e_3} - 8\mathbf{e_1e_3e_1e_2} - 16\mathbf{e_1e_3e_1e_3} \\
&= -8 - 16\mathbf{e_2e_3} - 12\mathbf{e_3e_1} - 24\mathbf{e_1e_2} - 8\mathbf{e_2e_3} - 16 \\
&= -24 - 24\mathbf{e_1e_2} - 24\mathbf{e_2e_3} + 12\mathbf{e_1e_3}
\end{aligned}$$

□

Exercise 3.1. Let $A = 4 + \mathbf{e_1e_3}$ and $B = 3\mathbf{e_1} + \mathbf{e_2}$. Calculate AB and BA.

□

Exercise 3.2. Let $A = 4 - 1\mathbf{e_3} + 2\mathbf{e_1e_2} + 3\mathbf{e_1e_3}$ and $B = 5 + 3\mathbf{e_1} + \mathbf{e_2e_3} + \mathbf{e_1e_2e_3}$. Calculate AB and BA.

□

We also get the following theorem:

Theorem 3.0.1. For vectors $\mathbf{a}, \mathbf{b}$ we have:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

Proof. If we have:

$$\mathbf{a} = \sum_{i=1}^{n} \alpha_i \mathbf{e_i} \text{ and } \mathbf{b} = \sum_{i=1}^{n} \beta_i \mathbf{e_i}$$

Then we have:

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} &= \sum_{i=1}^{n} \alpha_i \beta_i + \left(\sum_{i=1}^{n} \alpha_i \mathbf{e_i} \right) \wedge \left(\sum_{i=1}^{n} \beta_i \mathbf{e_i} \right) \\
&= \sum_{i=1}^{n} \alpha_i \beta_i + \sum_{i,j} \alpha_i \beta_j \mathbf{e_i} \wedge \mathbf{e_j} \\
&= \sum_{i=1}^{n} \alpha_i \beta_i + \sum_{i=j} \alpha_i \beta_j \mathbf{e_i} \wedge \mathbf{e_j} + \sum_{i \neq j} \alpha_i \beta_j \mathbf{e_i} \wedge \mathbf{e_j} \\
&= \sum_{i=1}^{n} \alpha_i \beta_i + \sum_{i=j} \alpha_i \beta_j (0) + \sum_{i \neq j} \alpha_i \beta_j \mathbf{e_i} \mathbf{e_j} \\
&= \sum_{i=1}^{n} \alpha_i \beta_i \mathbf{e_i} \mathbf{e_i} + \sum_{i \neq j} \alpha_i \beta_j \mathbf{e_i} \mathbf{e_j} \\
&= \sum_{i,j} \alpha_i \beta_j \mathbf{e_i} \mathbf{e_j} \\
&= \left(\sum_{i=1}^{n} \alpha_i \mathbf{e_i} \right) \left(\sum_{i=1}^{n} \beta_i \mathbf{e_i} \right) \\
&= \mathbf{ab}
\end{aligned}$$

□

From this theorem we can then also define the inner and outer products in terms of the geometric product.

Theorem 3.0.2. For vectors $\mathbf{a}$ and $\mathbf{b}$ we have:

(a) $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$

(b) $\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$

Proof. This follows immediately from:

(a) $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$

(b) $\mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$

□

We also get:

Theorem 3.0.3. For a vector $\mathbf{v}$ we have $\mathbf{vv} = \mathbf{v} \cdot \mathbf{v}$.

Proof. Since $\mathbf{v} \wedge \mathbf{v} = 0$ the result follows. $\square$

4 Additional Definitions

4.1 Grade Extraction from a Multivector

Definition 4.1.1. For a multivector A and for $k \in \mathbb{Z}$ we use the notation $\langle A \rangle_k$ to denote the extraction of the k-vector from A. If $k < 0$ the result will be 0.

$\square$

Example 4.1. If $A = 2 + 3\mathbf{e_1} - 1\mathbf{e_2} + 5\mathbf{e_1 e_2} + 3\mathbf{e_1 e_3} + 7\mathbf{e_1 e_2 e_3}$ then:

(a) $\langle A \rangle_0 = 2$

(b) $\langle A \rangle_1 = 3\mathbf{e_1} - 1\mathbf{e_2}$

(c) $\langle A \rangle_2 = 5\mathbf{e_1 e_2} + 3\mathbf{e_1 e_3}$

(d) $\langle A \rangle_3 = 7\mathbf{e_1 e_2 e_3}$

(e) $\langle A \rangle_4 = \langle A \rangle_5 = ... = 0$

(f) $\langle A \rangle_{-1} = \langle A \rangle_{-2} = ... = 0$

$\square$

4.2 Blades

Definition 4.2.1. A *k-blade* B is the outer product of k vectors. Here k may be 0 (a scalar) and 1 (a single vector) as well.

$\square$

Example 4.2. The following are blades:

(a) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is a 3-blade.

(b) $\mathbf{e_1 e_2} = \mathbf{e_1} \wedge \mathbf{e_2}$ is a 2-blade.

(c) $\mathbf{a}$ is a 1-blade.

(d) 7 is a 0-blade.

It's not clear that all bivectors are blades. For example in $\mathbb{R}^3$ clearly a single outer product of two vectors is a 2-blade but is the sum of two or more outer products of two vectors a 2-blade?

$\square$

Given $\mathbf{v_1}, ..., \mathbf{v_k}$ with $\mathbf{v_i} \in \mathbb{R}^n$ the k-blade $B = \mathbf{v_1} \wedge ... \wedge \mathbf{v_k}$ will be thought of as representing the subspace: span $\{\mathbf{v_1}, ..., \mathbf{v_k}\}$.

Example 4.3. In $\mathbb{R}^3$ the blade $\mathbf{e_1 e_2} = \mathbf{e_1} \wedge \mathbf{e_2}$ will be thought of as representing the k-dimensional subspace: span $\{\mathbf{e_1}, \mathbf{e_2}\}$, essentially meaning the xy-plane.

$\square$

Keep in mind that the blade is not equal to the subspace but rather that it will be thought of as representing it. The reason for packaging the vectors together in an outer product rather than just as a set is that it creates a single entity.

Note that any given subspace is represented by many different blades.

Example 4.4. In $\mathbb{R}^3$ the two blades $B_1 = \mathbf{e_1} \wedge \mathbf{e_2}$ and $B_2 = (\mathbf{e_1} + \mathbf{e_2}) \wedge (\mathbf{e_1} - \mathbf{e_2})$ represent the same suspace of $\mathbb{R}^3$, that being the set span$\{\mathbf{e_1}, \mathbf{e_2}\}$.

$\square$

Theorem 4.2.1. In $\mathbb{R}^2$ and $\mathbb{R}^3$ all bivectors are 2-blades.

Take a moment to appreciate what this theorem states, primarily in $\mathbb{R}^3$. It states that any linear combination of outer products of two vectors equals a single outer product of two vectors. This will be extremely useful when we are proving things about bivectors.

Proof. In $\mathbb{R}^2$ observe that all bivectors have the form $\lambda \mathbf{e_1 e_2}$ for $\lambda \in \mathbb{R}$ and note that:

$$\lambda \mathbf{e_1 e_2} = \lambda(\mathbf{e_1} \wedge \mathbf{e_2}) = (\lambda \mathbf{e_1}) \wedge \mathbf{e_2}$$

and we're done.

In $\mathbb{R}^3$ we know every bivector has the form $\alpha \mathbf{e_1 e_2} + \beta \mathbf{e_2 e_3} + \gamma \mathbf{e_1 e_3}$ for $\alpha, \beta, \gamma \in \mathbb{R}$.

If $\beta = \gamma = 0$ then we're done. Otherwise $\beta^2 + \gamma^2 \neq 0$ and observe that if we assign:

$$\begin{aligned}
\mathbf{a} &= \frac{1}{\beta^2 + \gamma^2}(-\alpha\beta \mathbf{e_1} + \alpha\gamma \mathbf{e_2} + (\beta^2 + \gamma^2)\mathbf{e_3}) \\
\mathbf{b} &= -\gamma \mathbf{e_1} - \beta \mathbf{e_2} + 0\mathbf{e_3}
\end{aligned}$$

Then we have:

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= \frac{1}{\beta^2 + \gamma^2}(-\alpha\beta \mathbf{e_1} + \alpha\gamma \mathbf{e_2} + (\beta^2 + \gamma^2)\mathbf{e_3}) \wedge (-\gamma \mathbf{e_1} - \beta \mathbf{e_2} + 0\mathbf{e_3}) \\
&= \frac{1}{\beta^2 + \gamma^2}\left[(\alpha\beta^2 + \alpha\gamma^2)\mathbf{e_1 e_2} + (\beta(\beta^2 + \gamma^2))\mathbf{e_2 e_3} + (\gamma(\beta^2 + \gamma^2))\mathbf{e_1 e_3}\right] \\
&= \alpha \mathbf{e_1 e_2} + \beta \mathbf{e_2 e_3} + \gamma \mathbf{e_1 e_3}
\end{aligned}$$

Note that it's fairly clear from the nature of the proof that there are multiple ways to choose/construct $\mathbf{a}$ and $\mathbf{b}$.

