Classical Steinberg Groups

Ivan Andrus.

The Periodic Table of Finite Simple Groups

GREEN FUNCTIONS IN THE COMPUTATION OF ORDER

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Lie Groups and Representation Theory Seminar

University of Maryland, 7 April 2021
• Fix a prime $p$ and a simple algebraic group $G$ over $k = \overline{\mathbb{F}}_p$.
• Assume $G$ defined over $\mathbb{F}_q \subseteq k$ ($q = p^f$), with Frobenius map $F: G \to G$.
• Obtain finite group $G(\mathbb{F}_q) = G^F = \{g \in G \mid F(g) = g\}$.
  \[ |G(\mathbb{F}_q)| = q^{\dim G} + \text{smaller powers of } q. \]

**General aim/program:**

Determine $\text{Irr}(G^F) = \text{complex irreducible characters of } G^F$.

• Parametrisation, values on semisimple elements: solved (Lusztig 1980s).
• Arbitrary elements: theory of character sheaves (Lusztig 1984–today) \ldots
• \ldots where certain normalisations (roots of unity) remain to be determined.

*And with this, create an electronic ATLAS of “generic” character tables, extending the famous Cambridge ATLAS of finite groups.*

(E.g., the latter contains character table of $F_4(\mathbb{F}_2)$; would like this for $F_4(\mathbb{F}_{2^f})$, all $f \geq 1$.)
In this talk:

- Special case for above program: “Green functions”.
- “Normalisation problem” for Green functions solved in most cases, mainly by work of Beynon–Spaltenstein, Shoji and Lusztig (1980s–2000s).
- Will explain how to solve last remaining open cases, for $G$ of exceptional types (uses computer calculations).

Platform for electronic ATLAS and computer calculations (ongoing project).

**CHEVIE:** M.Geck, G.Hiss, F.Luebeck, G.Malle, J.Michel, G.Pfeiffer

http://www.math.rwth-aachen.de/~CHEVIE/

Implemented in **GAP3**; 64-bit version, with many extensions, see:

http://webusers.imj-prg.fr/~jmichel/gap3

Latest development: Port to Julia language (J. Michel).
Let $T \subseteq G$ be an $F$-stable maximal torus and $\theta \in \text{Irr}(T^F)$

$\sim$  $R_{T,\theta}$ virtual character of $G^F$  (Deligne and Lusztig 1970s).

Let $G_{uni}$ be the variety of unipotent elements of $G$

$\sim$  Green function $Q_T: G_{uni}^F \rightarrow \overline{\mathbb{Q}_\ell}$,  $u \mapsto R_{T,\theta}(u)$.

- $Q_T$ has values in $\mathbb{Z}$, and does not depend on $\theta$.
- Character formula: Get all values of $R_{T,\theta}$ from $Q_T$ and inductive procedure.

Lusztig 1984: Knowledge of all $Q_T$'s  $\sim$  “average value” character table of $G^F$.

Let $\rho \in \text{Irr}(G^F)$ and $C$ be an $F$-stable conjugacy class of $G$. Then $C^F$ splits into finitely many classes in $G^F$, with representatives $g_1, \ldots, g_r \in C^F$ say.

$\sim$  “average value”  $\text{AV}(\rho, C) := \sum_{1 \leq i \leq r} [A_i : A_i^F] \rho(g_i)$, where $A_i = C_G(g_i)/C_G^o(g_i)$ finite group (with induced action of $F$).
Assume from now on: \( G \) of adjoint type, defined and split over \( \mathbb{F}_q \).

The \( G^F \)-conjugacy classes of \( F \)-stable maximal tori \( T \subseteq G \) are parametrised by the conjugacy classes of \( W \) (Weyl group of \( G \)); write \( T = T_w \) for \( w \in W \).

- “New” Green functions \( Q_\phi := |W|^{-1} \sum_{w \in W} \phi(w) Q_{T_w} \) for \( \phi \in \text{Irr}(W) \).
- Values of \( Q_\phi \) are given by \( m \times n \) table, where \( m = |\text{Irr}(W)| \) and \( n = \) number of unipotent classes of \( G^F \).

