Incoherent definite orthogonal spaces and Shimura varieties

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In the first half of this talk, I will review the theory of quadratic forms over number fields (Hilbert, Minkowski, Hasse, Witt, ...).

I will use this to define an incoherent definite orthogonal space.

In the second half of the talk, I will show how incoherent definite spaces can be used to study orthogonal Shimura varieties.

Let k be a field, not of characteristic 2.

An orthogonal space V over k is a finite dimensional vector space equipped with a non-degenerate symmetric bilinear form

$$\langle , \rangle : V \times V \to k.$$

Non-degenerate means that the linear map $V \to V^*$ defined by $v \to f_v(w) = \langle v, w \rangle$ is an isomorphism.

We obtain a quadratic form $q : V \to k$ defined by

$$q(v) = \langle v, v \rangle / 2 \quad \langle v, w \rangle = q(v + w) - q(v) - q(w).$$

There is an orthogonal basis $\{v_1, v_2, \ldots, v_n\}$ of V.

Let $a_i = q(v_i)$, then $a_i \neq 0$ and

$$q(\sum x_i v_i) = \sum a_i . x_i^2 .$$

.

We say two orthogonal spaces V and W over k are isomorphic if there is a linear isomorphism $T : V \to W$ which satisfies $q_W(Tv) = q_V(v)$.

Writing $q \sim \sum a_i.x_i^2$ we define three invariants.

- the dimension $n = \mathrm{dim}(V)$.
- the determinant

$$d(V) = a_1.a_2...a_n \in k^*/k^{*2} = H^1(k, \mu_2).$$

- the Witt invariant

$$w(V) = \prod_{i<j} a_i \cup a_j \in Br_2(k) = H^2(k, \mu_2).$$

If k is the reals or a p-adic field, then $Br_2(k) = \langle \pm 1 \rangle$. The cup product $a \cup b$ is given by the Hilbert symbol (a, b), which is $+1$ if the quadratic form $ax^2 + by^2 - z^2$ represents zero and -1 if not.

When $k = \mathbb{C}$, the space V is determined up to isomorphism by its dimension.

$$d(V) \equiv 1 \qquad w(V) = 1.$$

When $k = \mathbb{R}$, there is an orthogonal basis with $q(v_i) = \pm 1$, and the space V is determined up to isomorphism by its signature (r, s).

$$\dim(V) = r + s \quad d(V) \equiv (-1)^s \quad w(V) = (-1)^{s(s-1)/2}.$$

When k is a p-adic field, the space V is determined up to isomorphism by $\dim(V), d(V)$, and $w(V)$.

All possible invariants occur once $\dim(V) \geq 3$.

If k is a number field, the Hasse-Minkowski theorem states that an orthogonal space V over k is determined up to isomorphism by its localizations $V_v = V \otimes k_v$.

The local invariants satisfy

- $\dim(V_v) = n$
- $d_v(V_v) \equiv d \in k^*/k^{*2}$
- $w_v(V_v) = +1$ for almost all v and $\prod_v w_v(V_v) = +1$.

Finally, the invariants d_v and w_v are trivial at each complex place, and determined by the signature (r_v, s_v) at each real place.

When $\dim(V) \geq 3$, these are the only restrictions on the local invariants.

In fact, for a and b in k^*, the local Hilbert symbols satisfy

$$\prod_v (a,b)_v = +1.$$

When $k = \mathbb{Q}$, this is Hilbert's restatement of the law of quadratic reciprocity.

if p and q are distinct odd primes, then

$$(p,q)_p = \left(\frac{q}{p}\right) \quad (p,q)_q = \left(\frac{p}{q}\right)$$

$$(p,q)_2 = +1$$

unless $p \equiv q \equiv 3 \pmod 4$ when $(p,q)_2 = -1$. Finally

$$(p,q)_v = +1$$

at all other places v.

We say that the local orthogonal data $\{V_v\}$ for k is **incoherent** if the local invariants satisfy

- $\dim(V_v) = n$
- $d(V_v) \equiv d \in k^*/k^{*2}$
- $w_v(V_v) = +1$ for almost all v and $\prod_v w_v(V_v) = -1$.

There is **no orthogonal space over** k with this local data.

When k is a totally real number field, we say the incoherent data is **definite** if the spaces V_v are positive definite at all real places v.