The fact that the bivector is a 2-blade follows immediately from the definition of a 2-blade.

$\square$

In friendlier terms in $\mathbb{R}^3$ bivectors are the same as parallelograms and every linear combination of parallelograms equals a parallelogram.

This is computationally important because it allows us to use the term bivector and the expression $\mathbf{a} \wedge \mathbf{b}$ interchangeably. We will do this constantly.

Exercise 4.1. Write the bivector $B = 2\mathbf{e_1}\mathbf{e_2} + 5\mathbf{e_2}\mathbf{e_3}$ as a 2-blade.

$\square$

Exercise 4.2. Write the bivector $B = 2\mathbf{e_1}\mathbf{e_2} + 5\mathbf{e_2}\mathbf{e_3} - 3\mathbf{e_3}\mathbf{e_1}$ as a 2-blade.

$\square$

Note: $\mathbb{R}^3$ is the highest dimension in which this is true. For example in $\mathbb{R}^4$ the bivector $\mathbf{e_1}\mathbf{e_2} + \mathbf{e_3}\mathbf{e_4}$ cannot be written as a blade. This is not obvious.

4.3 Norm

Definition 4.3.1. The *norm of a multivector* A is defined as the square root of the sum of the squares of the coefficients of the basis k-vectors making up A. This corresponds with the regular definition of the norm when we look at vectors, and the absolute value when we look at scalars.

$\square$

Example 4.5. If $A = 2 + 3\mathbf{e_1} + 4\mathbf{e_2} - 5\mathbf{e_1}\mathbf{e_2}$ then:

$$|A| = \sqrt{4 + 9 + 16 + 25} = \sqrt{54}$$

$\square$

If a multivector is not written this way then we must expand it first.

4.4 Reversion

Computation using multivectors (specifically blades) is something we will be doing. It is made easier by the introduction of a concept called reversion. Reversion essentially plays an analogous role to conjugation.

Definition 4.4.1. We define *reversion*, denoted by † (sometimes by $\sim$), by the following rules:

(a) For a scalar α we put $\alpha^\dagger = \alpha$

(b) For a vector $\mathbf{a}$ we put $\mathbf{a}^\dagger = \mathbf{a}$

(c) For multivectors A, B we put $(AB)^\dagger = B^\dagger A^\dagger$

(d) For multivectors A, B we put $(A + B)^\dagger = A^\dagger + B^\dagger$

□

Note 4.4.1. Note that for a bivector B we have $B^\dagger = -B$ because a bivector is a sum of $\mathbf{e_i e_j}$ and that sum will be negated.

Practical Calculation 0.4. To calculate $A^\dagger$ use the formulas above. These formulas cannot inherently manage outer products so we must rewrite first.

□

Example 4.6. For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we have:

$$(\mathbf{ab} + \mathbf{c})^\dagger = (\mathbf{ab})^\dagger + \mathbf{c}^\dagger = \mathbf{ba} + \mathbf{c}$$

□

Example 4.7. Consider $B = 2\mathbf{e_1} \wedge (\mathbf{e_2} + 6\mathbf{e_3})$. We rewrite the original B as $B = 2\mathbf{e_1 e_2} + 12\mathbf{e_1 e_3}$ and then:

$$\begin{aligned} B^\dagger &= (2\mathbf{e_1 e_2})^\dagger + (12\mathbf{e_1 e_3})^\dagger \\ &= 2\mathbf{e_2 e_1} + 12\mathbf{e_3 e_1} \\ &= -2\mathbf{e_1 e_2} - 12\mathbf{e_1 e_3} \end{aligned}$$

□

Exercise 4.3. Find the reversion of each of the following:

(a) $\mathbf{abc} + \mathbf{b} + \mathbf{ac}$

(b) $5\mathbf{e_1 e_2 e_3} - 2\mathbf{e_1 e_2}$

(c) $5\mathbf{e_2} \wedge (\mathbf{e_1} + 6\mathbf{e_3})$

□

4.5 Connection Between Norm and Reversion

Reversion plays a similar role to conjugation in the following sense:

Theorem 4.5.1. For any multivector A we have:

$$\left\langle AA^\dagger \right\rangle_0 = |A|^2$$

Proof. If we write A as a linear combination of geometric products of basis vectors then $A^\dagger$ reverses the order of those gometric products. When we multiply $AA^\dagger$ the only constant terms emerging from the products will emerge from the multiplication of a summand in the multivector with its reversion, since all the other multiplications will yield residual basis vectors. Those emerging constant terms will be the squares of the coefficients and the sum of those is precisely the square of the norm. □

Note 4.5.1. Note that if $AA^\dagger$ is a scalar then we get $AA^\dagger = |A|^2$. This appears in two special cases we will need.

Theorem 4.5.2. For $v \in \wedge^1 \mathbb{R}^n$ we have:

$$\mathbf{v}\mathbf{v}^\dagger = |\mathbf{v}|^2$$

Proof. Observe that $\mathbf{v}\mathbf{v}^\dagger$ is a scalar:

$$\mathbf{v}\mathbf{v}^\dagger = \mathbf{v}\mathbf{v} = \mathbf{v} \cdot \mathbf{v}$$

The result follows. □

Theorem 4.5.3. For $B \in \wedge^2 \mathbb{R}^3$ we have:

$$BB^\dagger = |B|^2$$

Proof. Observe that $BB^\dagger$ is a scalar. If we set:

$$B = \alpha \mathbf{e_1 e_2} + \beta \mathbf{e_2 e_3} + \gamma \mathbf{e_1 e_3}$$

Then we have:

$$\begin{aligned}
BB^\dagger &= -\left(\alpha \mathbf{e_1 e_2} + \beta \mathbf{e_2 e_3} + \gamma \mathbf{e_1 e_3}\right)\left(-\alpha \mathbf{e_1 e_2} - \beta \mathbf{e_2 e_3} - \gamma \mathbf{e_1 e_3}\right) \\
&= -\alpha^2 \mathbf{e_1 e_2 e_1 e_2} - \alpha\beta \mathbf{e_1 e_2 e_2 e_3} - \alpha\gamma \mathbf{e_1 e_2 e_1 e_3} \\
&- \alpha\beta \mathbf{e_2 e_3 e_1 e_2} - \beta^2 \mathbf{e_2 e_3 e_2 e_3} - \beta\gamma \mathbf{e_2 e_3 e_1 e_3} \\
&- \alpha\gamma \mathbf{e_1 e_3 e_1 e_2} - \beta\gamma \mathbf{e_1 e_3 e_2 e_3} - \gamma^2 \mathbf{e_1 e_3 e_1 e_3} \\
&= \alpha^2 - \alpha\beta \mathbf{e_1 e_3} + \alpha\gamma \mathbf{e_2 e_3} \\
&+ \alpha\beta \mathbf{e_1 e_3} + \beta^2 - \beta\gamma \mathbf{e_1 e_2} \\
&- \alpha\gamma \mathbf{e_2 e_3} + \beta\gamma \mathbf{e_1 e_2} + \gamma^2 \\
&= \alpha^2 + \beta^2 + \gamma^2
\end{aligned}$$

The result follows. □

4.6 Inversion

Definition 4.6.1. We say that a multivector A is *invertible* if there is some multivector B with $AB = BA = 1$. We write A^{-1} in place of B. In this case the multivectors are the *inverse* of one another.

□

Theorem 4.6.1. For a nonzero vector $\mathbf{v}$ we have:

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{|\mathbf{v}|^2}$$

Proof. We have:

$$\mathbf{v}\mathbf{v} = |\mathbf{v}|^2$$

And so:

$$\mathbf{v}\left(\frac{\mathbf{v}}{|\mathbf{v}|^2}\right) = 1$$

The result follows. □

And:

Theorem 4.6.2. For a nonzero bivector B we have:

$$B^{-1} = -\frac{B}{|B|^2}$$

Proof. From earlier we have:

$$BB^\dagger = |B|^2$$

However $B^\dagger = -B$ and hence:

$$B(-B) = |B|^2$$

And so:

$$B\left(-\frac{B}{|B|^2}\right) = 1$$

The result follows. □

Example 4.8. Consider $B = 2\mathbf{e_1} \wedge (\mathbf{e_2} + 6\mathbf{e_3})$. We rewrite $B = 2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}$ and then we have:

$$\begin{aligned} B^{-1} &= -\frac{B}{|B|^2} \\ &= -\frac{2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}}{|2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}|} \\ &= \frac{-2\mathbf{e_1}\mathbf{e_2} - 12\mathbf{e_1}\mathbf{e_3}}{(2)^2 + (12)^2} \\ &= \frac{-2\mathbf{e_1}\mathbf{e_2} - 12\mathbf{e_1}\mathbf{e_3}}{148} \end{aligned}$$

□

Exercise 4.4. Find the inverse of each of the following multivectors:

(a) $5\mathbf{e_1}\mathbf{e_2}$

(b) $(2\mathbf{e_1} + 3\mathbf{e_2}) \wedge (-5\mathbf{e_1} + 4\mathbf{e_3})$

□

4.7 Inner and Outer Products of Multivectors

We defined the geometric product of multivectors by extending the geometric product of vectors which itself was based on the definition of the inner and outer products of vectors.