Example: \( G \) of type \( E_8 \).

\[
m = |\text{Irr}(W)| = 112 \quad \text{and} \quad n = \begin{cases} 146 & \text{if } p = 2, \\ 127 & \text{if } p = 3, \\ 117 & \text{if } p = 5, \\ 113 & \text{if } p > 5, \end{cases} \quad (\text{Mizuno 1980}).
\]

Above assumption implies: If \( C \) is a unipotent class of \( G \), then \( F(C) = C \) and there exists \( u \in C^F \) such that \( F \) acts trivially on \( A(u) = C_G(u)/C_G^o(u) \).
Values of $Q_\phi$ “almost known” by a general, purely combinatorial algorithm.

- For $\phi \in \text{Irr}(W)$ there is a unique unipotent class $C = C_\phi$ of $G$ such that
  \[ \{g \in G_{\text{uni}}^F \mid Q_\phi(g) \neq 0 \} \subseteq \overline{C} \quad \text{and} \quad Q_\phi|_{C^F} \neq 0. \]
  Thus, obtain a map $\phi \mapsto C_\phi$ (= Springer correspondence).

- Set $d_\phi := (\dim G - \dim C_\phi - \text{rank}(G))/2 \in \mathbb{Z}_{\geq 0}$. Define $Y_\phi : G_{\text{uni}}^F \to \mathbb{Q}$ by
  \[ Q_\phi|_{C^F_\phi} = q^{d_\phi} Y_\phi \quad \text{(and extend by 0 outside } C^F_\phi). \]

- Lusztig, Shoji, ... (1976–2012): There are unique $p_{\phi',\phi} \in \mathbb{Z}$ such that
  \[ Q_\phi = \sum_{\phi' \in \text{Irr}(W)} q^{d_\phi} p_{\phi',\phi} Y_{\phi'} \quad \text{for all } \phi \in \text{Irr}(W). \]
  Matrix $(p_{\phi',\phi})$ is triangular with 1 on diagonal.
  It can be computed by a purely combinatorial algorithm, which relies on a priori knowledge of the map $\phi \mapsto C_\phi$, i.e., Springer correspondence for $G$.

- $\rightsquigarrow$ Function $\text{ICCTable}$ in J. Michel’s version of CHEVIE.
“Almost known”? Let $\phi \in \text{Irr}(W)$, $C = C_\phi$, and $Y_\phi$ be the corresponding function. The remaining problem is to determine the values of $Y_\phi$ on $C_\phi^F$.

- Let $u_\phi \in C_\phi^F$ be such that $F$ acts trivially on $A(u_\phi) = C_G(u_\phi)/C^G_G(u_\phi)$.
- Springer correspondence also associates to $\phi$ a character $\psi_\phi \in \text{Irr}(A(u_\phi))$.
- Let $a_1, \ldots, a_r \in A(u_\phi)$ be representatives of the conjugacy classes of $A(u_\phi)$.
- There are corresponding representatives $u_1, \ldots, u_r \in C_\phi^F$ of the conjugacy classes of $G^F$ into which $C_\phi^F$ splits.

Then there exists a sign $\delta_\phi = \pm 1$ such that $Y_\phi(u_i) = \delta_\phi \psi_\phi(a_i)$ for all $i$.

$\sim$ Everything is reduced to the — tricky! — task of determining the signs $\delta_\phi$.

- For $G$ of classical type, signs are determined by Shoji (1980s, 2007).
- For $G$ of exceptional type, Beynon–Spaltenstein (1984), except for cases where $p$ is small.
- For $G_2, ^3D_4, F_4, E_6, ^2E_6$ and $p$ small, various explicit computations by Enomoto, Enomoto–Yamada, Spaltenstein, Malle, Porsch, Marcelo–Shinoda (1970s–1990s).
Open cases: \( 2E_6(F_{3^f}), \ E_7(F_{2^f}), \ E_7(F_{3^f}), \ E_8(F_{2^f}), \ E_8(F_{3^f}), \ E_8(F_{5^f}). \)

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**Theorem** (G., 2020) Let \( \phi \in \text{Irr}(W) \) and \( \delta_\phi \) be the corresponding sign.

For \( n \geq 1 \), consider the group \( G^F_n = G(F_{q^n}) \) and the Green function \( Q_{\phi,n}: G^F_n_{\text{uni}} \rightarrow \mathbb{Q} \), with corresponding sign \( \delta_{\phi,n} \). Then we have \( \delta_{\phi,n} = \delta^n_\phi \).

Proof uses interpretation of Green functions in terms of character sheaves, work of Lusztig and Shoji; and there are no restrictions on the characteristic \( p \).

