MATH431: Gimbal Lock

Justin Wyss-Gallifent

November 3, 2021

1 Introduction

To achieve all possible orientations of an object in $\mathbb{R}^3$ there are three degrees of freedom that must be accounted for. For example in aircraft control these are typically called roll, pitch and yaw. Roll is rotation along the front-to-back axis. Pitch is rotation along the side-to-side axis, this corresponds to the plane pointing upwards or downwards. Yaw is rotation along a vertical axis. All of these are necessary for the proper functioning of the aircraft.

2 Euler Angles

One way to achieve all possible orientations is to allow rotations in each of the major axes and then combine these rotations to orient our object. It's certainly possible to achieve any orientation this way. This method is nice because it's intuitive.

The matrix rotating about the x-axis by an angle of α equals:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

The matrix rotating about the y-axis by an angle of β equals:

$$R_y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

The matrix rotating about the z-axis by an angle of γ equals:

$$R_z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each of these the direction of rotation follows the right-hand-rule with the thumb pointing in each axis' positive direction.

In order to achieve any orientation of our object we'll probably need all three of these rotations. But how to combine them?

Typically an order is chosen and then the result can be used to obtain any orientation we wish.

If pick an order, say z then y then x and we define:

$$R_{xyz}(\alpha, \beta, \gamma) = R_x(\alpha)R_y(\beta)R_z(\gamma)$$

Then the resulting transformation matrix is:

$R_{xyz}(\alpha, \beta, \gamma) = R_x(\alpha)R_y(\beta)R_z(\gamma) =$

$$\begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha \\ \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{bmatrix}$$

Think about R_{xyz} as a control unit. We input three angles α, β, and γ and it returns an output matrix which we apply to our points.

So now for example if we start with a rocket which is pointing upwards and we want it to point towards the positive x-axis. We feed the control unit $(\alpha, \beta, \gamma) = (0, \pi/2, 0)$.

We can track the point $[0; 0; 1]$ to see:

$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let's be even more robust. Suppose we have a rocket whose nose is $[0; 0; 1]$ and which has a logo on the side $[0; 1; 0]$. and suppose we want the rocket to point along the positive ray $z = x$ with the logo below it. We feed the control unit $(\alpha, \beta, \gamma) = (0, \pi/4, -\pi/2)$.

We can track the points to see:

$$R_{xyz}(0, \pi/4, -\pi/2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

3 Gimbal Lock

Suppose we set $(\alpha, \beta, \gamma) = (0, \pi/2, 0)$ in our control unit. The resulting rotation matrix is:

$$R_{xyz}(\alpha, \pi/2, \gamma) = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ -\cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \end{bmatrix}$$

At this point our control unit has lost some functionality because altering two inputs α and γ has the same effect on the output matrix. Moreover if we increase α and decrease γ by the same amount, the output matrix doesn't change!

This gets even worse. From this particular setting of $(\alpha, \beta, \gamma) = (0, \pi/2, 0)$ is it possible to alter any combination of α, β, and γ in a way which results in a slight rotation about the z-axis?

Consider an object with two noted points, $P = [0; 1; 0]$ and $Q = [-1; 0; 0]$.

Observe that:

$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =: P' \text{ (on the } xy\text{-plane)}$$
$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =: Q' \text{ (on the } z\text{-axis)}$$

Suppose at this instant we want to make changes to the control unit input angles so as to effectively rotate the resulting P' slightly about the z-axis. This means that P' needs to move (specifically y needs to decrease while x increases or decreases) while Q' needs to remain fixed.

Consider what happens at this instant if we modify α, β and γ by $a \approx 0$, $b \approx 0$, and $c \approx 0$ respectively:

$$R_{xyz}(a, \pi/2 + b, c) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin b \sin c \\ \cos a \cos c - \sin a \cos b \sin c \\ \sin a \cos c + \cos a \cos b \sin c \end{bmatrix} =: P''$$

and

$$R_{xyz}(a, \pi/2 + b, c) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin b \cos c \\ -\cos a \sin c - \sin a \cos b \cos c \\ \cos a \cos b \cos c - \sin a \sin c \end{bmatrix} =: Q''$$

Our wish is to choose a, b, c so that $Q'' = Q'$ and so that P'' is P' undergoing a slight rotation about the z-axis.

To keep $Q'' = Q' = [0; 0; 1]$ we need the first entry $\sin b \cos c$ to remain at 0. This requires b to remain at 0. Note that since $c \approx 0$ we know that $\cos c \approx 1$.

However if b remains at 0 then the first entry of P'' which is $\sin b \sin c$ remains at 0 too, meaning we can't change the x-coordinate of P'. We can't rotate - this is our lock.

4 Avoiding Gimbal Lock

The only way to avoid gimbal lock is to take a different approach to describing rotations. Quaternions are one of the most popular ways.