Cohomology of Shimura varieties and Hodge–Tate weights

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Background: cohomology with trivial coefficients

Given a proper smooth variety X over $\mathbb{Q}$, we have (compatibly):

- $H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$;
- $H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$;
- $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} B_{\mathrm{dR}}$.

Here B_{dR} is Fontaine's p-adic de Rham period ring, which is a filtered field with a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action such that $\mathrm{gr}^i B_{\mathrm{dR}} \cong \mathbb{C}_p(i)$.

The last isomorphism above is compatible with the filtrations and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-actions on both sides, and induces

- $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \oplus_{a+b=i}\big(H^b(X, \Omega^a_X) \otimes_{\mathbb{Q}} \mathbb{C}_p(-a)\big)$.

We say a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation V over $\mathbb{Q}_p$ is *de Rham* if $D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ and V have same dimensions. Its multiset $\mathrm{HT}(V)$ of *Hodge–Tate weights* contains $a \in \mathbb{Z}$ with multiplicity $\dim_{\mathbb{Q}_p}\big(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(a)\big)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$. Then $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is de Rham, with $\dim_{\mathbb{Q}} H^b(X, \Omega^a_X)$ times of a in $\mathrm{HT}\big(H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)\big)$.

Today, we would like to allow *nontrivial* coefficients on *nonproper* X, in the context of the cohomology of (general) *Shimura varieties*.

Shimura data and Shimura varieties

Notation: $\widehat{\mathbb{Z}} := \varprojlim_N (\mathbb{Z}/N\mathbb{Z})$, $\mathbb{A}^\infty := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{A} := \mathbb{R} \times \mathbb{A}^\infty$

Suppose we have a *Shimura datum* (G, X), where:

- G is a reductive algebraic group; and
- X is a $\mathrm{G}(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{C}^\times \to \mathrm{G}(\mathbb{R})$ satisfying certain axioms, which is a finite disjoint union of Hermitian symmetric domains.

Consider the tower $\{\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}} := \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)\}$ of complex manifolds indexed by *neat* open compact subgroups K of $\mathrm{G}(\mathbb{A}^\infty)$ (called the *levels*) with its right action by $\mathrm{G}(\mathbb{A}^\infty)$.

- Baily–Borel and Borel: This is the analytification of a canonical tower $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ of quasi-projective varieties.
- Shimura, ..., Deligne, ..., Milne, Borovoi: $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ has a *canonical model* $\{\mathrm{Sh}_K\}_K$ over a number field E (*reflex field*).

Hence, $\varinjlim_K H^\bullet_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ has commuting actions of $\mathrm{G}(\mathbb{A}^\infty)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. (Useful in Langlands program. "Trivial weight".)

For the main theme today, we need "*nontrivial weights*".

Betti local systems: complex analytic construction

Let $\mathrm{G}^c =$ quotient of G by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of G.

Let $V \in \mathrm{Rep}_F(\mathrm{G}^c)$ (algebraic representations of G^c over F), where F is any coefficient field. (Nontrivial weights!)

Then the compatible double coset construction

$$\mathrm{G}(\mathbb{Q})\backslash\big((\mathsf{X} \times V) \times \mathrm{G}(\mathbb{A}^\infty)/K\big) \to \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)$$

defines a (Betti) *local system* ${}_{\mathrm{B}}\underline{V}$ over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$.

(For general $V \in \mathrm{Rep}_F(\mathrm{G})$ not factoring through G^c, the construction breaks down due to infinite stabilizers.)

We will introduce certain étale and de Rham analogues of ${}_{\mathrm{B}}\underline{V}$, but we will not assume that they are (known to be) of geometric origin.

For simplicity of exposition, let us assume from now on that:

- (reflex field) $E = \mathbb{Q}$.
- $\mathrm{G} = \mathrm{G}^c$, and coefficient fields will be mainly $\mathbb{Q}$, $\mathbb{C}$, and $\mathbb{Q}_p$.