Theorem motivated by general character theory of finite groups. Let \( \Gamma, S \) be finite groups of coprime order such that \( S \) is solvable and acts by automorphisms on \( \Gamma \).

**Glauberman correspondence:** \( \text{Irr}_S(\Gamma) \xleftrightarrow{1-1} \text{Irr}(C_\Gamma(S)), \chi \leftrightarrow \chi^* \).

**Problem/Conjecture** (1990s): The degree of \( \chi^* \) divides the degree of \( \chi \).
Hartley–Turull (1994): (1) Reduction to finite simple groups; (2) by classification: difficult case are groups of Lie type; and (3) for these, it is enough to show:

**Congruence condition for Green functions.**

Let $T \subseteq G$ be an $F$-stable maximal torus and $u \in G^F$ be unipotent. Let $r \in \mathbb{N}$ be a prime such that $r \nmid |G^F|$. Then $Q_{T,F}(u) \equiv Q_{T,Fr}(u) \mod r$.

In G. 2020, this is deduced from the theorem; so, $\chi^*(1) | \chi(1)$ holds in general!

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**Back to problem of determining the signs $\delta_\phi$ for $\phi \in \text{Irr}(W)$.**

Recall $G^F = G(\mathbb{F}_q)$ where $q = p^f$ with $f \geq 1$.

- Theorem implies that it is enough to compute $\delta_\phi$ assuming $f = 1$.
- Hence, “only” need to compute values of $Q_\phi$ for the 6 individual groups $^2E_6(\mathbb{F}_3)$, $E_7(\mathbb{F}_2)$, $E_7(\mathbb{F}_3)$, $E_8(\mathbb{F}_2)$, $E_8(\mathbb{F}_3)$, $E_8(\mathbb{F}_5)$. 

Open case challenge: \( G(\mathbb{F}_q) = E_8(\mathbb{F}_2); \) 112 functions \( Q_\phi. \)

If character table of \( E_8(\mathbb{F}_2) \) was known (like for other groups as in the Cambridge ATLAS), then we could easily determine the 112 missing signs.

But: \(|E_8(\mathbb{F}_2)| \approx 3 \cdot 10^{73}\) and size of character table is \(1156 \times 1156\)
(see http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/index.html)

Brute force methods won’t work. Solved by alternative methods, as follows.

Recall: \( Q_\phi \)'s linear combinations of \( Y_\phi \)'s, \( Y_\phi \) known up to \( \delta_\phi \). Further information:
Let \( B \subseteq G \) be an \( F \)-stable Borel subgroup and \( T_1 \subseteq B \) an \( F \)-stable maximal torus.

\[ R_{T_1,1} = \text{permutation character of } G^F \text{ on cosets of } B^F. \]

So, if \( u \in G^F \) is unipotent, then
\[
\sum_{\phi \in \text{Irr}(W)} \phi(1) Q_\phi(u) = Q_{T_1}(u) = R_{T_1,1}(u) = |\{ gB^F \in G^F/B^F \mid g^{-1}ug \in B^F \}|.
\]

Hence, if we can compute the right hand side, then we may get information on \( \delta_\phi \)'s.
Example: \( G^F = E_8(\mathbb{F}_q) \) with \( q = p^f \) where \( p \neq 3 \)

Let \( C = \text{unipotent class } E_8(b_6) \) with representative \( u = z_{77} \) (Mizuno 1980, \( D_8(a_3) \)).

We have \( A(u) = C_G(u)/C_G^\circ(u) \cong S_3 \) and \( |C_G(u)^F| = 6q^{28} \).

There are three \( \phi \in \text{Irr}(W) \) such that \( C_\phi = C \), denoted \( 2240_{10}, 175_{12} \) and \( 840_{13} \).

(Springer correspondence known by Spaltenstein 1982, 1985.)

Want to determine \( \delta_{2240_{10}} = \pm 1, \delta_{175_{12}} = \pm 1, \delta_{840_{13}} = \pm 1 \).

Beynon–Spaltenstein (1984): If \( p > 5 \), then \( \delta_{2240_{10}} = \delta_{175_{12}} = 1, \delta_{840_{13}} \equiv q \mod 3 \).