What we will do now is use the geometric product of multivectors to define the inner and outer products of multivectors.

Definition 4.7.1. If A is a j-multivector and B is a k-multivector then:

$$A \cdot B = \langle AB \rangle_{k-j}$$
$$A \wedge B = \langle AB \rangle_{k+j}$$

□

It's critical to note that the dot product of two multivectors does not necessarily yield a constant. Rather it is a grade-lowering operation. Similarly the outer product is a grade-raising operation.

Practical Calculation 0.5. To calculate either of these use the definition.

□

Example 4.9. Consider $\mathbf{a} = 7\mathbf{e_1} - 5\mathbf{e_3}$ and $B = 2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}$. Since $\mathbf{a}$ is a $j = 1$-vector and B is a $k = 2$ vector it follows that $\mathbf{a} \cdot B$ is a $2 - 1 = 1$-vector and $\mathbf{a} \wedge B$ is a $2 + 1 = 3$-vector. We have:

$$\begin{aligned}
\mathbf{a}B &= (7\mathbf{e_1} - 5\mathbf{e_3})(2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}) \\
&= 14\mathbf{e_1}\mathbf{e_1}\mathbf{e_2} + 84\mathbf{e_1}\mathbf{e_1}\mathbf{e_3} - 10\mathbf{e_3}\mathbf{e_1}\mathbf{e_2} - 60\mathbf{e_3}\mathbf{e_1}\mathbf{e_3} \\
&= 14\mathbf{e_2} + 84\mathbf{e_3} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 60\mathbf{e_1} \\
&= 60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}
\end{aligned}$$

Hence we have:

$$\mathbf{a} \cdot B = \left\langle 60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} \right\rangle_1 = 60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3}$$

and

$$\mathbf{a} \wedge B = \left\langle 60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} \right\rangle_3 = -10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}$$

□

Exercise 4.5. Given $\mathbf{a} = 2\mathbf{e_1} + 5\mathbf{e_2}$ and $B = -4\mathbf{e_3}\mathbf{e_1} + \mathbf{e_1}\mathbf{e_2}$, find $\mathbf{a}B$, $B\mathbf{a}$, $\mathbf{a} \cdot B$, $\mathbf{a} \wedge B$, $B \cdot \mathbf{a}$, and $B \wedge \mathbf{a}$.

□

First note that we must ensure that this is compatible with these products for vectors as given initially.

Theorem 4.7.1. If $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}$ then:

$$\begin{aligned}
&\left\langle \mathbf{ab} \right\rangle_0 = a_1b_1 + a_2b_2 + a_3b_3 \\
&\left\langle \mathbf{ab} \right\rangle_2 = (a_1b_2 - a_2b_1)(\mathbf{e_1}\mathbf{e_2}) + (a_2b_3 - a_3b_2)(\mathbf{e_2}\mathbf{e_3}) + (a_1b_3 - a_3b_1)(\mathbf{e_1}\mathbf{e_3})
\end{aligned}$$

Proof. Observe that:

$$\begin{aligned}
\mathbf{ab} &= (a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3})(b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}) \\
&= a_1b_1 + a_1b_2\mathbf{e_1}\mathbf{e_2} + a_1b_3\mathbf{e_1}\mathbf{e_3} \\
&- a_2b_1\mathbf{e_1}\mathbf{e_2} + a_2b_2 + a_2b_3\mathbf{e_2}\mathbf{e_3} \\
&- a_3b_1\mathbf{e_1}\mathbf{e_3} - a_3b_2\mathbf{e_2}\mathbf{e_3} + a_3b_3 \\
&= a_1b_1 + a_2b_2 + a_3b_3 \\
&+ (a_1b_2 - a_2b_1)\mathbf{e_1}\mathbf{e_2} + (a_2b_3 - a_3b_2)\mathbf{e_2}\mathbf{e_3} + (a_1b_3 - a_3b_1)\mathbf{e_1}\mathbf{e_3}
\end{aligned}$$

And the results follow immediately. □

There is one special case of this which will be useful as we proceed. Notice that the following is different (the signs) from the case of two vectors.

Theorem 4.7.2. If $\mathbf{a} \in \wedge^1\mathbb{R}^3$ and $B \in \wedge^2\mathbb{R}^3$ then:

$$\begin{aligned}
\mathbf{a} \cdot B &= \frac{1}{2}(\mathbf{a}B - B\mathbf{a}) \\
\mathbf{a} \wedge B &= \frac{1}{2}(\mathbf{a}B + B\mathbf{a})
\end{aligned}$$

Proof. We know that $B = \alpha\mathbf{e_1}\mathbf{e_2} + \beta\mathbf{e_2}\mathbf{e_3} + \gamma\mathbf{e_3}\mathbf{e_1}$ for $\alpha, \beta, \gamma \in \mathbb{R}$. Observe that for $\mathbf{e_1}$ we have:

$$\begin{aligned}
\mathbf{e_1}B &= \mathbf{e_1}(\alpha\mathbf{e_1}\mathbf{e_2} + \beta\mathbf{e_2}\mathbf{e_3} + \gamma\mathbf{e_3}\mathbf{e_1}) = \alpha\mathbf{e_2} + \beta\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} - \gamma\mathbf{e_3}) \\
B\mathbf{e_1} &= (\alpha\mathbf{e_1}\mathbf{e_2} + \beta\mathbf{e_2}\mathbf{e_3} + \gamma\mathbf{e_3}\mathbf{e_1})\mathbf{e_1} = -\alpha\mathbf{e_2} + \beta\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + \gamma\mathbf{e_3})
\end{aligned}$$

Then note:

$$\frac{1}{2}(\mathbf{e_1}B - B\mathbf{e_1}) = \alpha\mathbf{e_1} - \gamma\mathbf{e_3} = \left\langle \mathbf{e_1}B \right\rangle_1 = \mathbf{e_1} \cdot B$$

and:

$$\frac{1}{2}(\mathbf{e_1}B + B\mathbf{e_1}) = \beta\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} = \left\langle \mathbf{e_1}B \right\rangle_3 = \mathbf{e_1} \wedge B$$

Similar results hold for $\mathbf{e_2}$ and $\mathbf{e_3}$, work omitted. Then for a general vector $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$ we have:

$$\mathbf{a}B = (a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3})B = a_1\mathbf{e_1}B + a_2\mathbf{e_2}B + a_3\mathbf{e_3}B$$

and then:

$$\begin{aligned}
\mathbf{a} \cdot B &= \langle \mathbf{a}B \rangle_1 \\
&= \langle a_1\mathbf{e_1}B + a_2\mathbf{e_2}B + a_3\mathbf{e_3}B \rangle_1 \\
&= a_1 \langle \mathbf{e_1}B \rangle_1 + a_2 \langle \mathbf{e_2}B \rangle_1 + a_1 \langle \mathbf{e_3}B \rangle_1 \\
&= a_1 \frac{1}{2}(\mathbf{e_1}B - B\mathbf{e_1}) + a_2 \frac{1}{2}(\mathbf{e_2}B - B\mathbf{e_2}) + a_3 \frac{1}{2}(\mathbf{e_3}B - B\mathbf{e_3}) \\
&= \frac{1}{2}\left((a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3})B - B(a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3})\right) \\
&= \frac{1}{2}(\mathbf{a}B - B\mathbf{a})
\end{aligned}$$

And similarly for $\mathbf{a} \wedge B$. □

Corollary 4.7.1. It follows immediately (by adding the two) that for a vector $\mathbf{a}$ and a bivector B that we have:

$$\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B$$

Exercise 4.6. Show the similar results for $\mathbf{e_2}$ and $\mathbf{e_3}$.

□

4.8 Dual of a Multivector

Definition 4.8.1. Given a multivector A we define the *dual* of A, denoted A^*, as:

$$A^* = AI^{-1}$$

where $I = \mathbf{e_1e_2}$ so $I^{-1} = \mathbf{e_2e_1}$ in $\mathbb{R}^2$ and $I = \mathbf{e_1e_2e_3}$ so $I^{-1} = \mathbf{e_3e_2e_1}$ in $\mathbb{R}^3$.