We will however emphasize nontrivial V and noncompact $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$.

Deligne's Riemann–Hilbert correspondence

Given an algebraic variety X over $\mathbb{C}$, there is a tensor equivalence of categories between the following:

$$\left\{\begin{matrix}\text{(Betti) local systems } \mathbb{L} \\ \text{with coefficient field } \mathbb{C} \\ \text{over } X^{\mathrm{an}}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{(algebraic) vector bundles} \\ \mathcal{E} \text{ over } X \\ \text{with integrable connections} \\ \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \\ \text{with regular singularities}\end{matrix}\right\}$$

The analytification $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ is just $(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d)$, where $d : \mathcal{O}_{X^{\mathrm{an}}} \to \Omega^1_{X^{\mathrm{an}}}$ is the usual derivation; and $\mathbb{L} = (\mathcal{E}^{\mathrm{an}})^{\nabla^{\mathrm{an}}}$ is the sheaf of horizontal sections. But Deligne's algebraic construction requires the *canonical extensions* of $(\mathcal{E}, \nabla)$ (over smooth compactifications of X with normal crossings boundary divisors, with eigenvalues of residues having real parts in $[0, 1)$).

Deligne also proved $H^i(X^{\mathrm{an}}, \mathbb{L}) \cong H^i_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}}) \cong H^i_{\mathrm{dR}}(X, \mathcal{E})$.

If $\mathcal{E}$ is equipped with a (decreasing) filtration $\mathrm{Fil}^\bullet$ satisfying the *Griffiths transversality* $\nabla(\mathrm{Fil}^i\mathcal{E}) \subset (\mathrm{Fil}^{i-1}\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X$, then we say (abusively, for simplicity) that $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$ is a *filtered connection*.

Filtered connections: complex analytic construction

Given $V \in \mathrm{Rep}_{\mathbb{Q}}(\mathrm{G})$ with ${}_{\mathrm{B}}\underline{V}$ over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$ as above, consider the *coefficient base change* ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}} := {}_{\mathrm{B}}\underline{V} \otimes_{\mathbb{Q}} \mathbb{C}$.

By Deligne's Riemann–Hilbert correspondence, we obtain $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$ over $\mathrm{Sh}_{K,\mathbb{C}}$, with analytification $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{an}}, \nabla^{\mathrm{an}})$.

Moreover, any $h \in \mathsf{X}$ induces $h_{\mathbb{C}} : \mathbb{C}^{\times} \times \mathbb{C}^{\times} \to \mathrm{G}(\mathbb{C})$, whose restriction to the first $\mathbb{C}^{\times}$ defines the *Hodge cocharacter* μ_h, inducing a filtration $\mathrm{Fil}^{\bullet}$ on ${}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{an}} = {}_{\mathrm{B}}\underline{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}}$ making $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{an}}, \nabla^{\mathrm{an}}, \mathrm{Fil}^{\bullet})$ a *filtered connection* over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$.

(In fact, we obtain a *variation of Hodge structures* over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$.)

The filtration $\mathrm{Fil}^{\bullet}$ extends to the canonical extension of $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ and hence algebraizes to a filtration of $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$, which we still denote by the same symbols. Thus, we obtain an *algebraic filtered connection* $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^{\bullet})$ over $\mathrm{Sh}_{K,\mathbb{C}}$.

(M. Harris, Milne: $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^{\bullet})$ also have *canonical models*.)

Next step: p-adic analytic constructions!

p-adic étale local systems

Fix a prime $p > 0$. Consider $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, with continuous action of $\mathrm{G}(\mathbb{A}^\infty)$ via the projection $\mathrm{G}(\mathbb{A}^\infty) \twoheadrightarrow \mathrm{G}(\mathbb{Q}_p)$. Let us define an *étale local system* (i.e., *lisse étale sheaf*) ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ over Sh_K:

Let $V_0 \subset V_{\mathbb{Q}_p}$ be any $\mathbb{Z}_p$-lattice stabilized by $K \subset \mathrm{G}(\mathbb{A}^\infty)$.

