Howe to transfer Harish-Chandra characters via Weil's representation

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Dual Pair

- F a p-adic field.
- W symplectic vector space of dimension $2n$, with isometry group $\mathrm{Sp}(W)$.
- $V = V^{\pm}$ the two $2m+1$ dimensional quadratic spaces of discriminant 1, with isometry group $\mathrm{O}(V^{\pm})$.

Have a dual pair

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(W \otimes V).$$

Let $\mathrm{Mp}(W)$ be the unique two-fold nonlinear cover of $\mathrm{Sp}(W)$.

Have:

$$\mathrm{Mp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Mp}(W \otimes V).$$

Weil Representation and Theta Correspondence

For a fixed nontrivial character ψ of F, let

$$\omega_\psi = \text{Weil rep. of } \mathrm{Mp}(W \otimes V).$$

Pulling back gives a representation $\omega_{V,W,\psi}$ of $\mathrm{Mp}(W) \times \mathrm{O}(V)$.

For $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, define a smooth rep. of $\mathrm{Mp}(W)$ by

$$\Theta(\pi) = (\omega_{V,W,\psi} \otimes \pi^\vee)_{\mathrm{O}(V)} \quad \text{(big theta lift)}.$$

Likewise, for $\tilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}(W))$, have smooth rep. $\Theta(\tilde{\pi})$ of $\mathrm{O}(V)$.

Theorem (Howe Duality)

(i) $\Theta(\pi)$ has finite length and a unique irreducible quotient $\theta(\pi)$.

(ii) If $\pi_1 \neq \pi_2$, then $\theta(\pi_1) \neq \theta(\pi_2)$ (if both nonzero).

Equal Rank Case

We shall focus on the special case $m = n$, so that $\dim V^{\pm} = \dim W + 1 = 2n + 1$.

Theorem (Local Shimura Correspondence)

The theta correspondence, together with the restriction from $O(V)$ to $SO(V)$, gives a bijection

$$\mathrm{Irr}(Mp(W)) \longleftrightarrow \mathrm{Irr}(SO(V^+)) \sqcup \mathrm{Irr}(SO(V^-)).$$

Moreover, under this bijection, discrete series representations correspond, and so do tempered representations.

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

Characters

If $\pi \in \mathrm{Irr}(G(F))$, set

$$\Theta_\pi = \text{ Harish-Chandra character of } \pi.$$

It is a conjugacy-invariant distribution on $G(F)$, which is given by a locally L^1 smooth function on the regular semisimple locus:

$$\Theta_\pi : C_c^\infty(G(F)) \twoheadrightarrow C_c^\infty(G(F))_{G(F)^\Delta} \to \mathbb{C}.$$

If π is unitary and $\{e_i\}$ is an orthonormal basis of π, then

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f)) = \sum_i \langle \pi(f)e_i, e_i \rangle.$$

If π is tempered, then Θ_π is a tempered distribution: it extends to a linear form on the Harish-Chandra-Schwarz space $\mathcal{C}(G(F)) \subset L^2(G(F))$.

The Question

Suppose $\tilde{\pi} \in \mathrm{Irr}(\mathsf{Mp}(W))$ and $\pi \in \mathrm{Irr}(\mathsf{SO}(V^\epsilon))$ satisfy

$$\tilde{\pi} = \theta(\pi).$$

Question: $\boxed{\text{How are the characters of } \pi \text{ and } \tilde{\pi} \text{ related?}}$

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- This question has been studied by T. Prezbinda when $F = \mathbb{R}$; he introduced a construction known as the Cauchy-Harish-Chandra integral, which transfers invariant eigendistributions from one group to another and conjectured that this relates the characters of representations in theta correspondence with each other.
- He verified his conjecture in the stable range. The analytic difficulties in working with this integral is one obstacle in extending his results beyond the stable range.

$\boxed{\text{We would like to propose a more conceptual approach.}}$

An Approach to Character Relations

We shall:

- introduce spaces of test functions on $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$;
- define a notion of transfer of test functions from one group to another;
- show that this transfer descends to a well-defined map on the level of coinvariant spaces (i.e. orbital integrals), thus inducing a transfer of invariant distributions.
- show that the transfer of Θ_π is equal to $\Theta_{\tilde{\pi}}$.
- describe the transfer in geometric terms (in terms of a moment map).