Then the locally compact group $\prod' \mathrm{SO}(V_v)$ has open compact subgroups (but perhaps no discrete co-compact subgroup).

Lett $\{V_v\}$ be incoherent definite data for k, of dimension $n \geq 3$ and determinant $d \in k_+^*/k^{*2}$.

For each place u of k, we will define a global orthogonal space $V(u)$ of dimension n and determinant d, up to isomorphism over k.

To do this, we insist that:

- The localization of $V(u)$ at the place v is isomorphic to V_v, for all places $v \neq u$.
- The Witt invariant of $V(u)$ at the place u is the negative of the Witt invariant of V_u.
- If u is a real place, the signature of $V(u)$ at the place u is $(n-2, 2)$.

The product of Witt invariants is now $+1$, so a global space with these invariants exists, and is unique up to isomorphism.

We call the global spaces $V(u)$, for each place u, the **neighbors** of the incoherent definite data $\{V_v\}$.

$V(u)$ is unique up to isomorphism over k, but **not** up to a unique isomorphism.

We can't speak of a vector or a subspace of $V(u)$, but we can consider an orbit of subspaces under the orthogonal group.

For example, when u is a real place the local orthogonal space $V(u)_u$ has signature $(n-2, 2)$. We can consider the set of negative definite planes $W \subset V(u)_u$

They form a single orbit under the orthogonal group, by Witt's extension theorem, and $W^\perp$ is positive definite.

An **orientation** of the negative definite plane W is a choice of an isomorphism of one dimensional tori

$$\mathrm{SO}(W) \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)/\mathbb{G}_m.$$

The real group $\mathrm{SO}(V(u)_u) \cong \mathrm{SO}(n-2,2)$ acts transitively on the space X_u of oriented negative definite planes $W \subset V(u)_u$, and the stabilizer of W is the compact subgroup $\mathrm{SO}(W) \times \mathrm{SO}(W^{\perp})$.

The tangent space to X_u at W is the orthogonal representation $W \otimes W^{\perp}$ of dimension $2(n-2)$. This has a complex structure and X_u is two copies of the Hermitian symmetric space of $\mathrm{SO}(n-2,2)$.

For $n=3$, $\mathrm{SO}(1,2) \cong \mathrm{PGL}_2(\mathbb{R})$ and X_u can be identified with the upper and lower half planes.

Following Deligne, a Shimura variety S_M is associated to a reductive group G over $\mathbb{Q}$, an open compact subgroup $M \subset G(\mathbb{A}^f)$, and a distinguished conjugacy class X of homomorphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to G_{\mathbb{R}}$$

which has three weights on the Lie algebra.

Each connected component X^+ is a Hermitian symmetric space and as a complex analytic orbifold

$$S_M(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)/M = \cup\, \Gamma_g \backslash X^+.$$

- S_M is a complex variety, which has a canonical model over a subfield $E \hookrightarrow \mathbb{C}$, which has finite degree over $\mathbb{Q}$. (The components of S_M are defined over an abelian extension of E.)
- If we choose a different embedding of E into $\mathbb{C}$, the complex points of S_M can be described using different Deligne data (G', X').

Let k be a totally real field.

Let V be an orthogonal space over k of dimension $n \geq 3$ with signature $(n-2, 2)$ over the real completion k_u and signature $(n, 0)$ over all other real completions of k.

$$G = \mathrm{Res}_{k/\mathbb{Q}} \mathrm{SO}(V) \qquad G_{\mathbb{R}} \cong SO(n-2, 2) \times K.$$

Then the conjugacy class X can be identified with the homogeneous space X_u of oriented, negative definite 2-planes W in the real orthogonal space $V \otimes k_u$.

In this case, S_M has dimension $n-2$. The canonical model of S_M is defined over k, embedded in $k_u = \mathbb{R} \hookrightarrow \mathbb{C}$ by the place u.

If we embed k into $\mathbb{C}$ by a different real place w, the complex variety $S_M(\mathbb{C}_w)$ is a Shimura variety for a different orthogonal group $G' = \mathrm{Res}_{k/\mathbb{Q}} \mathrm{SO}(V')$.

The orthogonal space V' over k has signature $(n-2, 2)$ at k_w and signature $(n, 0)$ at all other real completions. At all finite places, V' is isomorphic to V.