Take open normal subgroups $K^{(j)}$ of K acting trivially on $V_0/p^j V_0$.

These define Galois finite étale coverings $\mathrm{Sh}_{K^{(j)}} \to \mathrm{Sh}_K$ over E (not just over $\mathbb{C}$!) with Galois groups $K/K^{(j)}$, and the sections of

$$\mathrm{Sh}_{K^{(j)}} \times^{K/K^{(j)}} (V_0/p^j V_0) \to \mathrm{Sh}_{K^{(j)}}/(K/K^{(j)}) \cong \mathrm{Sh}_K$$

define a locally constant étale sheaf ${}_{\text{ét}}\underline{(V_0/p^j V_0)}$ over Sh_K.

Then we set ${}_{\text{ét}}\underline{V_0} := \varprojlim_j {}_{\text{ét}}\underline{(V_0/p^j V_0)}$ and ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p} := {}_{\text{ét}}\underline{V_0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Note: Well-defined, functorial, etc. Pullback of ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ to $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ can be compared with ${}_{\mathrm{B}}\underline{V}_{\mathbb{Q}_p} := {}_{\mathrm{B}}\underline{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ in a precise sense.

Natural questions: (also for $H^i_{\text{ét,c}}$, $H^i_{\text{ét,int}}{:=}\mathrm{Im}(H^i_{\text{ét,c}} {\to} H^i_{\text{ét}})$, $H^i_{\text{ét},\partial}$, "$IH^i_{\text{ét}}$")

- Is $H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})$ *de Rham*?
- How do we describe or compute $\mathrm{HT}\big(H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})\big)$?

p-adic (arithmetic) Riemann–Hilbert correspondence

Suppose X is a smooth algebraic variety over a finite extension k of $\mathbb{Q}_p$. We no longer have an equivalence, but only a functor (by Diao–Lan–Liu–Zhu, based on Scholze and Liu–Zhu):

$$D_{\mathrm{dR}}^{\mathrm{alg}} : \left\{ \begin{matrix} \text{étale } \mathbb{Q}_p\text{-local} \\ \text{systems } \mathbb{L} \text{ over } X \end{matrix} \right\} \to \left\{ \begin{matrix} \text{filtered connections} \\ (\mathcal{E}, \nabla, \mathrm{Fil}^\bullet) \text{ over } X \\ \text{with regular singularities} \end{matrix} \right\}$$

Some good properties of $D_{\mathrm{dR}}^{\mathrm{alg}}$:

- If X is a classical point (defined by a finite extension of $\mathbb{Q}_p$), then it is Fontaine's D_{dR} functor (already not an equivalence).
- Compatible with pullbacks in X (e.g., to a classical point).
- If X is connected and $\mathbb{L}$ is *de Rham* at one classical point of X, then it is de Rham at all other classical points of X.
- When restricted to de Rham étale $\mathbb{Q}_p$-local systems, $D_{\mathrm{dR}}^{\mathrm{alg}}$ is compatible with tensors and duals, and with proper smooth pushforwards. Also, we have the *p-adic de Rham comparison* $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X, D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})) \otimes_k B_{\mathrm{dR}}$ for de Rham $\mathbb{L}$, compatible with $\mathrm{Gal}(\overline{k}/k)$-actions and filtrations. (Will explain: filtration on H^i_{dR} uses smooth compactifications.)