Spaces of Test Functions

Consider the diagram

$$\begin{tikzcd}[column sep=small] & \Omega^\epsilon = \omega_{W,V^\epsilon} \otimes \overline{\omega_{W,V^\epsilon}} \arrow[dl, "p^\epsilon"'] \arrow[dr, "q^\epsilon"] & \\ C^\infty(\mathrm{Mp}(W)) & & C^\infty(\mathrm{SO}(V^\epsilon)) \end{tikzcd}$$

The two maps are defined by matrix coefficients:

$$p^\epsilon(\phi_1 \otimes \phi_2)(g) = \langle \phi_1, g\phi_2 \rangle \quad \text{and} \quad q^\epsilon(\phi_1 \otimes \phi_2)(h) = \langle \phi_1, h\phi_2 \rangle.$$

for $\phi_1 \otimes \overline{\phi_2} \in \Omega^\epsilon$. Set

$$\mathcal{S}^\epsilon(\mathrm{Mp}(W)) = \mathrm{Image}(p^\epsilon) \quad \text{and} \quad \mathcal{S}(\mathrm{SO}(V^\epsilon)) = \mathrm{Image}(q^\epsilon).$$

These are the spaces of test functions.

Transfer of Test Functions

We say that

$$f^{\epsilon} \in \mathcal{S}(\mathrm{SO}(V^{\epsilon})) \quad \text{and} \quad \tilde{f}^{\epsilon} \in \mathcal{S}^{\epsilon}(\mathrm{Mp}(W))$$

are transfer of each other if there exists $\Phi \in \Omega^{\epsilon}$ such that

$$p^{\epsilon}(\Phi) = f^{\epsilon} \quad \text{and} \quad q^{\epsilon}(\Phi) = \tilde{f}^{\epsilon}.$$

More generally, say that

$$f = (f^{+}, f^{-}) \in \mathcal{S}(\mathrm{SO}(V^{\pm})) := \mathcal{S}(\mathrm{SO}(V^{+})) \oplus \mathcal{S}(\mathrm{SO}(V^{-}))$$

and

$$\tilde{f} = (\tilde{f}^{+}, \tilde{f}^{-}) \in \mathcal{S}(\mathrm{Mp}(W)) := \mathcal{S}^{+}(\mathrm{Mp}(W)) \oplus \mathcal{S}^{-}(\mathrm{Mp}(W))$$

are in correspondence if the $\pm$-components correspond.
Transfers always exist, by definition.

Properties of Test Functions

Lemma

$$C_c^\infty(Mp(W)) \subset \mathcal{S}(Mp(W)) \subset \mathcal{C}(Mp(W))$$

and

$$C_c^\infty(SO(V^\epsilon)) \subset \mathcal{S}(SO(V^\epsilon)) \subset \mathcal{C}(SO(V^\epsilon)).$$

Corollary

For $\pi \in \mathrm{Irr}_{temp}(SO(V^\epsilon))$ *and* $f \in \mathcal{S}(SO(V^\epsilon))$ *the operator* $\pi(f)$ *is defined and so is its trace*

$$\Theta_\pi(f) = \sum_{v \in ONB(\pi)} \langle \pi(f)v, v \rangle$$

Equivariance Properties

Lemma

(i) The map

$$p^\epsilon : \Omega^\epsilon \longrightarrow \mathcal{C}(Mp(W))$$

is $Mp(W) \times Mp(W)$-equivariant and $O(V^\epsilon)^\Delta$-invariant.