In particular, $G(\mathbb{A}^f) \cong G'(\mathbb{A}^f)$ and $M \subset G(\mathbb{A}^f)$ gives a subgroup M' of $G'(\mathbb{A}^f)$.

At the real place w we have

$$S_M(\mathbb{C}_w) = G'(\mathbb{Q}) \backslash X' \times G'(\mathbb{A}^f)/M'.$$

What determines the variety S_M over k, **without choosing a real embedding**?

The data $\{V_v\}$ of an **incoherent definite orthogonal space**.

At each real place u, the neighboring space $V(u)$ is used to describe the Shimura variety S_M over the completion $k_u = \mathbb{R} \hookrightarrow K_u = \mathbb{C}$.

Let $G(u) = \mathrm{SO}(V(u))$ and let X_u be the space of oriented, negative definite planes in $V(u)_u$. Then

$$S_M(K_u) = G(u)(k)\backslash X_u \times G(u)(\mathbb{A}_k^f)/M.$$

Can we use the neighboring orthogonal spaces $V(\mathfrak{p})$ at finite primes $\mathfrak{p}$ of k to study the variety S_M over the completions $k_{\mathfrak{p}}$?

We need an analog of the homogeneous space X_u of oriented, negative definite planes W. Note that W is isomorphic to $\mathbb{C}$ with quadratic form given by -1 times the norm, and $w(W) = (-1, -1) = -1$.

Let $\mathfrak{p}$ be a finite place of k (not dividing 2) where the discriminant d is a unit and where $w(V_\mathfrak{p}) = +1$.

Let $W = K_\mathfrak{p}$ be the unramified quadratic extension of $k_\mathfrak{p}$, with quadratic form given by $\pi_\mathfrak{p}$ times the norm. Then $d(W)$ is a unit and $w(W) = -1$.

An orientation of W is the choice of an isomorphism

$$\mathrm{SO}(W) \cong \mathrm{Res}_{K_\mathfrak{p}/k_\mathfrak{p}}(\mathbb{G}_m)/\mathbb{G}_m.$$

The group $\mathrm{SO}(V(\mathfrak{p})_\mathfrak{p})$ acts transitively on the set of oriented planes in $V(\mathfrak{p})_\mathfrak{p}$, with stabilizer isomorphic to $\mathrm{SO}(W) \times \mathrm{SO}(W^\perp)$.

One difference from the real case is that the stabilizer $\mathrm{SO}(W) \times \mathrm{SO}(W^\perp)$ is only compact when $\dim W^\perp = 1$.

The orthogonal space $W^\perp$ has unit discriminant and Witt invariant $+1$. Hence there are integral lattices $L \subset W^\perp$ which are self-dual for the bilinear form. They form a single orbit for the group $\mathrm{SO}(W^\perp)$.

We let $X_{\mathfrak{p}}$ be the set of pairs (W, L), where $W \subset V(\mathfrak{p})_{\mathfrak{p}}$ is an oriented plane and L is a self-dual lattice in the orthogonal complement $W^\perp$.

The group $\mathrm{SO}(V(\mathfrak{p})_{\mathfrak{p}})$ acts transitively on $X_{\mathfrak{p}}$, with compact stabilizer isomorphic to $\mathrm{SO}(W) \times \mathrm{SO}(L)$.

This stabilizer is contained in a unique maximal compact subgroup, which we will now describe.

Let e be a vector in W with $q(e) = \pi_{\mathfrak{p}}$, and let $A_{\mathfrak{p}}$ be the ring of integers in $K_{\mathfrak{p}}$.

Then $A_{\mathfrak{p}}.e$ is the unique maximal integral lattice in W and $\Lambda = A_{\mathfrak{p}}.e + L$ is a maximal lattice in $V(\mathfrak{p})_{\mathfrak{p}}$, whose stabilizer is a maximal compact subgroup.

The subgroup (of index 2) which stabilizes the maximal integral lattice $\Lambda = A_{\mathfrak{p}}.e + L$ and preserves an orientation of its discriminant lattice Λ^*/Λ is a maximal parahoric subgroup.