□

Example 4.10. For example in $\mathbb{R}^2$ if $A = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ then

$$A^* = (a_1\mathbf{e_1} + a_2\mathbf{e_2})\mathbf{e_2e_1} = a_2\mathbf{e_1} - a_1\mathbf{e_2}$$

□

Exercise 4.7. Find the duals of the following multivectors.

(a) $2\mathbf{e_3e_1} + 8\mathbf{e_1e_2}$

(b) $3 + \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_1e_2} - \mathbf{e_2e_3} + 3\mathbf{e_1e_2e_3}$

(c) $(3\mathbf{e_1} + 2\mathbf{e_3}) \wedge (-\mathbf{e_2} + 6\mathbf{e_3})$

□

It's tempting to believe that the dual of a dual yields the original multivector but this is in fact not the case. Rather there are some sign issues that arise. We will only need this case:

Theorem 4.8.1. In $\mathbb{R}^3$ for any multivector A we have $(A^*)^* = -A$.

Proof. Observe that for a given A we simply write down the calculation and switch and cancel the various $\mathbf{e_i}$:

$$(A^*)^* = (A\mathbf{e_3}\mathbf{e_2}\mathbf{e_1})\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} = (A(-\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}))\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} = -A$$

□

The primary use of the dual is in referring to perpendicular subspaces. We have the following:

Theorem 4.8.2. If B is a blade representing a subspace U of $\mathbb{R}^n$ then B^* represents the perpendicular complement $U^\perp$ of the subspace.

Proof. Omit. □

Rather than giving the proof of this in general, we'll give two specific cases which will be useful to us.

Here's the case for a bivector.

Theorem 4.8.3. In $\mathbb{R}^3$ the dual of a bivector $(\mathbf{a} \wedge \mathbf{b})^*$ represents the subspace perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. More explicitly we have:

$$(\mathbf{a} \wedge \mathbf{b})^* = \mathbf{a} \times \mathbf{b}$$

Proof. This is simply calculation. We have:

$$\begin{aligned}
(\mathbf{a} \wedge \mathbf{b})^* &= \left[\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}(\mathbf{e_1} \wedge \mathbf{e_2}) + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}(\mathbf{e_2} \wedge \mathbf{e_3}) + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}(\mathbf{e_3} \wedge \mathbf{e_1})\right]^* \\
&= \left[\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{e_1}\mathbf{e_2} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{e_2}\mathbf{e_3} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{e_3}\mathbf{e_1}\right](\mathbf{e_3}\mathbf{e_2}\mathbf{e_1}) \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{e_1}\mathbf{e_2}\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{e_2}\mathbf{e_3}\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{e_3}\mathbf{e_1}\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{e_3} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{e_1} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{e_2} \\
&= \mathbf{a} \times \mathbf{b}
\end{aligned}$$

□

The following corollary will be critical when we discuss rotations.

Corollary 4.8.1. We have:

$$(\mathbf{a} \times \mathbf{b})^* = ((\mathbf{a} \wedge \mathbf{b})^*)^* = -(\mathbf{a} \wedge \mathbf{b})$$

Proof. Immediate. □

Theorem 4.8.4. In $\mathbb{R}^3$ the dual of a vector $\mathbf{v}^*$ represents the perpendicular subspace $\mathbf{v}^\perp$. In other when $\mathbf{v}^*$ is written in the form $\mathbf{a} \wedge \mathbf{b}$ then all vectors in $\text{span}\{\mathbf{a}, \mathbf{b}\}$ are perpendicular to $\mathbf{v}$.

Proof. First observe that if $\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3}$ then:

$$\begin{aligned}\mathbf{v}^* &= (v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3})\mathbf{e_3}\mathbf{e_2}\mathbf{e_1} \\ &= -v_3\mathbf{e_1}\mathbf{e_2} - v_1\mathbf{e_2}\mathbf{e_3} - v_2\mathbf{e_3}\mathbf{e_1}\end{aligned}$$

Now then we can rewrite this in the form $\mathbf{a} \wedge \mathbf{b}$ using a previous theorem and using:

$$\begin{aligned}\mathbf{a} &= \frac{1}{v_1^2 + v_2^2}\left(-v_1v_3\mathbf{e_1} - v_2v_3\mathbf{e_2} + (v_1^2 + v_2^2)\mathbf{e_3}\right) \\ \mathbf{b} &= -v_2\mathbf{e_1} + v_1\mathbf{e_2} + 0\mathbf{e_3}\end{aligned}$$

Since $\mathbf{a}\cdot\mathbf{v} = \mathbf{b}\cdot\mathbf{v} = 0$ (just straight calculation), both $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to $\mathbf{v}$ and thus so is every linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Our specific decomposition of $\mathbf{v}^*$ doesn't matter since by the lemma above any other $\mathbf{c} \wedge \mathbf{d}$ represents the same subspace. Thus the proof is complete.

□

The following will be useful in the section on projection and rejection later.

Theorem 4.8.5. If B is a bivector then $BB^* = B^*B$.

Proof. This is just brute force. □

5 Computation in 3D

5.1 Summary of Previous

For organizational purposes here is a list of the critical essentials from the additional definitions which we will need going forward.

(a) For a multivector A the norm $|A|$ is the square root of the sum of the coefficients of the basis k-vectors.

(b) For a vector $\mathbf{v}$ we have $\mathbf{v}^{-1} = \frac{\mathbf{v}}{|\mathbf{v}|^2}$.

(c) For a bivector B we have $B^{-1} = -\frac{B}{|B|^2}$.

(d) For a multivector A we have $\langle A \rangle_k$ extracting the k-vector from A.

(e) For a j-vector A and k-vector B we define $A \cdot b = \langle AB \rangle_{k-j}$ and $A \wedge B = \langle AB \rangle_{k+j}$.

(f) For a vector $\mathbf{a}$ and a bivector B we have $\mathbf{a} \cdot B = \frac{1}{2}(\mathbf{a}B - B\mathbf{a})$.

(g) For a vector $\mathbf{a}$ and a bivector B we have $\mathbf{a} \wedge B = \frac{1}{2}(\mathbf{a}B + B\mathbf{a})$.

(h) For a vector $\mathbf{a}$ and a bivector B we have $\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B$.

(i) In $\mathbb{R}^3$ the dual of a multivector A is defined by $A^* = A\mathbf{e_3e_2e_1}$.

(j) In $\mathbb{R}^3$ for a multivector A we have $(A^*)^* = -A$.

(k) If a blade B represents a subspace V of $\mathbb{R}^n$ then B^* represents $V^\perp$.

(l) In $\mathbb{R}^3$ we have $(\mathbf{a} \wedge \mathbf{b})^* = \mathbf{a} \times \mathbf{b}$ and hence $(\mathbf{a} \times \mathbf{b})^* = \left((\mathbf{a} \wedge \mathbf{b})^*\right)^* = -\mathbf{a} \wedge \mathbf{b}$.

(m) If B is a bivector then $BB^* = B^*B$.

5.2 Projection and Rejection

The formula for the projection of **a** onto **b** is familiar from calculus:

$$\mathrm{Proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$$

Notice that this can be rewritten in the language of geometric algebra:

$$\mathrm{Proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \mathbf{a}\cdot\mathbf{b}\left(\frac{\mathbf{b}}{|\mathbf{b}|^2}\right) = (\mathbf{a}\cdot\mathbf{b})\mathbf{b}^{-1}$$

In addition to that (and not so familiar) is the rejection, which is the part of **a** which is perpendicular to **b**. If this is ever needed in calculus we typically do:

$$\mathrm{Rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$$

This could be done more elegantly by taking the subspace of $\mathbb{R}^3$ which is perpendicular to **b** and projecting to that, but calculus doesn't make this computationally easy. Geometric algebra does, however.

We know that:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{ab} - \mathbf{a}\wedge\mathbf{b}$$

Hence:

$$\begin{aligned}\mathrm{Rej}_{\mathbf{b}}\mathbf{a} &= \mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}^{-1}\\ &= \mathbf{a} - (\mathbf{ab} - \mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1}\\ &= \mathbf{a} - \mathbf{abb}^{-1} + (\mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1}\\ &= (\mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1}\end{aligned}$$

That's pretty cool. Now we'll see that a similar thing happens for other projections and rejections.

Lemma 5.2.1. Let **a** be a vector and B be a 2-blade. Let $\mathrm{Proj}_B\mathbf{a}$ be the projection of **a** onto the subspace represented by B and let $\mathrm{Rej}_B\mathbf{a}$ be the rejection of **a** relative to the subspace represented by B. Then:

$$\begin{aligned}(\mathrm{Proj}_B\mathbf{a})\wedge B &= 0\\ (\mathrm{Rej}_B\mathbf{a})\cdot B &= 0\end{aligned}$$

Proof. For the first, note that the projection is in the subspace represented by B and thus $(\mathrm{Proj}_B\mathbf{a}) \wedge B$ creates a parallelepipid with volume 0, hence equals 0.