Indeed, $p = p^+ \oplus p^-$ induces an isomorphism

$$\bigoplus_\epsilon \Omega^\epsilon_{O(V^\epsilon)^\Delta} \cong \mathcal{S}(Mp(W)).$$

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(ii) The map

$$q^{\epsilon} : \Omega^{\epsilon} \longrightarrow \mathcal{C}(SO(V^{\epsilon}))$$

is $SO(V^{\epsilon}) \times SO(V^{\epsilon})$-equivariant and $Mp(W)^{\Delta}$-invariant. We have an isomorphism

$$\oplus_{\epsilon}(\Omega^{\epsilon})_{Mp(W)^{\Delta}} \cong \oplus_{\epsilon}\mathcal{S}(SO(V^{\epsilon})) =: \mathcal{S}(SO(V^{\pm})).$$

Isomorphism of Spaces of Orbital Integrals

Consider the composite:

$$\Omega = \oplus_\epsilon \Omega^\epsilon \to \mathcal{S}(\mathrm{Mp}(W)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}.$$

This map is $\mathrm{Mp}(W)^\Delta$-invariant and thus factors as:

$$\Omega \to \Omega_{\mathrm{Mp}(W)^\Delta} \cong \mathcal{S}(\mathrm{SO}(V^\pm)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}$$

Since the last arrow is also $\mathrm{SO}(V^\epsilon)^\Delta$-invariant, it further descends to

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Lemma

This construction gives an isomorphism

$$\mathcal{S}(SO(V^+))_{SO(V^+)^\Delta} \oplus \mathcal{S}(SO(V^-))_{SO(V^-)^\Delta} \cong \mathcal{S}(Mp(W))_{Mp(W)^\Delta}.$$

A Character Identity

The previous lemma allows one to transfer invariant distributions between $\mathcal{S}(\mathrm{Mp}(W))$ and $\mathcal{S}(\mathrm{SO}(V^{\pm})$.

Theorem

Suppose that $\pi \in \mathrm{Irr}_{temp}(SO(V^{\epsilon}))$*, so that* $\tilde{\pi} = \theta(\pi) \in \mathrm{Irr}_{temp}(Mp(W))$*. Then for* f *and* $\tilde{f}$ *in correspondence,*

$$\Theta_{\pi}(f) = \Theta_{\tilde{\pi}}(\tilde{f}).$$

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$$\Theta_{\pi}(f) = \Theta_{\tilde{\pi}}(\tilde{f}).$$

$\boxed{\text{What we will discuss in the rest of this talk}}$

- a sketch proof of the character identity.
- a geometric description of the transfer of test functions.

Sketch Proof via the Plancherel Theroem

For $\Phi = \phi_1 \otimes \phi_2 \in \Omega^\epsilon$, observe that

$$p(\Phi)(1) = \langle \phi_1, \phi_2 \rangle = q(\Phi)(1).$$

Since $p(\Phi) \in \mathcal{C}(\mathrm{Mp}(W))$, the Harish-Chandra-Plancherel theorem gives:

$$p(\Phi)(1) = \int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}).$$

Likewise,

$$q(\Phi)(1) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

So we get the equality of both RHS's.

We have shown:

$$\int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

Under the local Shimura correspondence

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \cup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

one has (by G.-Ichino)

$$\theta_*(d\mu_{\mathrm{SO}(V^+)}) + \theta_*(d\mu_{\mathrm{SO}(V^-)}) = d\mu_{\mathrm{Mp}(W)}$$

This gives

$$\int_{\widehat{\mathrm{SO}(V)}} \Theta_{\theta(\pi)}(p(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

One can peel off the integrals on both sides using a Bernstein center argument.

Doubling

The key for understanding the transfer is to interpret everything in terms of the doubling see-saw.

Let

$$\mathbb{W} = W \oplus (-W)$$

be the doubled symplectic space. This contains

$$W^{\Delta} = \{(w, w) : w \in W\} \quad \text{and} \quad W^{\nabla} = \{(w, -w) : w \in W\}$$

as maximal isotropic subspaces.

Likewise,

$$\mathbb{V} = V \oplus (-V)$$

which contains V^{Δ} and V^{∇} as maximal isotropic subspaces.

