Mathematics in the Computer

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April 26, 2021
Who am I?

Github: digama0
Zulip: Mario Carneiro

- PhD student in Logic at CMU
- Proof engineering since 2013
  - Metamath (maintainer)
  - Lean 3 (maintainer)
  - Dabbled in Isabelle, HOL Light, Coq, Mizar
  - Metamath Zero (author)
- Proved 37 of Freek’s 100 theorems list in Metamath
- Lots of library code in `set.mm` and `mathlib`
- Say hi at https://leanprover.zulipchat.com
How I got involved in formalization

- Undergraduate at Ohio State University
  - Math, CS, Physics
- Reading Takeuti & Zaring, *Axiomatic Set Theory*

→ Found Metamath via a random internet search
  - → they already formalized half of the book!
  - ... and there is some stuff on cofinality they don't have yet, maybe I can help
- Got involved, did it as a hobby for a few years
- Got a job as an android developer, kept on the hobby
- Norm Megill suggested that I submit to a (Mizar) conference, it went well
- Met Leo de Moura (Lean author) at a conference, he got me in touch with Jeremy Avigad (my current advisor)
- Now I'm a PhD at CMU philosophy!
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  - It is a real eye opener when you realize how much you skipped over when reading a proof “normally” vs when you have to convince a computer

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What is it like to formalize mathematics?
Sidebar: Metamath architecture

Metamath is a specification for .mm files that contain definitions, theorems, and proofs. Metamath has many (many!) verifiers, written in dozens of languages. Proofs in an .mm file are expressed in a compressed format that is not intended to be written by humans. To produce proofs, you need a proof assistant, and there are several of these (not as many as the verifiers). I used for this for many years (and now I'm the maintainer).
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I used mmj2\(^1\) for this for many years (and now I’m the maintainer).

\(^1\)https://github.com/digama0/mmj2/
Sidebar: Metamath architecture

Search “mmj2 tutorial” to see it in action
What is it like to formalize mathematics?

In Metamath:

▶ You state a theorem in the formal language
▶ You apply theorems from the library, and the computer prompts you for the subgoals
▶ In some cases the computer can automatically prove some of the subgoals
▶ When you finish the proof, it gives you a big blob to stick in the .mm file and celebrate
▶ You PR your changes to the set.mm repository
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In Lean:

- You state a theorem in the formal language
- You apply tactics, which transform the goals into subgoals in some defined way
- Often the tactic is simply apply `thm` where `thm` is a lemma from the library
- When you finish the proof, you leave the tactic script in the file
- You PR your changes to the `mathlib` repository
What is it like to formalize mathematics?

From Kevin Buzzard’s “10 minute Lean tutorial: proving logical propositions”
How do Metamath and Lean differ?

- Lean has a lot more institutional support (MSR, CMU)
- Lean has a great user experience and is generally better suited to mathematician users
- Metamath is deliberately as simple as it can be, and writing a verifier for Metamath is a weekend project
  - “Batteries not included”: tooling is decentralized and DIY
- There is essentially only one Lean verifier
  - Parsing lean files is nearly impossible for anything other than lean
  - Lean has an export format that has a few alternate verifiers
- Compiling set.mm is about $1000 \times$ faster than compiling mathlib
How do we get closer to air tight mathematics?

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1. The verifier should be as simple as it can be
2. This isn’t good enough, because the chance of a bug in the program will still be larger than the computer’s own error rate
How do we get closer to air tight mathematics?

- How do we ensure there are no bugs in a program?
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- Hammer, meet nail
Self verifying theorem provers

Wait, didn’t Gödel prove this is impossible?

Well, yes and no.

The important observation is that we want to prove “implementation correctness,” not consistency.

We will not be able to completely eliminate the circularity, though, so we still need to rely on human verification to some extent.
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The metamathematics of theorem provers

- Let $\mathcal{M} \subseteq \{0, 1\}^* \times \{0, 1\}^*$ be a machine semantics, where $\mathcal{M}(P, x)$ means that program $P$ on input $x$ terminates and indicates success.
  - For example, $\mathcal{M}(P, x)$ if $P$ encodes a Turing machine that when run on input $x$ stops in finitely many steps with a 1 on the tape
- Let $\mathcal{L} \subseteq \{0, 1\}^*$ be a language of assertions
  - For example, $\varphi \in \mathcal{L}$ if $\varphi$ encodes a statement in FOL
  - We generally want $\varphi \in \mathcal{L}$ to be decidable
- Let $T$ be a theory of interest
  - For example, $T = \text{ZFC}$

Implementation correctness for a theorem prover

Program $V$ is a theorem prover (for $T$ in $\mathcal{M}$ and $\mathcal{L}$) if for all $\varphi \in \mathcal{L}$, if there exists $p$ such that $\mathcal{M}(V, (\varphi, p))$, then $T \vdash \varphi$. 
1. Suppose $V$ is a correct verifier, i.e. if $T \vdash_V A$ then $T \vdash A$ for all $T, A$.