For the second, note that since $\mathrm{Rej}_B\mathbf{a}$ is perpendicular to B it represents the dual $(\mathrm{Rej}_B\mathbf{a})^*$ and thus $\mathrm{Rej}_B\mathbf{a} = \alpha B^*$ for some $\alpha \in \mathbb{R}$. Then observe that:

$$(\mathrm{Rej}_B\mathbf{a})B = \alpha B^* B = \alpha B B^* = B\alpha B^* = B(\mathrm{Rej}_B\mathbf{a})$$

From here we get:

$$(\mathrm{Rej}_B\mathbf{a}) \cdot B = \frac{1}{2}\left((\mathrm{Rej}_B\mathbf{a})B - B(\mathrm{Rej}_B\mathbf{a})\right) = 0$$

□

Theorem 5.2.1. Let $\mathbf{a}$ be a vector and B be a 2-blade then the projection and rejection of $\mathbf{a}$ in relation to the subspace represented by B are:

$$\begin{aligned} \mathrm{Proj}_B\mathbf{a} &= (\mathbf{a} \cdot B)B^{-1} \text{ which we know } = \langle \mathbf{a}B \rangle_1 B^{-1} \\ \mathrm{Rej}_B\mathbf{a} &= (\mathbf{a} \wedge B)B^{-1} \text{ which we know } = \langle \mathbf{a}B \rangle_3 B^{-1} \end{aligned}$$

Proof. We have:

$$\begin{aligned} (\mathrm{Proj}_B\mathbf{a})\, B &= (\mathrm{Proj}_B\mathbf{a}) \cdot B + (\mathrm{Proj}_B\mathbf{a}) \wedge B \\ &= (\mathrm{Proj}_B\mathbf{a}) \cdot B + 0 \\ &= (\mathrm{Proj}_B\mathbf{a}) \cdot B + (\mathrm{Rej}_B\mathbf{a}) \cdot B \\ &= (\mathrm{Proj}_B\mathbf{a} + \mathrm{Rej}_B\mathbf{a}) \cdot B \\ &= \mathbf{a} \cdot B \\ \mathrm{Proj}_B\mathbf{a} &= (\mathbf{a} \cdot B)B^{-1} \end{aligned}$$

The second equation is similar. □

Exercise 5.1. Prove the second equation.

□

Practical Calculation 0.6. To use this formula just dig in and do the calculations.

□

Example 5.1. Suppose we wish to project the vector $\mathbf{a} = 7\mathbf{e_1} - 5\mathbf{e_3}$ onto the subspace represented by the blade $B = 2\mathbf{e_1} \wedge (\mathbf{e_2} + 6\mathbf{e_3})$. The result is:

$$\mathrm{Proj}_B\mathbf{a} = (\mathbf{a} \cdot B)B^{-1}$$

We'll work this out bit by bit. Note:

$$B = 2\mathbf{e_1} \wedge (\mathbf{e_2} + 6\mathbf{e_3}) = 2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}$$

First, noting that $\mathbf{a}$ is a 1-vector and B is a 2-blade, the inner product extracts the $2 - 1 = 1$-grade.

$$\begin{aligned}
\mathbf{a} \cdot B &= (7\mathbf{e_1} - 5\mathbf{e_3}) \cdot (2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}) \\
&= \big\langle (7\mathbf{e_1} - 5\mathbf{e_3})(2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}) \big\rangle_1 \\
&= \big\langle 14\mathbf{e_1}\mathbf{e_1}\mathbf{e_2} + 84\mathbf{e_1}\mathbf{e_1}\mathbf{e_3} - 10\mathbf{e_3}\mathbf{e_1}\mathbf{e_2} - 60\mathbf{e_3}\mathbf{e_1}\mathbf{e_3} \big\rangle_1 \\
&= \big\langle 14\mathbf{e_2} + 84\mathbf{e_3} - 10\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 60\mathbf{e_1} \big\rangle_1 \\
&= 60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3}
\end{aligned}$$

Second:

$$\begin{aligned}
B^{-1} &= -\frac{B}{|B|^2} \\
&= -\frac{2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}}{|2\mathbf{e_1}\mathbf{e_2} + 12\mathbf{e_1}\mathbf{e_3}|^2} \\
&= \frac{-2\mathbf{e_1}\mathbf{e_2} - 12\mathbf{e_1}\mathbf{e_3}}{148}
\end{aligned}$$

Then:

$$\begin{aligned}
\mathrm{Proj}_B\mathbf{a} &= (60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3}) \left(\frac{-2\mathbf{e_1}\mathbf{e_2} - 12\mathbf{e_1}\mathbf{e_3}}{148} \right) \\
&= \frac{1}{148}(60\mathbf{e_1} + 14\mathbf{e_2} + 84\mathbf{e_3})(-2\mathbf{e_1}\mathbf{e_2} - 12\mathbf{e_1}\mathbf{e_3}) \\
&= \frac{1}{148}(-120\mathbf{e_1}\mathbf{e_1}\mathbf{e_2} - 720\mathbf{e_1}\mathbf{e_1}\mathbf{e_3} - 28\mathbf{e_2}\mathbf{e_1}\mathbf{e_2} - 168\mathbf{e_2}\mathbf{e_1}\mathbf{e_3} - 168\mathbf{e_3}\mathbf{e_1}\mathbf{e_2} - 1008\mathbf{e_3}\mathbf{e_1}\mathbf{e_3}) \\
&= \frac{1}{148}(-120\mathbf{e_2} - 720\mathbf{e_3} + 28\mathbf{e_1} + 168\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} - 168\mathbf{e_1}\mathbf{e_2}\mathbf{e_3} + 1008\mathbf{e_1}) \\
&= \frac{1}{148}(1036\mathbf{e_1} - 120\mathbf{e_2} - 720\mathbf{e_3}) \\
&= \frac{1}{37}(259\mathbf{e_1} - 30\mathbf{e_2} - 180\mathbf{e_3})
\end{aligned}$$

It's worth noting that this can be done fairly quickly in $\mathbb{R}^3$. Notice how the vectors in the following correspond to the data given above. The subspace B has orthogonal basis $\{[2;0;0],[0;1;6]\}$ and so the projection of $[7;0;-5]$ onto B can be found by adding the projections onto the basis vectors. We have:

$$\begin{aligned}
\mathrm{Proj}_{[2;0;0]}[7;0;-5] &= [7;0;0] \\
\mathrm{Proj}_{[0;1;6]}[7;0;-5] &= -\frac{30}{37}[0;1;6]
\end{aligned}$$

Thus the overall projection is:

$$[7; 0; 0] - \frac{30}{37}[0; 1; 6] = \frac{1}{37}[259; -30; -180]$$

You might wonder what is gained in the geometric algebra way. The answer is that it doesn't care if we have an orthogonal basis and it's formulaic in all cases.

$\square$

Example 5.2. Note that for the previous example to do the rejection we've done most of the work already. Instead of $\mathbf{a} \cdot B$ we find $\mathbf{a} \wedge B$ but this just involves extracting the grade 3 component $\langle \mathbf{a}B \rangle_3$ and multiplying by B^{-1}:

$$(-10\mathbf{e_1e_2e_3})\left(\frac{-2\mathbf{e_1e_2} - 12\mathbf{e_1e_3}}{148}\right) = \frac{30}{37}\mathbf{e_2} - \frac{5}{37}\mathbf{e_3}$$

Note that the projection and the rejection add up to $\mathbf{a}$, which makes sense.

$\square$

Exercise 5.2. Find the projection and rejection of $\mathbf{a} = 2\mathbf{e_1} - 3\mathbf{e_2} + 5\mathbf{e_3}$ relative to the subspace span $\{4\mathbf{e_1} + 1\mathbf{e_2} - 1\mathbf{e_3}, 2\mathbf{e_1} + 2\mathbf{e_3}\}$.