Observe that one has isomorphisms of symplectic spaces:

$$V \otimes \mathbb{W} = (V \otimes W) \oplus (V \otimes (-W)) \cong (V \otimes W) \oplus ((-V) \otimes W) = \mathbb{V} \otimes W.$$

Doubling See-Saw

In $\mathrm{Sp}(V \otimes \mathbb{W}) = \mathrm{Sp}(\mathbb{V} \otimes W)$, there are 2 dual pairs fitting in a see-saw:

$$\begin{tikzcd}
\mathrm{Mp}(\mathbb{W}) \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(V) \times \mathrm{O}(V) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{Mp}(W) \times \mathrm{Mp}(-W) & \mathrm{O}(V)^{\Delta}
\end{tikzcd}$$

Moreover, if $\Omega_{\mathbb{W},V,\psi}$ denotes the Weil rep. for $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$, then

$$\Omega_{\mathbb{W},V,\psi} \cong \omega_{W,V,\psi} \otimes \overline{\omega_{W,V,\psi}}$$

when restricted to $\mathrm{Mp}(W) \times \mathrm{Mp}(W) \times \mathrm{O}(V)^{\Delta}$.

$$\Omega_{\mathbb{W},V,\psi} \cong \omega_{W,V,\psi} \otimes \overline{\omega_{W,V,\psi}}$$

RHS is the domain Ω of the maps p and q in the definition of transfer; we see now that it is a Weil rep. for $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$.

We want to interpret the map

$$p : \Omega \longrightarrow \mathcal{S}(\mathrm{Mp}(W))$$

through the lens of the other dual pair $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$.

Since p is $\mathrm{O}(V)^{\Delta}$-invariant, it descends to

$$p : \Omega_{\mathrm{O}(V)^{\Delta}} \longrightarrow \mathcal{S}(\mathrm{Mp}(W))$$

Now the LHS is a rep. of $\mathrm{Mp}(\mathbb{W})$ which has been described by Rallis.

A Result of Rallis

Theorem

There is a natural morphism of $Mp(\mathbb{W})$-modules

$$\iota : \oplus_\epsilon \Omega^\epsilon \twoheadrightarrow \oplus_\epsilon \Omega^\epsilon_{O(V^\epsilon)} \cong I_{P(W^\Delta)}(0)$$

where the RHS is a degenerate principal series rep. of $Mp(\mathbb{W})$ induced from the Siegel parabolic stabilizing W^Δ.

The morphism ι is described as follows:

- The rep. Ω^ϵ can be realized on $\mathcal{S}(V^\epsilon \otimes W^\nabla)$.
- For $\Phi \in \mathcal{S}(V^\epsilon \otimes W)$,

$$\iota(\Phi)(g) = (g \cdot \Phi)(0) \quad \text{for } g \in \mathrm{Mp}(\mathbb{W}).$$

Degenerate P.S.

Now let's examine the degenerate p.s. $I_{P(W^\Delta)}(0)$ from other points of view. By definition, elements of $I_{P(W^\Delta)}(0)$ are smooth sections of a line bundle on the partial flag variety

$$P\backslash \mathrm{Sp}(\mathbb{W}) = P(W^\Delta)\backslash \mathrm{Sp}(\mathbb{W}),$$

parametrizing maximal isotropic subspaces of $\mathbb{W}$. Such sections are determined by their restrictions to open dense subsets.

There are 2 such open dense subsets we will use:

- the open Bruhat cell:

$$X_1 = P\backslash P \cdot \overline{N} \subset P\backslash \mathrm{Sp}(\mathbb{W})$$

  and

$$\overline{N} \cong \mathrm{Sym}^2(W^\nabla) \cong \mathfrak{sp}(W).$$

- $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$ has an open dense orbit (orbit of W^Δ):

$$X_2 = \mathrm{Sp}(W)^\Delta\backslash \mathrm{Sp}(W) \times \mathrm{Sp}(W) \subset P\backslash \mathrm{Sp}(\mathbb{W}).$$

So we have two injective restriction maps

$$\mathrm{rest}_{X_2} : I_P(0) \hookrightarrow C^\infty(\mathrm{Mp}(W))$$

and

$$\mathrm{rest}_{X_1} : I_P(0) \hookrightarrow C^\infty(N) = C^\infty(\mathfrak{sp}(W)).$$

Lemma

The map $p : \Omega \longrightarrow \mathcal{S}(Mp(W))$ *is given by:*

$$p = \mathrm{rest}_{X_2} \circ \iota.$$

Hence

$$\mathrm{rest}_{X_2} : I_P(0) \cong \mathcal{S}(Mp(W)).$$

Denoting the image of rest_{X_1} by $\mathcal{S}(\mathfrak{sp}(W))$, we have an isomorphism

$$j : \mathcal{S}(\mathrm{Mp}(W)) \cong I_P(0) \cong \mathcal{S}(\mathfrak{sp}(W)).$$

What is this isomorphism?