2. Suppose we prove, in $V + PA$, the correctness theorem for $W$, that is, $PA \vdash_V \forall T, A: (T \vdash_W A \rightarrow T \vdash A)$

3. Then $PA \vdash \forall T, A: (T \vdash_W A \rightarrow T \vdash A)$

4. If $PA$ is sound, then $\forall T, A: (T \vdash_W A \rightarrow T \vdash A)$, that is, $T \vdash_W A$ implies $T \vdash A$, so $W$ is a correct verifier

- Bootstrapping: set $W := V$ in the above
- Note that this does not run afoul of Gödel incompleteness
- Circular proof! Need a backup to ground the argument
  - $\Rightarrow$ small verifier
  - independent bootstraps
From theory to practice

- Can we use Lean to prove Lean correct?

...
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  - Maybe, although it’s kind of painful
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Metamath Zero Architecture

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- The plan: write a verifier for MM0 in the MMC language
From “Metamath Zero (MM0/MM1) tutorial”
Metamath Zero Architecture

MM0 input is split into two parts:

- **.mm0 specification file**
  - trusted, human readable
  - contains the statement of axioms and assertions

- **.mmb proof file**
  - untrusted, binary
  - fully elaborated, designed for efficient checking by the verifier

Underlined components are trusted.
delimeter $( ] \sim , ] \sim \) $; -- these tokens don't need a space after them
$ ) $; -- these tokens don't need a space before them

-- Propositional logic
strict provable sort wff;

term im: wff > wff > wff;
infixr im: $\rightarrow$ prec 25;

term not: wff > wff;
prefix not: $\sim$ prec 41;

axiom ax_1 (ph ps: wff): $ ph \rightarrow ps \rightarrow ph $;
axiom ax_2 (ph ps ch: wff): $ (ph \rightarrow ps \rightarrow ch) \rightarrow (ph \rightarrow ps) \rightarrow ph \rightarrow ch $;
axiom ax_3 (ph ps: wff): $ (\sim ph \rightarrow \sim ps) \rightarrow ps \rightarrow ph $;
axiom ax_mp (ph ps: wff): $ ph \rightarrow ps $ > $ ph $ > $ ps $;

def an (a b: wff): wff = $ \sim (a \rightarrow \sim b) $;
infixl an: $/\$ prec 34;

def iff (a b: wff): wff = $ (a \rightarrow b) /\ (b \rightarrow a) $;
infixl iff: $<\rightarrow$ prec 20;

def or (a b: wff): wff = $ \sim a \rightarrow b $;
infixl or: $/\$ prec 30;
Predicate logic (on nat)

```
sort nat;
term al {x: nat} (ph: wff x): wff;
prefix al: $A.$ prec 41;

def ex {x: nat} (ph: wff x): wff = $
\neg (A. x \neg ph)$ $;
prefix ex: $E.$ prec 41;

term eq (a b: nat): wff;
infixl eq: $= $ prec 50;

axiom ax_gen {x: nat} (ph: wff x): $ ph $ > $ A. x ph $;
axiom ax_4 {x: nat} (ph ps: wff x): $ A. x (ph -> ps) -> A. x ph -> A. x ps $;
axiom ax_5 {x: nat} (ph: wff): $ ph -> A. x ph $;
axiom ax_6 (a: nat) {x: nat}: $ E. x x = a $;
axiom ax_7 (a b c: nat): $ a = b -> a = c -> b = c $;
axiom ax_10 {x: nat} (ph: wff x): $ \neg A. x ph -> A. x \neg A. x ph $;
axiom ax_11 {x y: nat} (ph: wff x y): $ A. x A. y ph -> A. y A. x ph $;
axiom ax_12 {x: nat} (a: nat) (ph: wff x): $ x = a -> ph -> A. x (x = a -> ph) $;

def sb (a: nat) {x .y: nat} (ph: wff x): wff = $ A. y (y = a -> A. x (x = y -> ph)) $;
notation sb (a: nat) {x: nat} (ph: wff x): wff = ($[$:41] a ($/$:0) x ($]$:$0) ph;
```
-- Peano's axioms

term d0: nat; prefix d0: $0$ prec max;
term suc (n: nat): nat;
axiom peano1 (a: nat): $\sim\text{suc } a = 0$;
axiom peano2 (a b: nat): $\text{suc } a = \text{suc } b \leftrightarrow a = b$;
axiom peano5 {x: nat} (P: wff x):
  $[0 / x] P \rightarrow A. x (P \rightarrow [\text{suc } x / x] P) \rightarrow A. x P$;