Differences between complex and p-adic analytic theories

- As we explained, Deligne's Riemann–Hilbert correspondence is based on an analytic correspondence and on the construction of *canonical extensions* over smooth compactifications with normal crossings boundary divisors (so that GAGA applies).
- But p-adically, while we also have an analytic functor D_{dR} based on Scholze and Liu–Zhu, general connections have no canonical extensions (and, accordingly, no algebraizations).
- Instead, we constructed a p-adic *log Riemann–Hilbert functor* $D_{\mathrm{dR,log}}$, which provides the canonical extensions of $D_{\mathrm{dR}}(\mathbb{L}^{\mathrm{an}})$, by working with pro-Kummer étale sites and log de Rham period sheaves over suitable (analytified) smooth compactifications $\overline{X}^{\mathrm{an}}$.
- Crucially, $\overline{\mathbb{L}}^{\mathrm{an}} = R(X^{\mathrm{an}}_{\mathrm{\acute{e}t}} \to \overline{X}^{\mathrm{an}}_{\mathrm{k\acute{e}t}})_*(\mathbb{L}^{\mathrm{an}})$ is a local system, and *eigenvalues of residues* of $D_{\mathrm{dR,log}}(\overline{\mathbb{L}}^{\mathrm{an}})$ are in $\mathbb{Q} \cap [0,1)$ (by theory of decompletions, and $[k : \mathbb{Q}_p] < \infty$). Then $D_{\mathrm{dR,log}}$ and its algebraization $D^{\mathrm{alg}}_{\mathrm{dR,log}}$ induce the desired $D^{\mathrm{alg}}_{\mathrm{dR}}$. Also:
  - E_1 degen. of log Hodge s.s. $E_1^{a,b} = H^{a,b}_{\log \mathrm{Hodge}}(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR,log}}(\overline{\mathbb{L}}))$ $\Rightarrow H^{a+b}_{\log \mathrm{dR}}(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR,log}}(\overline{\mathbb{L}})) \cong H^i_{\mathrm{dR}}(X, D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L}))$ ($\cong$ uses residues).
  - p-adic generations of Kodaira vanishing (also using residues).

Filtered connections: p-adic analytic construction

Back to Shimura varieties. Given $V \in \mathrm{Rep}_{\mathbb{Q}}(\mathrm{G})$ with the étale local system ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ over Sh_K as before, we now have the following:

- By p-adic Riemann–Hilbert, the pullback of ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ to the variety $\mathrm{Sh}_{K,\mathbb{Q}_p} := \mathrm{Sh}_K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ defines a filtered connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{Q}_p}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{Q}_p}$ with regular singularities.
  (Note that ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ is de Rham because it is so at special points, which exist in abundance on each connected component.)
- By base change from $\mathbb{Q}_p$ to $\mathbb{C}$, we obtain a filtered connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$ with regular singularities.
  (Note that this base change from $\mathbb{Q}_p$ to $\mathbb{C}$ makes sense because we are working with *algebraic* filtered connections! It wouldn't work if we only had a rigid analytic construction.)
- By Deligne's Riemann–Hilbert correspondence, $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla)$ defines a Betti local system ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$.
  (The filtration is not used in the definition of ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$. But ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ still remembers the existence of filtrations—via the Simpson correspondence, the Higgs field is *nilpotent*.)

Summary of constructions, and comparisons of local systems

$$
\begin{tikzcd}
{}_{\mathrm{B}}\underline{V}_{\mathbb{C}} \arrow[dr, dash, dotted, "?"] & V \in \mathrm{Rep}_{\mathbb{Q}}(\mathrm{G}) \arrow[l] \arrow[r] & {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p} \arrow[d, "p\text{-adic RH}"] \\
({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet) \arrow[u, "\text{classical RH}"] \arrow[dr, dash, dotted, "?"] & {}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}} & ({}_{p\text{-}\mathrm{dR}}\underline{V}_{\mathbb{Q}_p}, \nabla, \mathrm{Fil}^\bullet) \arrow[dl, bend left, "\substack{\text{(barbaric)} \\ \text{base change from } \mathbb{Q}_p \text{ to } \mathbb{C}}"] \\
& ({}_{p\text{-}\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet) \arrow[u, "\text{classical RH}"] &
\end{tikzcd}
$$

- Do we have ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}} \cong {}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$?
- Do we have $({}_{p\text{-}\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet) \cong ({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$?

Theorem: Both answers are *yes*!

Not surprising for Shimura varieties of Hodge type, when the local systems are (Tate twists of) summands of the relative cohomology of abelian schemes (thanks to Deligne's and Blasius's works).