Lemma

Given $f \in \mathcal{S}(Mp(W))$,

$$j(f)(x) = f(c(x)) \cdot |\det(1 - c(x))|^{\frac{\dim V}{2}}.$$

where

$$c : \mathfrak{sp}(W) \longrightarrow Mp(W)$$

which is the "birational map" given by the Cayley transform

$$c(x) = (x-1)(x+1)^{-1}$$

(when projected to $Sp(W)$).

Character of Weil representation

The factor $|\det(1-c(x))|^{\frac{\dim V}{2}}$ which appears in the previous lemma can be interpreted in terms of the character of the Weil representation. One has the following result of Teruji Thomas:

Theorem

As a generalized function on $Mp(V \otimes W)$, the character of the Weil representation $\omega_{V,W,\psi}$ is given by

$$\mathrm{Tr}(\omega_{V,W,\psi}(g)) = \gamma_\psi(g) \cdot |\det_{V\otimes W}(g-1)|^{-1/2}.$$

For $g \in \mathrm{Mp}(W)$, one has

$$\mathrm{Tr}(\omega_{V,W,\psi}(g \otimes 1_V)) = \gamma_\psi(g) \cdot |\det_W(g-1)|^{-\dim V/2}.$$

So

$$j(f)(x) = f(c(x)) \cdot \mathrm{Tr}(\omega_{V,W,\psi}(c(x) \otimes 1_V))^{-1}.$$

Summary

At this point, we have the following diagram:

$$\begin{tikzcd}
 & \Omega^\epsilon = \mathcal{S}(V^\epsilon \otimes W) \arrow[dl, "p^\epsilon"'] \arrow[dr, "q^\epsilon"] & \\
\mathcal{S}(\mathrm{Mp}(W)) \arrow[d, "j_W"'] & & \mathcal{S}(\mathrm{SO}(V^\epsilon)) \arrow[d, "j_V"] \\
\mathcal{S}(\mathfrak{sp}(W)) & & \mathcal{S}(\mathfrak{so}(V^\epsilon))
\end{tikzcd}$$

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\mathcal{S}(\mathfrak{sp}(W)) & & \mathcal{S}(\mathfrak{so}(V^\epsilon))
\end{tikzcd}$$

This diagram arises in another context: the moment map associated to the Hamiltonian $\mathrm{O}(V) \times \mathrm{Sp}(W)$-variety $V \otimes W$.

Moment Map

The moment map is a double fibration

$$\begin{tikzcd}
 & V \otimes W \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\
\mathfrak{sp}(W)^* \cong \mathrm{Sym}^2 W^* & & \mathfrak{so}(V)^* \cong \wedge^2 V^*
\end{tikzcd}$$

The maps are given by:

$$p'(T) = T \circ T^* \quad \text{and} \quad q'(T) = T^* \circ T.$$

- The map p' is $\mathrm{Sp}(W)$-equivariant and $\mathrm{SO}(V)$-invariant, whereas q' is $\mathrm{SO}(V)$-equivariant and $\mathrm{Sp}(W)$-invariant.
- It induces a correspondence of orbits between $\mathfrak{so}(V)$ and $\mathfrak{sp}(W)$, giving a bijection

$$\mathfrak{so}(V)^{\heartsuit}//\mathrm{SO}(V) \longleftrightarrow \mathfrak{sp}(W)^{\heartsuit}//\mathrm{Sp}(W),$$

where $\mathfrak{sp}(W)^{\heartsuit}$ correspond to maximally nondegenerate maps.

Moment Map Correspondence

The moment map diagram

$$\begin{tikzcd} & V \otimes W \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ \mathfrak{sp}(W) & & \mathfrak{so}(V) \end{tikzcd}$$

induces by integration along the fibers:

$$\begin{tikzcd} & \mathcal{S}(V \otimes W) \arrow[dl, "p'_*"'] \arrow[dr, "q'_*"] & \\ \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) & & \mathcal{S}(\mathfrak{so}(V)^{\heartsuit}) \end{tikzcd}$$

This defines a "moment map correspondence" of the two spaces of test functions, which descends to give an isomorphism of orbital integrals

$$\mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}//\mathrm{Sp}(W)^{\Delta}) \cong \mathcal{S}(\mathfrak{so}(V)^{\heartsuit}//\mathrm{SO}(V^{\pm}))$$

Transfer and Moment Map

We may ask if the maps p and p'_* are related?