term add: nat > nat > nat; infixl add: $+$ prec 64;
term mul: nat > nat > nat; infixl mul: $*$ prec 70;

axiom addeq (a b c d: nat): $a = b \rightarrow c = d \rightarrow a + c = b + d$;
axiom muleq (a b c d: nat): $a = b \rightarrow c = d \rightarrow a * c = b * d$;
axiom add0 (a: nat): $a + 0 = a$;
axiom addS (a b: nat): $a + \text{suc } b = \text{suc } (a + b)$;
axiom mul0 (a: nat): $a * 0 = 0$;
axiom mulS (a b: nat): $a * \text{suc } b = a * b + a$;
This is a complete .mm0 file that asserts that Goldbach’s conjecture (GC) is derivable from the axioms of Peano Arithmetic.

The correctness theorem for MM0 implies that if the MM0 verifier accepts any .mmb proof of this .mm0 file, then GC is in fact provable in PA.
The MM0 toolchain

- MM0 verifier is currently implemented in C (2000 LOC)
  - The MM1 proof assistant is in Rust
  - MMC implementation in progress (in Rust)
- Scales well to large developments
  - Can verify set.mm library of ~30000 proofs in 200 ms, faster than metamath itself (metamath.exe – 8 s, smm – 900 ms)
    - That’s 10,000× faster than lean, although this is an incredibly unfair comparison for a number of reasons
  - The library of supporting material from PA for this project checks in 2 ms
- Proof terms are stored in the binary format as fully elaborated and fully deduplicated terms
  - Essentially linear time verification
- Proof size is comparable to compiled proof formats in other languages (.mm, .olean)
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- Proving Lean correct in MM0 is no easier than Lean in Lean, but we have another alternative: proof translation
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- Even after the MM0 self-verification theorem is complete, it is only useful if people use it... or is it?
- A theorem prover that can bootstrap itself can also be the foundation for others
- Proving Lean correct in MM0 is no easier than Lean in Lean, but we have another alternative: proof translation
- If all theorems in Lean can be translated in a proof preserving way to MM0 theorems, then Lean can be used as an MM0 IDE
  - MM0 gains all the benefits of Lean’s user interface
  - We don’t need to convince anyone to switch
  - Work on Lean → MM0 translation can proceed independently of new theorems to mathlib
Translation

- The MM0 formal system lies at the intersection of Metamath and second order logic, and so it has easy translation paths to each
  - MM $\rightarrow$ MM0
  - MM0 $\rightarrow$ OpenTheory
  - MM0 $\rightarrow$ Lean

- The MM $\rightarrow$ MM0 translator has been used to losslessly translate the entire Metamath ZFC library into MM0

```lean
theorem dirith' {n : ℕ} {a : ℤ} (n0 : n ≠ 0) (g1 : int.gcd a n = 1) :
  ¬ set.finite {x | nat.prime x ∧ ↑n | ↑x - a}
```
The Rosy Future

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- The bottleneck on interesting formalized mathematics today is lack of mathematicians who know and care to formalize interesting mathematics.
- There are no serious technical limitations on proving any branch of mathematics that I am aware of.
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  - Also, by translation *from* MM0, libraries can communicate formal content and build on material proved in other theorem provers.
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  - Also, by translation from MM0, libraries can communicate formal content and build on material proved in other theorem provers.
- The MM0 bootstrap itself can be improved independently, for example by verifying the electronic model of the hardware.
All together, these points mean that anyone will have easy access to mechanical means to verify proofs to the quality of the physical hardware (up to the probability of e.g. cosmic ray interference and our understanding of physical laws).
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Then:

- Make it easy for programmers to write bug-free software through better verified-programming language design
- Make it easy for mathematicians to write formal mathematics, so that all the new mathematics that comes out is also verified
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**Correctness: solved!**
Conclusion

▶ Verification systems are the means by which we check mathematics in the computer
▶ Metamath and Lean are near polar opposite designs for a theorem prover
  ▶ (I recommend Lean for mathematicians)
▶ To improve on the correctness frontier, we need to verify the verifier
▶ The goal of the MM0 project is to prove a correctness theorem for the verifier of the form “if execution of verifier $V$ according to the semantics of x86 machine code reports that theorem $\varphi$ follows from axioms $T$, then $T \vdash \varphi$”
▶ Large parts of the project are already complete, and you can play with the MM1 proof assistant today
Resources

- Metamath: http://us.metamath.org/
- Lean/mathlib: http://leanprover-community.github.io/
- Metamath Zero: https://github.com/digama0/mm0
- Lean Zulip: https://leanprover.zulipchat.com/
  - Ask me anything on Zulip, I’m there a lot

Thanks!