But it's unclear that such families of varieties exist in general. Our indirect proof used almost all tools we know over general Shimura varieties, including Hecke symmetry, Margulis superrigidity, and a construction of Piatetski-Shapiro's in Borovoi's and Milne's works.

De Rham and Hodge–Tate comparison isomorphisms

Note that $\iota^{-1} : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ induces $\mathbb{Q}_p \hookrightarrow \mathbb{C} \hookrightarrow B_{\mathrm{dR}}$ and $\mathbb{C} \hookrightarrow \mathbb{C}_p$.

With Liu and Zhu, we have established comparison isomorphisms forming the following commutative diagram

$$\begin{array}{ccc} H^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} & \xrightarrow{\sim} & H^i_{\mathrm{dR},\mathrm{c}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}) \otimes_{\mathbb{C}} B_{\mathrm{dR}} \\ \text{can.}\downarrow & & \downarrow\text{can.} \\ H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} & \xrightarrow{\sim} & H^i_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}) \otimes_{\mathbb{C}} B_{\mathrm{dR}} \end{array}$$

(strictly compatible with filtrations) and (by taking graded pieces)

$$\begin{array}{ccc} H^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \xrightarrow{\sim} & \oplus_a \big(H^{a,i-a}_{\mathrm{Hodge},\mathrm{c}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}_p(-a)\big) \\ \text{can.}\downarrow & & \downarrow\text{can.} \\ H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \xrightarrow{\sim} & \oplus_a \big(H^{a,i-a}_{\mathrm{Hodge}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}_p(-a)\big) \end{array}$$

These induce similar comparison for the interior cohomology.

Then $a \in \mathbb{Z}$ has multiplicity $\dim_{\mathbb{C}} H^{a,i-a}_{\mathrm{Hodge}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}})$ in $\mathrm{HT}\big(H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})\big)$, and similar for other cohomology.

We can more explicitly describe these dimensions, in terms of those of the so-called *coherent cohomology* (generalizing spaces of classical modular forms), using Faltings's dual BGG complexes.

Automorphic vector bundles and their extensions

Consider projective smooth *toroidal compactifications* $\mathrm{Sh}_K \hookrightarrow \mathrm{Sh}_K^{\mathrm{tor}}$ with normal crossings boundary divisors D (by AMRT and Pink).

Any $h : \mathbb{C}^\times \to \mathrm{G}(\mathbb{R})$ parameterized by X induces a homomorphism $h_\mathbb{C} : \mathbf{G}_{\mathrm{m},\mathbb{C}} \times \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_\mathbb{C}$, whose restriction to the first factor $\mathbf{G}_{\mathrm{m},\mathbb{C}}$ defines the so-called *Hodge cocharacter* $\mu_h : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_\mathbb{C}$. Up to modifying h in X, we may assume that μ_h is induced by some homomorphism $\mathbf{G}_{\mathrm{m},\overline{\mathbb{Q}}} \to \mathrm{G}_{\overline{\mathbb{Q}}}$, which we still denote by μ_h.

Let P denote the parabolic subgroup of $\mathrm{G}_{\overline{\mathbb{Q}}}$ defined by μ_h, with Levi subgroup M the centralizer of μ_h. We view any representation of M as one of P via the canonical homomorphism $\mathrm{P} \twoheadrightarrow \mathrm{M}$.

M. Harris: There is a tensor functor assigning to $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{P})$ the *automorphic vector bundle* ${}_{\mathrm{coh}}\underline{W}_\mathbb{C}$ over $\mathrm{Sh}_{K,\mathbb{C}}$, canonically isomorphic to ${}_{\mathrm{dR}}\underline{V}_\mathbb{C}$ when $W_\mathbb{C} \cong V_\mathbb{C}|_{\mathrm{P}_\mathbb{C}}$ for some $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G})$. Moreover, this extends to a tensor functor assigning to W the *canonical extension* ${}_{\mathrm{coh}}\underline{W}_\mathbb{C}^{\mathrm{can}}$ of ${}_{\mathrm{coh}}\underline{W}$ over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{tor}}$, canonically isomorphic to ${}_{\mathrm{dR}}\underline{V}_\mathbb{C}^{\mathrm{can}}$ when $W_\mathbb{C} \cong V_\mathbb{C}|_{\mathrm{P}_\mathbb{C}}$ for some $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G})$.