Proposition

$$j_W \circ p = \mathcal{F}^{\heartsuit} \circ p'_* \circ \mathcal{F}_{V\otimes W}.$$

where $\mathcal{F}_{V\otimes W}$ is the Fourier transform on $V \otimes W$ and

$$\mathcal{F}^{\heartsuit} : \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \longrightarrow \mathcal{S}(\mathfrak{sp}(W))$$

is the Fourier transform (of distributions) on $\mathfrak{sp}(W)$.

So have commutative diagram:

$$\begin{tikzcd}
\mathcal{S}(V\otimes W) \arrow[r, "\mathcal{F}_{V\otimes W}"] \arrow[d, "p"'] & \mathcal{S}(V\otimes W) \arrow[r, "p'_*"] & \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \arrow[d, "\mathcal{F}^{\heartsuit}"] \\
\mathcal{S}(\mathrm{Mp}(W)) \arrow[r, "j_W"'] & \mathcal{S}(\mathfrak{sp}(W)) \arrow[r, equal] & \mathcal{S}(\mathfrak{sp}(W))
\end{tikzcd}$$

Geometric Description of Transfer

Here is our geometric description of the transfer of test functions:

- given $\tilde{f} \in \mathcal{S}(\mathrm{Mp}(W))$ and $f \in \mathcal{S}(\mathrm{SO}(V))$, we consider

$$j_W(\tilde{f}) \in \mathcal{S}(\mathfrak{sp}(W)) \quad \text{and} \quad j_V(f) \in \mathcal{S}(\mathfrak{so}(V)).$$

- Then $\tilde{f}$ and f correspond if the Fourier transforms $\mathcal{F}^{\heartsuit}(j_W(\tilde{f}))$ and $\mathcal{F}^{\heartsuit}(j_V(f))$ correspond under the moment map correspondence.
- In that case, $\mathcal{F}^{\heartsuit}(j_W(\tilde{f}))$ and $\mathcal{F}^{\heartsuit}(j_V(f))$ have equal orbital integrals under the bijection of nondegenerate orbits induced by the moment map.

Corollary

If $\tilde{\mathcal{O}}$ is an $Sp(W)$-orbit in $\mathfrak{sp}(W)^{\heartsuit}$ with corresponding $SO(V)$-orbit $\mathcal{O}$ in $\mathfrak{so}(V)^{\heartsuit}$, then the transfer map identifies the Fourier transform of the orbital integrals $\mu_{\tilde{\mathcal{O}}}$ and $\mu_{\mathcal{O}}$.

Periods and Theta Correspondence

The techniques for deriving the main character identity can be applied in the setting of the relative Langlands program to give relative character identities.

In theta correspondence, given a dual pair $G \times H$, it is typical for one to relate a period $\mathcal{P}$ on G with another period $\mathcal{P}'$ on H. Given a period $\mathcal{P}$, one can associate a relative character. Thus, in the above setting, one may ask if there is an identity relating the relative character for $\mathcal{P}$ and that for $\mathcal{P}'$.

For example:

- The theory discussed before corresponds to the group case, where $G = G_0 \times G_0$ and the period $\mathcal{P}$ is the G_0^{Δ}-period.
- For $\mathrm{O}_{n-1} \backslash \mathrm{O}_n$ vs. $(N, \psi) \backslash \mathrm{SL}_2$, see paper with Xiaolei Wan.
- Wan's thesis deal with $\mathrm{U}_2 \backslash \mathrm{SO}_5$ v.s. $(N, \psi) \backslash \mathrm{PGL}_2 \times T \backslash \mathrm{PGL}_2$, using the theta correspondence for $\mathrm{PGSp}_4 \times \mathrm{PGSO}_4$.

THANK YOU FOR YOUR ATTENTION!