$\square$

Exercise 5.3. Show that if $\mathbf{a}$ is perpendicular to the subspace represented by B then $\mathrm{Proj}_B\mathbf{a} = \mathbf{0}$

$\square$

Exercise 5.4. Show that if $\mathbf{a}$ is parallel to (inside) the subspace represented by B then $\mathrm{Rej}_B\mathbf{a} = \mathbf{0}$

$\square$

5.3 Reflection

Theorem 5.3.1. Given a vector $\mathbf{v}$ and a 2-blade B the result of reflecting $\mathbf{v}$ in the subspace represented by B is:

$$\mathbf{v} \mapsto -B\mathbf{v}B^{-1}$$

Proof. To see that this does the required job, we decompose $\mathbf{v}$ into two parts:

$$\mathbf{v} = \underbrace{(\mathbf{v} \cdot B)B^{-1}}_{\mathrm{Proj}_B\mathbf{v}} + \underbrace{(\mathbf{v} \wedge B)B^{-1}}_{\mathrm{Rej}_B\mathbf{v}}$$

A reflection should negate the second part and leave the first part alone. Thus:

$$\begin{aligned}
\mathrm{Refl}_B\mathbf{v} &= (\mathbf{v}\cdot B)B^{-1} - (\mathbf{v}\wedge B)B^{-1}\\
&= (\mathbf{v}\cdot B - \mathbf{v}\wedge B)B^{-1}\\
&= \left(\frac{1}{2}(\mathbf{v}B - B\mathbf{v}) - \frac{1}{2}(\mathbf{v}B + B\mathbf{v})\right)B^{-1}\\
&= (-B\mathbf{v})B^{-1}\\
&= -B\mathbf{v}B^{-1}
\end{aligned}$$

□

Example 5.3. Suppose we wish to reflect the vector $\mathbf{v} = 2\mathbf{e_1} + 5\mathbf{e_2} + 7\mathbf{e_3}$ in the subspace spanned by $\mathbf{e_1} - \mathbf{e_2}$ and $\mathbf{e_1} + 3\mathbf{e_3}$. We set:

$$B = (\mathbf{e_1} - \mathbf{e_2}) \wedge (\mathbf{e_1} + 3\mathbf{e_3}) = \mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 3\mathbf{e_3e_1}$$

We note:

$$B^{-1} = -\frac{B}{|B|^2} = -\frac{\mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 3\mathbf{e_3e_1}}{|\mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 3\mathbf{e_3e_1}|^2} = \frac{-\mathbf{e_1e_2} + 3\mathbf{e_2e_3} + 3\mathbf{e_3e_1}}{19}$$

Then we calculate:

$$\begin{aligned}
\mathrm{Refl}_B\mathbf{v} &= -B\mathbf{v}B^{-1}\\
&= -(\mathbf{e_1e_2} - 3\mathbf{e_2e_3} - 3\mathbf{e_3e_1})(2\mathbf{e_1} + 5\mathbf{e_2} + 7\mathbf{e_3})\left(\frac{-\mathbf{e_1e_2} + 3\mathbf{e_3e_1} + 3\mathbf{e_2e_3}}{19}\right)\\
&= \text{...Python...}\\
&= -\frac{1}{19}(-46\mathbf{e_1} + 11\mathbf{e_2} + 161\mathbf{e_3})\\
&= \frac{46}{19}\mathbf{e_1} - \frac{11}{19}\mathbf{e_2} + \frac{161}{19}\mathbf{e_3}
\end{aligned}$$

□

Exercise 5.5. Find the result of reflecting the vector $-4\mathbf{e_1} + 2\mathbf{e_2} - 7\mathbf{e_3}$ in the subspace spanned by $3\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$ and $\mathbf{e_1} - 2\mathbf{e_2}$.

□

Exercise 5.6. Show mathematically that if $\mathbf{a}$ is perpendicular to the subspace represented by B then the reflection formula just yields $-\mathbf{a}$ as expected.

□

5.4 Rotation

Before diving into rotations, let's define the exponential of a multivector. This is not technically necessary for what follows but it brings back some familiar notation.

Definition 5.4.1. For any multivector A we can define the *exponential of the multivector*

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

□

It turns out that this converges for all multivectors A and moreover for a unit bivector $\hat{\mathbf{B}}$ we have:

$$\exp(\theta\hat{\mathbf{B}}) = \cos\theta + \hat{\mathbf{B}}\sin\theta$$

Now then, in order to see how rotations are constructed recall from previous chapters that if we take two planes which intersect in a line and if we reflect first in one and then in the other, the result is a rotation around the intersecting lines. Not just that, but any rotation around the axis may be constructed using two such reflections.

So now suppose we're given a unit axis $\hat{\mathbf{r}}$ that we wish to rotate around and an angle θ according to the right hand rule.

Let $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ be any two unit vectors perpendicular to $\hat{\mathbf{r}}$ and with an angle of $\theta/2$ between them as measured from $\hat{\mathbf{m}}$ to $\hat{\mathbf{n}}$ according to the right hand rule with respect to $\hat{\mathbf{r}}$. In other words $\hat{\mathbf{r}} = \alpha(\hat{\mathbf{m}} \times \hat{\mathbf{n}})$ for some $\alpha > 0$.

Noting that $\hat{\mathbf{m}} \perp \hat{\mathbf{r}}$ and $\hat{\mathbf{n}} \perp \hat{\mathbf{r}}$, consider now the two planes:

$$\hat{\mathbf{m}} \wedge \hat{\mathbf{r}} = \hat{\mathbf{m}} \wedge \hat{\mathbf{r}} + 0 = \hat{\mathbf{m}} \wedge \hat{\mathbf{r}} + \hat{\mathbf{m}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{m}}\,\hat{\mathbf{r}}$$

and

$$\hat{\mathbf{n}} \wedge \hat{\mathbf{r}} = \hat{\mathbf{n}} \wedge \hat{\mathbf{r}} + 0 = \hat{\mathbf{n}} \wedge \hat{\mathbf{r}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{n}}\,\hat{\mathbf{r}}$$

Note that $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are in these two planes, they are not the normal vectors for these two planes!

It's clear that reflection in $\hat{\mathbf{m}}\,\hat{\mathbf{r}}$ followed by reflection in $\hat{\mathbf{n}}\,\hat{\mathbf{r}}$ performs the desired rotation. Since these are both bivectors we can use the previous section to calculate this.

Before the calculation, note the following:

$$(\hat{\mathbf{m}}\,\hat{\mathbf{r}})^{-1} = \hat{\mathbf{r}}\,\hat{\mathbf{m}}$$
$$(\hat{\mathbf{n}}\,\hat{\mathbf{r}})^{-1} = \hat{\mathbf{r}}\,\hat{\mathbf{n}}$$

and

$$\hat{\mathbf{m}}\,\hat{\mathbf{r}} = \hat{\mathbf{m}} \wedge \hat{\mathbf{r}} = -\hat{\mathbf{r}} \wedge \hat{\mathbf{m}} = -\hat{\mathbf{r}}\,\hat{\mathbf{m}}$$

Now we calculate:

$$\begin{aligned}
\mathbf{a} &\mapsto -(\hat{\mathbf{n}}\,\hat{\mathbf{r}})(-(\hat{\mathbf{m}}\,\hat{\mathbf{r}})\mathbf{a}(\hat{\mathbf{m}}\,\hat{\mathbf{r}})^{-1})(\hat{\mathbf{n}}\,\hat{\mathbf{r}})^{-1} \\
&\mapsto (\hat{\mathbf{n}}\,\hat{\mathbf{r}})(\hat{\mathbf{m}}\,\hat{\mathbf{r}})\mathbf{a}(\hat{\mathbf{r}}\,\hat{\mathbf{m}})(\hat{\mathbf{r}}\,\hat{\mathbf{n}}) \\
&\mapsto (\hat{\mathbf{n}}\,\hat{\mathbf{r}})(-\hat{\mathbf{r}}\,\hat{\mathbf{m}})\mathbf{a}(-\hat{\mathbf{m}}\,\hat{\mathbf{r}})(\hat{\mathbf{r}}\,\hat{\mathbf{n}}) \\
&\mapsto (\hat{\mathbf{n}}\,\hat{\mathbf{r}})(\hat{\mathbf{r}}\,\hat{\mathbf{m}})\mathbf{a}(\hat{\mathbf{m}}\,\hat{\mathbf{r}})(\hat{\mathbf{r}}\,\hat{\mathbf{n}}) \\
&\mapsto \hat{\mathbf{n}}\,\hat{\mathbf{m}}\,\mathbf{a}\,\hat{\mathbf{m}}\,\hat{\mathbf{n}} \\
&\mapsto (\hat{\mathbf{n}}\,\hat{\mathbf{m}})\,\mathbf{a}\,(\hat{\mathbf{n}}\,\hat{\mathbf{m}})^{-1}
\end{aligned}$$

Now then, $\hat{\mathbf{n}}\,\hat{\mathbf{m}}$ can be thought of as packaging together both the plane of rotation and the angle of rotation. Technically it's called a *rotor*.

In other words we can think of rotating a plane (and all parallel planes) by assigning that plane using a geometric product of two vectors which span the plane and meet at an angle of $\theta/2$.