We also have the *subcanonical extension* ${}_{\mathrm{coh}}\underline{W}_\mathbb{C}^{\mathrm{sub}} := {}_{\mathrm{coh}}\underline{W}_\mathbb{C}^{\mathrm{can}}(-D)$.

Positive roots and weights, and Weyl actions

- Let us fix a maximal torus T of M and hence of $\mathrm{G}_{\overline{\mathbb{Q}}}$, with roots $\Phi_{\mathrm{G}_{\overline{\mathbb{Q}}}} \supset \Phi_{\mathrm{M}}$ and weights $\mathrm{X}_{\mathrm{G}_{\overline{\mathbb{Q}}}} = \mathrm{X}_{\mathrm{M}}$. Let us also choose compatibly positive roots $\Phi^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ and Φ^+_{M}, and dominant weights $\mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ and $\mathrm{X}^+_{\mathrm{M}}$, so that $\Phi^+_{\mathrm{M}} \subset \Phi^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ and $\mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}} \subset \mathrm{X}^+_{\mathrm{M}}$.
- For an irreducible $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}_{\overline{\mathbb{Q}}})$ of highest weight $\lambda \in \mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$, we write $V = V_\lambda$, $V_{\mathbb{C}} = V_{\lambda,\mathbb{C}}$, ${}_{\mathrm{dR}}\underline{V}_{\mathbb{C}} = {}_{\mathrm{dR}}\underline{V}_{\lambda,\mathbb{C}}$, etc. Similarly, for an irreducible $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{M})$ of highest weight $\nu \in \mathrm{X}^+_{\mathrm{M}}$, we write $W = W_\nu$, $W_{\mathbb{C}} = W_{\nu,\mathbb{C}}$, ${}_{\mathrm{coh}}\underline{W}_{\mathbb{C}} = {}_{\mathrm{coh}}\underline{W}_{\nu,\mathbb{C}}$, etc.
- Let $\rho = \rho_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ denote the usual half-sums of positive roots.
- Let $\mathrm{W}_{\mathrm{M}} \subset \mathrm{W}_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ denote the Weyl groups with respect to T.
- In addition to the natural action, there is also the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$, for all $w \in \mathrm{W}_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ and $\lambda \in \mathrm{X}_{\mathrm{G}_{\overline{\mathbb{Q}}}}$.
- Let W^{M} denote the subset of $\mathrm{W}_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ mapping $\mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$ into $\mathrm{X}^+_{\mathrm{M}}$, which are the minimal length representatives of $\mathrm{W}_{\mathrm{M}} \backslash \mathrm{W}_{\mathrm{G}_{\overline{\mathbb{Q}}}}$.
- Let H denote the coweight of $\mathrm{T} \subset \mathrm{M} \subset \mathrm{G}_{\overline{\mathbb{Q}}}$ induced by μ_h.

Faltings's dual BGG complexes

While the Hodge cohomology (as hypercohomology) is difficult to compute in general, $\mathrm{gr}^a\, DR_{\log}({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}})$ (the a-th graded piece of the log de Rham complex) has a miraculous quasi-isomorphic direct summand, called the *graded dual BGG complex*, whose differentials are *zero* and whose terms are direct sums of ${}_{\mathrm{coh}}\underline{W}_{\mathbb{C}}^{\mathrm{can}}$ for some representations W determined explicitly by V. Then the hypercohomology of this graded dual BGG complex is just a direct sum of coherent cohomology of ${}_{\mathrm{coh}}\underline{W}_{\mathbb{C}}^{\mathrm{can}}$ up to degree shifting.