For \( p \in \{2, 5\} \), run the GAP algorithm \textsc{ICCTable}. This yields the following identity:

\[
| \{ gB(\mathbb{F}_q) \in G(\mathbb{F}_q)/B(\mathbb{F}_q) \mid g^{-1}z_{77}g \in B(\mathbb{F}_q) \} | = R_{T_1,1}(z_{77}) = Q_{T_1}(z_{77}) \\
= (2240q^{10} + 3688q^9 + 3444q^8 + 2360q^7 + 1351q^6 + 672q^5 + 294q^4 + 112q^3 \\
+ 35q^2 + 8q + 1)\delta_{2240_{10}} + 350q^{10}\delta_{175_{12}} + (840q^{10} + 650q^9 + 160q^8)\delta_{840_{13}}
\]

By Theorem, we only need to consider the cases where \( q = p \).
For $q = p = 2$, obtain identity

$$\left| \{ gB(F_2) \in G(F_2)/B(F_2) \mid g^{-1}z_{77}g \in B(F_2) \} \right| = 5,479,485 \delta_{2240_{10}} + 358,400 \delta_{175_{12}} + 1,233,920 \delta_{840_{13}}.$$ 

For $q = p = 5$, obtain identity

$$\left| \{ gB(F_5) \in G(F_5)/B(F_5) \mid g^{-1}z_{77}g \in B(F_5) \} \right| = 30,631,220,541 \delta_{2240_{10}} + 3,417,968,750 \delta_{175_{12}} + 9,535,156,250 \delta_{840_{13}}.$$ 

In both cases, we can already conclude that $\delta_{2240_{10}} = 1$.

Now, total number of cosets $G(F_p)/B(F_p)$ is still huge, roughly $\sqrt{|G(F_p)|}$. So practically impossible to create actual permutation representation.

Consider Bruhat decomposition $G(F_p) = \bigsqcup_{w \in W} B(F_p)wB(F_p)$;

each double coset $B(F_p)wB(F_p)$ contains exactly $p^{\ell(w)}$ cosets of $B(F_p)$.

~ Systematic way of enumerating (in principle) all coset representatives for $G(F_p)/B(F_p)$, proceeding by increasing $\ell(w)$. 


Can use matrix realization of $G(\mathbb{F}_p)$ to perform these computations. This is not very efficient but it was good enough to obtain:

**Proposition** (G. 2020). Re-compute, or compute for the first time, the signs $\delta_\phi$ for all exceptional $G \neq E_8$ and $p = 2, 3$. In all these cases, we always have $\delta_\phi = 1$.

- Suggestions of F. Lübeck: Instead of matrices, work with Chevalley generators $x_\alpha(t)$ of $G$ (where $\alpha$ is a root, $t \in k$) and commutator relations.
- Number of cosets fixed by an element can be obtained as number of $\mathbb{F}_p$-solutions of system of polynomial equations in several variables.
- Julia package **ChevLie1.1** (G.; independent GAP programs by Lübeck).

Back to $E_8$: For $p = 2$, number of cosets fixed by $z_{77}$ should equal

$$5,479,485 + 358,400 \delta_{175_{12}} + 1,233,920 \delta_{840_{13}}.$$ 

With **ChevLie1.1**, find exact number of cosets fixed by $z_{77}$: 4,603,965.

(This takes < 2 mins on my laptop.) Hence, $\delta_{175_{12}} = 1$ and $\delta_{840_{13}} = -1$. 


For \( p = 5 \), number of cosets fixed by \( z_{77} \) should equal
\[
30,631,220,541 + 3,417,968,750 \delta_{17512} + 9,535,156,250 \delta_{84013}.
\]
With ChevLie1.1, find all 24,514,033,041 (!!!) cosets fixed by \( z_{77} \) in < 4 mins.
(Note \([G(\mathbb{F}_5) : B(\mathbb{F}_5)] \approx 4 \cdot 10^{84}\).) Hence, again, \( \delta_{17512} = 1 \) and \( \delta_{84013} = -1 \).

Lübeck (2021) has now worked through all remaining classes in \( E_8 \) and \( p = 2, 3, 5 \).

**Corollary.** The Green functions are explicitly known in all cases (all \( G \)). The class \( C = E_8(b_6) \) considered above is the only example where we can have \( \delta_\phi = -1 \).

**Next steps:**
- Compute Lusztig’s **generalised** Green functions.
- Determine complete tables of values (not just average values) of unipotent characters of \( G(\mathbb{F}_q) \), for \( G \) of exceptional type and all \( q = p^f, f \geq 1 \).