Moreover if we choose any *unit bivector/blade* $\hat{\mathbf{B}}$ (with area 1) which spans the same plane as $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ and has the same orientation as $\hat{\mathbf{m}} \wedge \hat{\mathbf{n}}$ then by the lemma earlier we have

$$\hat{\mathbf{m}} \wedge \hat{\mathbf{n}} = |\hat{\mathbf{m}}||\hat{\mathbf{n}}| \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}$$

From there since $|\hat{\mathbf{m}}| = |\hat{\mathbf{n}}| = 1$ we have:

$$\begin{aligned}
\hat{\mathbf{m}}\hat{\mathbf{n}} &= \hat{\mathbf{m}} \cdot \hat{\mathbf{n}} + \hat{\mathbf{m}} \wedge \hat{\mathbf{n}} \\
&= |\hat{\mathbf{m}}||\hat{\mathbf{n}}| \cos\left(\frac{\theta}{2}\right) + |\hat{\mathbf{m}}||\hat{\mathbf{n}}| \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}} \\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}} \\
&= \exp\left(\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right)
\end{aligned}$$

and:

$$\begin{aligned}
\hat{\mathbf{n}}\hat{\mathbf{m}} &= \hat{\mathbf{n}} \cdot \hat{\mathbf{m}} + \hat{\mathbf{n}} \wedge \hat{\mathbf{m}} \\
&= |\hat{\mathbf{n}}||\hat{\mathbf{m}}| \cos\left(-\left(\frac{\theta}{2}\right)\right) + |\hat{\mathbf{n}}||\hat{\mathbf{m}}| \sin\left(-\left(\frac{\theta}{2}\right)\right)\hat{\mathbf{B}} \\
&= \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}} \\
&= \exp\left(-\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right)
\end{aligned}$$

Thus we can write our rotation as:

$$\mathbf{a} \mapsto \left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right) \mathbf{a} \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right)$$

or as:

$$\mathbf{a} \mapsto \exp\left(-\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right) \mathbf{a} \exp\left(\left(\frac{\theta}{2}\right)\hat{\mathbf{B}}\right)$$

Exercise 5.7. Suppose $C = (2\mathbf{e_1} + 2\mathbf{e_3}) \wedge (\mathbf{e_1} - 3\mathbf{e_3})$ represents a plane. Write down the mapping which rotates C (and all parallel planes) by $\theta = 1.8$ radians. Note: What can $\hat{\mathbf{B}}$ be? Then rotate $\mathbf{a} = 10\mathbf{e_1} + 13\mathbf{e_2} - 20\mathbf{e_3}$.

□

However lastly and most carefully we may in fact want to calculate the rotation via the right hand rule given an axis $\hat{\mathbf{r}}$ and an angle θ.

It's tempting to simply set $\hat{\mathbf{B}} = \hat{\mathbf{r}}^*$ but this does not quite work. The reason is that although the subspace represented by $\hat{\mathbf{B}}$ is the correct plane of rotation, it has the incorrect orientation.

The reason for this is as follows. In the above argument $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ were chosen so that the direction of $\hat{\mathbf{m}} \times \hat{\mathbf{n}}$ is that of $\hat{\mathbf{r}}$. In other words recall that we had:

$$\hat{\mathbf{r}} = \alpha(\hat{\mathbf{m}} \times \hat{\mathbf{n}}) \text{ for some } \alpha > 0$$

Then we have:

$$\hat{\mathbf{r}}^* = \alpha(\hat{\mathbf{m}} \times \hat{\mathbf{n}})^* = \alpha(-(\hat{\mathbf{m}} \wedge \hat{\mathbf{n}})) = -\alpha(\hat{\mathbf{m}} \wedge \hat{\mathbf{n}})$$

This is the wrong orientation for the necessary bivector.

The solution is to set $\hat{\mathbf{B}} = -\hat{\mathbf{r}}^*$ and then the rotation becomes:

$$\begin{aligned}\mathbf{a} &\mapsto \left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)(-\hat{\mathbf{r}}^*)\right) \mathbf{a} \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(-\hat{\mathbf{r}}^*)\right) \\ &\mapsto \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right) \mathbf{a} \left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right)\end{aligned}$$

which can be rewritten as:

$$\mathbf{a} \mapsto \exp\left(\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right) \mathbf{a} \exp\left(-\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right)$$

We close with the previous results packaged into a theorem:

Theorem 5.4.1. Given a unit vector $\mathbf{\hat{r}}$ designating an axis of rotation and an angle θ, rotation by θ radians about $\mathbf{\hat{r}}$ in accordance to the right-hand rule may be achieved by:

$$\mathbf{a} \mapsto \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)\,\mathbf{a}\,\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)$$

which is the same as:

$$\mathbf{a} \mapsto \exp\left(\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)\mathbf{a}\exp\left(-\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)$$

Alternately we can perform the rotation by assigning two unit vectors $\mathbf{\hat{m}}$ and $\mathbf{\hat{n}}$ such that the angle between them is $\theta/2$ and then using the mapping:

$$\mathbf{a} \mapsto \mathbf{\hat{n}\hat{m}a\hat{m}\hat{n}}$$

Alternately if $\mathbf{\hat{B}}$ is any unit bivector spanning the same plane as $\mathbf{\hat{m}}$ and $\mathbf{\hat{n}}$ with the same orientation as $\mathbf{\hat{m}} \wedge \mathbf{\hat{n}}$ then we may use the mapping:

$$\mathbf{a} \mapsto \exp\left(-\left(\frac{\theta}{2}\right)\mathbf{\hat{B}}\right)\mathbf{a}\exp\left(\left(\frac{\theta}{2}\right)\mathbf{\hat{B}}\right)$$

which is the same as:

$$\mathbf{a} \mapsto \left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbf{\hat{B}}\right)\,\mathbf{a}\,\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{\hat{B}}\right)$$

Example 5.4. Suppose we wish to rotate the plane by $\theta = 2.4$ radians about the axis $\mathbf{\hat{r}} = \frac{1}{\sqrt{14}}(2\mathbf{e_1} + 3\mathbf{e_2} - \mathbf{e_3})$. We find:

$$\begin{aligned}
\mathbf{\hat{r}}^* &= \frac{1}{\sqrt{14}}(2\mathbf{e_1} + 3\mathbf{e_2} - \mathbf{e_3})\mathbf{e_3e_2e_1} \\
&= \frac{1}{\sqrt{14}}(-2\mathbf{e_2e_3} - 3\mathbf{e_3e_1} + \mathbf{e_1e_2}) \\
&= \frac{1}{\sqrt{14}}(\mathbf{e_1e_2} - 2\mathbf{e_2e_3} - 3\mathbf{e_3e_1})
\end{aligned}$$

The mapping is then given by:

$$\mathbf{a} \mapsto \exp\left(\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)\mathbf{a}\exp\left(-\left(\frac{\theta}{2}\right)\mathbf{\hat{r}}^*\right)$$

For $\theta = 2.4$ and $\hat{\mathbf{r}}^*$ above.

To rotate the point $\mathbf{a} = -2\mathbf{e_1} + 2\mathbf{e_2} + 7\mathbf{e_3}$ we calculate:

$$\begin{aligned}
\mathbf{a} &\mapsto \exp\left(\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right)(-2\mathbf{e_1} + 2\mathbf{e_2} + 7\mathbf{e_3})\exp\left(-\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}^*\right) \\
&\mapsto \text{...Python...} \\
&\mapsto 4.3859\mathbf{e_1} - 5.5026\mathbf{e_2} - 2.7360\mathbf{e_3}
\end{aligned}$$

□

Exercise 5.8. What is the mapping which rotates around the axis $\mathbf{r} = \mathbf{e_1} + \mathbf{e_3}$ by $\theta = 4$ radians? Make sure you normalize $\mathbf{r}$ first. Then find the result of rotating the point $\mathbf{a} = 2\mathbf{e_1} + 5\mathbf{e_2} - 3\mathbf{e_3}$.

□

6 Generalization of $\mathbb{H}$

Consider the following correspondance:

$$\begin{aligned}
\mathbf{e_3}\mathbf{e_2} &\leftrightarrow \hat{\boldsymbol{\imath}} \\
\mathbf{e_1}\mathbf{e_3} &\leftrightarrow \hat{\boldsymbol{\jmath}} \\
\mathbf{e_2}\mathbf{e_1} &\leftrightarrow \hat{\boldsymbol{k}}
\end{aligned}$$

Then multivectors of the form:

$$\alpha + \beta\mathbf{e_3}\mathbf{e_2} + \delta\mathbf{e_1}\mathbf{e_3} + \gamma\mathbf{e_2}\mathbf{e_1}$$

correspond to quaternions of the form:

$$\alpha + \beta\hat{\boldsymbol{\imath}} + \delta\hat{\boldsymbol{\jmath}} + \gamma\hat{\boldsymbol{k}}$$

This correspondance is not meaningless. For example from it we get the following correspondances:

$$\begin{aligned}
(\mathbf{e_3}\mathbf{e_2})(\mathbf{e_1}\mathbf{e_3}) = \mathbf{e_2}\mathbf{e_1} &\leftrightarrow \hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \\
(\mathbf{e_1}\mathbf{e_3})(\mathbf{e_2}\mathbf{e_1}) = \mathbf{e_3}\mathbf{e_2} &\leftrightarrow \hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} \\
(\mathbf{e_2}\mathbf{e_1})(\mathbf{e_3}\mathbf{e_2}) = \mathbf{e_1}\mathbf{e_3} &\leftrightarrow \hat{\boldsymbol{k}}\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}
\end{aligned}$$

It follows that multivectors of this form form a closed structure with the geometric product acting in a corresponding manner to the quaternion product.