More precisely, suppose $V \cong V_\lambda^\vee$ for some $\lambda \in \mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$. Then

$$\mathrm{gr}^a\, BGG^j_{\log}\big({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}\big) \cong \oplus_{w\in \mathrm{W}^{\mathrm{M}},\, l(w)=j,\, (w\cdot\lambda)(H)=-a}\ \big({}_{\mathrm{coh}}\underline{W}_{w\cdot\lambda,\mathbb{C}}^\vee\big)^{\mathrm{can}}$$

Consequently, we have

$$\begin{aligned} H^{a,i-a}_{\mathrm{Hodge}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}) &:= H^i\big(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, \mathrm{gr}^a\, DR_{\log}\big({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}\big)\big) \\ &\cong \oplus_{w\in \mathrm{W}^{\mathrm{M}},\, (w\cdot\lambda)(H)=-a}\ H^{i-l(w)}\big(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, \big({}_{\mathrm{coh}}\underline{W}_{w\cdot\lambda,\mathbb{C}}^\vee\big)^{\mathrm{can}}\big)\end{aligned}$$

We have similar decomposition for $H^{a,i-a}_{\mathrm{Hodge,c}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}})$ in terms of the coherent cohomology of subcanonical extensions.

Example: classical modular curves

Consider $\mathrm{G} = \mathrm{SL}_2$ (up to cocenter), and identify $\mathrm{X}_{\mathrm{G}_{\overline{\mathbb{Q}}}} = \mathrm{X}_{\mathrm{M}} = \mathbb{Z}$.

$\lambda = k, \quad W_\nu \cong W_{w\cdot\lambda}^\vee$
$l(w) = ? \quad \nu = ?$

(non-cohomological)
low weight
$\downarrow$

$l(w) = 0$ $\quad$ $-k$ $\qquad$ $0 \quad 1 \quad 2$ $\qquad$ $l(w) = 1$ $\quad$ $k+2$

dual BGG = Eichler–Shimura isomorphism (for weights ≥ 2 or ≤ 0):

$$\mathrm{gr}\, H^1_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{k,\mathbb{C}}) \cong H^0(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{k+2,\mathbb{C}}) \oplus H^1(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{-k,\mathbb{C}})$$

$${}_{\mathrm{dR}}\underline{V}_{k,\mathbb{C}} = \mathrm{Sym}^k(\mathbb{C}^{\oplus 2}), \quad {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{k,\mathbb{C}} = ({}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{-k,\mathbb{C}})^\vee = \omega^k$$

Weight 1 modular forms do not contribute to de Rham cohomology (but can be studied by congruences with cohomological weights).

Later when computing Hodge–Tate weights, need to use the full $\mathrm{G} = \mathrm{GL}_2$:

$$\mathrm{gr}\, H^1_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{(0,-k),\mathbb{C}}) \cong H^0(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(1,-k-1),\mathbb{C}}) \oplus H^1(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(-k,0),\mathbb{C}})$$

(Evaluate the weights $(1, -k-1)$ and $(-k, 0)$ on the coweight $H = (0, -1)$.)

Example: Siegel modular threefolds

Consider $\mathrm{G} = \mathrm{Sp}_4$ (up to cocenter), and identify $\mathrm{X}_{\mathrm{G}_{\overline{\mathbb{Q}}}} = \mathrm{X}_{\mathrm{M}} = \mathbb{Z}^2$.