Let $Y_{\mathfrak{p}}$ be the homogenous space of maximal integral lattices in $V(\mathfrak{p})_{\mathfrak{p}}$ together with an orientation of the discriminant lattice, and let $\phi : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ be the equivariant covering map, taking the pair (W, L) to the maximal lattice $\Lambda = A_{\mathfrak{p}}.e + L$.

The fibers of $\phi : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ can be identified with open unit polydiscs of dimension $n - 2$ over $A_{\mathfrak{p}}$.

With these two analogs of the Hermitian symmetric space, we can return to the Shimura variety S_M over k. We wish to parametrize some points over the quadratic extension $K_{\mathfrak{p}}$ of the completion $k_{\mathfrak{p}}$.

Assume that M has the form $M = M_{\mathfrak{p}} \times M^{\mathfrak{p}}$, with $M_{\mathfrak{p}}$ a hyperspecial maximal compact subgroup.

Then S_M has a (canonical) smooth model over the ring of integers of $k_{\mathfrak{p}}$ with good reduction modulo $\mathfrak{p}$.

Let $G(\mathfrak{p}) = \mathrm{SO}(V(\mathfrak{p}))$ and consider the map

$$\phi : G(\mathfrak{p})(k)\backslash X_{\mathfrak{p}} \times G(\mathfrak{p})(\mathbb{A}_k^{f,\mathfrak{p}})/M^{\mathfrak{p}} \to G(\mathfrak{p})(k)\backslash Y_{\mathfrak{p}} \times G(\mathfrak{p})(\mathbb{A}_k^{f,\mathfrak{p}})/M^{\mathfrak{p}}$$

The target

$$G(\mathfrak{p})(k)\backslash G(\mathfrak{p})(\mathbb{A}^f)/N_{\mathfrak{p}} \times M^{\mathfrak{p}}$$

is a finite set, which parametrizes the special supersingular points on S_M modulo $\mathfrak{p}$.

These points are all rational over the quadratic extension $F_{\mathfrak{p}^2}$ of the residue field $F_{\mathfrak{p}}$.

The source has the structure of an $A_{\mathfrak{p}}$-orbifold of dimension $n-2$, which parametrizes the points of S_M over $A_{\mathfrak{p}}$ which have special supersingular reduction modulo $\mathfrak{p}$.

In some cases when $k = \mathbb{Q}$ and $n \leq 21$, S_M is the moduli space of polarized $K3$ surfaces with level structure.

Artin introduced a stratification of the supersingular locus.

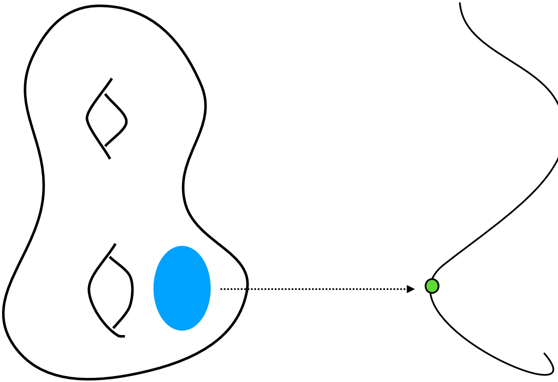
A $K3$ surface N in characteristic p is supersingular if

$$\Lambda = H^2(N, W)^{\phi = p}$$

is a $\mathbb{Z}_p$ lattice of rank 22.

N is in the special supersingular locus when Λ^*/Λ is a non-split quadratic space of rank 2 over $\mathbb{Z}/p\mathbb{Z}$.

$$\phi : G(\mathfrak{p})(k)\backslash X_{\mathfrak{p}} \times G(\mathfrak{p})(\mathbb{A}_k^{f,\mathfrak{p}})/M^{\mathfrak{p}} \to G(\mathfrak{p})(k)\backslash Y_{\mathfrak{p}} \times G(\mathfrak{p})(\mathbb{A}_k^{f,\mathfrak{p}})/M^{\mathfrak{p}}$$



For proofs of these statements, see:

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Benjamin Howard and Georgios Pappas, “Rapoport–Zink spaces for spinor groups”, Compositio 153 (2017)

Mark Kisin, ”Integral models for Shimura varieties of abelian type”, JAMS 23 (2010).

Pierre Deligne et alia, ”Hodge cycles, motives, and Shimura varieties”. Springer LNM 900 (1982).

Thank you.