For example the geometric product:

$$(7+2\mathbf{e_3e_2}+5\mathbf{e_1e_3}+8\mathbf{e_2e_1})(2-7\mathbf{e_3e_2}+2\mathbf{e_1e_3}-6\mathbf{e_2e_1}) = 66-91\mathbf{e_3e_2}-20\mathbf{e_1e_3}+13\mathbf{e_2e_1}$$

corresponds to the quaternion product:

$$(7+2\hat{\boldsymbol{\imath}}+5\hat{\boldsymbol{\jmath}}+8\hat{\boldsymbol{k}})(2-7\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}-6\hat{\boldsymbol{k}}) = 66-91\hat{\boldsymbol{\imath}}-20\hat{\boldsymbol{\jmath}}+13\hat{\boldsymbol{k}}$$

What's happening here is that the quaternions appear structurally as a subset of the geometric algebra.

We can see this parallel even more clearly when we look at a computation such as rotation.

In the previous example we rotated the point $-2\mathbf{e_1}+2\mathbf{e_2}+7\mathbf{e_3}$ by $\theta = 2.4$ radians about the axis $\hat{\mathbf{r}} = \frac{1}{\sqrt{14}}(2\mathbf{e_1}+3\mathbf{e_2}-\mathbf{e_3})$.

Here is that calculation reorganized and approximated so that we can easily trace the values.

We have:

$$\hat{\mathbf{r}} = 0.5345\mathbf{e_1}+0.8018\mathbf{e_2}-0.2673\mathbf{e_3}$$

and then:

$$\hat{\mathbf{r}}^* = 0.5345\mathbf{e_3e_2}+0.8018\mathbf{e_1e_3}-0.2673\mathbf{e_2e_1}$$

We have:

$$\exp((2.4/2)\hat{\mathbf{r}}^*) = 0.36+0.50\mathbf{e_3e_2}+0.75\mathbf{e_1e_3}-0.25\mathbf{e_2e_1}$$

and:

$$\exp((-2.4/2)\hat{\mathbf{r}}^*) = 0.36-0.50\mathbf{e_3e_2}-0.75\mathbf{e_1e_3}+0.25\mathbf{e_2e_1}$$

and then the product is

$$\begin{gathered}\exp((2.4/2)\hat{\mathbf{r}}^*)\mathbf{a}\exp((-2.4/2)\hat{\mathbf{r}}^*)\\ = (0.36+0.50\mathbf{e_3e_2}+0.75\mathbf{e_1e_3}-0.25\mathbf{e_2e_1})(-2\mathbf{e_1}+2\mathbf{e_2}+7\mathbf{e_3}(0.36-0.50\mathbf{e_3e_2}-0.75\mathbf{e_1e_3}+0.25\mathbf{e_2e_1})\\ = 4.39\mathbf{e_1}-5.50\mathbf{e_2}-2.74\mathbf{e_3}\end{gathered}$$

If we approached this as a quaternion problem we would assign:

$$p = \cos(2.4/2)+\sin(2.4/2)\hat{\mathbf{r}} = 0.36+0.50\hat{\boldsymbol{\imath}}+0.75\hat{\boldsymbol{\jmath}}-0.25\hat{\boldsymbol{k}}$$

Then:

$$p^* = \cos(2.4/2)+\sin(2.4/2)\hat{\mathbf{r}} = 0.36-0.50\hat{\boldsymbol{\imath}}-0.75\hat{\boldsymbol{\jmath}}+0.25\hat{\boldsymbol{k}}$$

Then the product is:

$$\begin{aligned}p\mathbf{a}p^* &= (0.36+0.50\hat{\boldsymbol{\imath}}+0.75\hat{\boldsymbol{\jmath}}-0.25\hat{\boldsymbol{k}})(-2\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}+7\hat{\boldsymbol{k}})(0.36-0.50\hat{\boldsymbol{\imath}}-0.75\hat{\boldsymbol{\jmath}}+0.25\hat{\boldsymbol{k}})\\ &= 4.39\hat{\boldsymbol{\imath}}-5.50\hat{\boldsymbol{\jmath}}-2.74\hat{\boldsymbol{k}}\end{aligned}$$

7 Generalization of $\mathbb{C}$

Given that $\mathbb{H}$ is an expansion of $\mathbb{C}$ we note that if we isolate our view to multivectors of the form

$$a + b\mathbf{e_3 e_2}$$

we arrive at the complex numbers. That is, we correspond:

$$a + b\mathbf{e_3 e_2} \leftrightarrow a + b\hat{\boldsymbol{\imath}}$$

This is instructive. Consider that rotation in $\mathbb{C}$ was performed by calculating:

$$z \mapsto (\cos\theta + \sin\theta\hat{\boldsymbol{\imath}})z$$

The analogous calculation here is then:

$$z \mapsto (\cos\theta + \mathbf{e_3 e_2}\sin\theta)z \text{ where } z = c + d\mathbf{e_3 e_2}$$

However observe that if we choose $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ meeting at an angle of θ and in the $\mathbf{e_3 e_2}$ plane then observe that:

$$\begin{aligned}\hat{\mathbf{a}}\hat{\mathbf{b}} &= \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} + \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \\ &= |\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos\theta + |\hat{\mathbf{a}}||\hat{\mathbf{b}}|\sin\theta\mathbf{e_3 e_2} \\ &= \cos\theta + \mathbf{e_3 e_2}\sin\theta\end{aligned}$$

Thus rotation in our model of $\mathbb{C}$ can be done with:

$$z \mapsto (\cos\theta + \mathbf{e_3 e_2}\sin\theta)z \text{ where } z = c + d\mathbf{e_3 e_2}$$

Or we can simply use the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ and calculate:

$$z \mapsto \hat{\mathbf{a}}\hat{\mathbf{b}}z \text{ where } z = c + d\mathbf{e_3 e_2}$$

Example 7.1. Consider the two vectors $\hat{\mathbf{a}} = \frac{1}{\sqrt{18}}(2\mathbf{e_2} + 4\mathbf{e_3})$ and $\hat{\mathbf{b}} = \frac{1}{\sqrt{37}}(6\mathbf{e_2} + 1\mathbf{e_3})$. These meet at an angle of 0.9420 radians in the $\mathbf{e_2 e_3}$-plane. Thus the operation:

$$c + d\mathbf{e_3 e_2} \mapsto \hat{\mathbf{a}}\hat{\mathbf{b}}(c + d\mathbf{e_3 e_2})$$

provides a geometric algebra model of the rotation of $\mathbb{C}$ by $\theta = 0.9420$ radians about the origin. For example to rotate $(10, 8)$ we calculate:

$$10 + 8\mathbf{e_3 e_2} \mapsto \hat{\mathbf{a}}\hat{\mathbf{b}}(10 + 8\mathbf{e_3 e_2}) = -0.5882 + 12.7923\mathbf{e_3 e_2}$$

yielding the point $(-0.5882, 12.7923)$

The analogous complex calculation would simply be:

$$10 + 8\hat{\boldsymbol{\imath}} \mapsto (\cos\theta + \hat{\boldsymbol{\imath}}\sin\theta)(10 + 8\hat{\boldsymbol{\imath}}) = -0.5882 + 12.7923\hat{\boldsymbol{\imath}}$$

$\square$

It's important to note that we can do rotations in $\mathbb{R}^2$ using only $\mathbf{e_1}$ and $\mathbf{e_2}$, something we did not do here because we worked mostly in 3D, so this section is not really about that, but rather about seeing at how geometric algebra subsumes $\mathbb{C}$.

Without much explanation notice the above calculation works without introducing any scalars into the mix and using $\mathbf{e_1}$ and $\mathbf{e_2}$ as:

Example 7.2. Consider the two vectors $\hat{\mathbf{a}} = \frac{1}{\sqrt{18}}(2\mathbf{e_1}+4\mathbf{e_2})$ and $\hat{\mathbf{b}} = \frac{1}{\sqrt{37}}(6\mathbf{e_1}+1\mathbf{e_2})$. These meet at an angle of 0.9420 radians in the $\mathbf{e_1}\mathbf{e_2}$-plane. Thus the operation:

$$c\mathbf{e_1} + d\mathbf{e_2} \mapsto \hat{\mathbf{a}}\hat{\mathbf{b}}(c\mathbf{e_1} + d\mathbf{e_2})$$

also rotates, this time the $\mathbf{e_1}\mathbf{e_2}$-plane. For example to rotate $(10, 8)$ we calculate:

$$10\mathbf{e_1} + 8\mathbf{e_2} \mapsto \hat{\mathbf{a}}\hat{\mathbf{b}}(10\mathbf{e_1} + 8\mathbf{e_2}) = -0.5882\mathbf{e_1} + 12.7923\mathbf{e_2}$$

yielding the point $(-0.5882, 12.7923)$

$\square$