$\lambda = (k_1, k_2), \quad W_\nu \cong W^\vee_{w\cdot\lambda}$

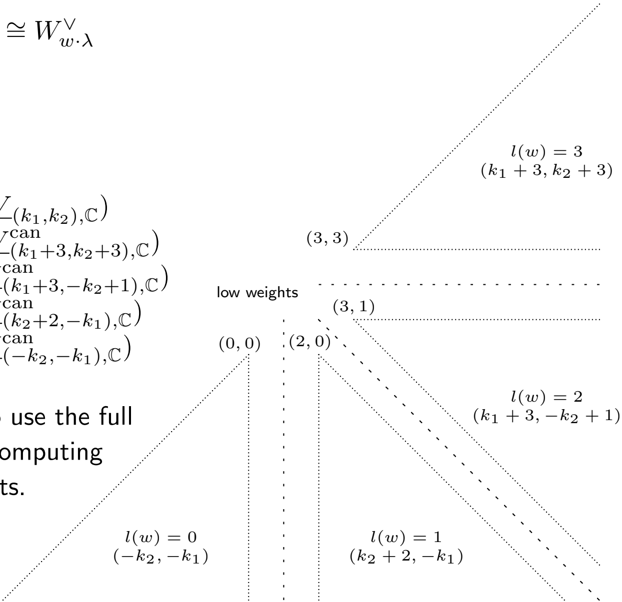
$l(w) = ?$

$\nu = ?$

dual BGG:

$$
\begin{aligned}
&\mathrm{gr}\, H^3_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{(k_1,k_2),\mathbb{C}}) \\
&\cong H^0(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(k_1+3,k_2+3),\mathbb{C}}) \\
&\oplus H^1(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(k_1+3,-k_2+1),\mathbb{C}}) \\
&\oplus H^2(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(k_2+2,-k_1),\mathbb{C}}) \\
&\oplus H^3(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{coh}}\underline{W}^{\mathrm{can}}_{(-k_2,-k_1),\mathbb{C}})
\end{aligned}
$$

Again, will need to use the full $G = \mathrm{GSp}_4$ when computing Hodge–Tate weights.



Hodge–Tate weights for Shimura varieties

Thus, when $V \cong V_\lambda^\vee$, for $\lambda \in \mathrm{X}^+_{\mathrm{G}_{\overline{\mathbb{Q}}}}$, each $a \in \mathbb{Z}$ has multiplicity

- $\sum_{w \in \mathrm{W}^{\mathrm{M}},\, (w \cdot \lambda)(H) = -a} \dim_{\mathbb{C}} H^{i - l(w)}\big(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, \big({}_{\mathrm{coh}}\underline{W}^\vee_{w \cdot \lambda, \mathbb{C}}\big)^{\mathrm{can}}\big)$ in $\mathrm{HT}\big(H^i_{\text{ét}}(\mathrm{Sh}_{K, \overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})\big)$;
- $\sum_{w \in \mathrm{W}^{\mathrm{M}},\, (w \cdot \lambda)(H) = -a} \dim_{\mathbb{C}} H^{i - l(w)}\big(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, \big({}_{\mathrm{coh}}\underline{W}^\vee_{w \cdot \lambda, \mathbb{C}}\big)^{\mathrm{sub}}\big)$ in $\mathrm{HT}\big(H^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}_{K, \overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})\big)$; and
- $\sum_{w \in \mathrm{W}^{\mathrm{M}},\, (w \cdot \lambda)(H) = -a} \dim_{\mathbb{C}} H^{i - l(w)}_{\mathrm{int}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, \big({}_{\mathrm{coh}}\underline{W}^\vee_{w \cdot \lambda, \mathbb{C}}\big)^{\mathrm{can}}\big)$ in $\mathrm{HT}\big(H^i_{\text{ét},\mathrm{int}}(\mathrm{Sh}_{K, \overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})\big)$, where "$\mathrm{int}$" means the image of the cohomology of "sub", by abuse of notation.

These dimensions can be computed in terms of relative Lie algebra cohomology (M. Harris, Jun Su, based on Borel, Franke, ...)

By results of Schwermer, Li–Schwermer, and Harris–Zucker, when λ is regular, the *interior cohomology* coincides (strictly!) with the *intersection cohomology*, so the above results also cover the latter.

When $\lambda = 0$, people have techniques showing that the intersection cohomology is de Rham. But, not just for other irregular λ, it is desirable to extend p-adic RH to perverse sheaves (beyond ICs).