On the Kottwitz conjecture for local shtuka spaces

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Introduction

Recall: A **Shimura datum** is (roughly) a pair (G, X) where $G/\mathbf{Q}$ is a connected reductive group and $X \simeq G(\mathbf{R})/K_\infty$ is a Hermitian symmetric domain.

Theorem (Baily-Borel, Shimura, Deligne, Borovoi, Milne)

Given (G, X) as above, and $K \subset G(\mathbf{A}_f)$ a sufficiently small open compact subgroup, the locally symmetric manifold

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$$

is $\cong \mathrm{Sh}(G, X)_K(\mathbf{C})$ for a certain smooth quasiprojective algebraic variety $\mathrm{Sh}(G, X)_K$ defined over a number field $E = E(G, X)$.

$\rightsquigarrow$ Get a tower $\{\mathrm{Sh}(G, X)_K\}_K$ with $G(\mathbf{A}_f)$-action.
When $(G, X) = (\mathrm{GL}_2, \mathfrak{H}^{\pm})$, this is the usual tower of modular curves.

Introduction (cont.)

Shimura varieties give a wonderful tool/playground for working out ideas in arithmetic geometry and the Langlands program. Some particular instances of this:

- $\mathrm{Sh}(G, X)_K$ can often (but not always!) be interpreted as a moduli space of abelian varieties w. extra structures. ($\sim$"Shimura varieties of Hodge type.") $\rightsquigarrow$ Canonical integral models / $\mathcal{O}_E[1/N]$ (Katz-Mazur, Kottwitz, Kisin, ...) w. concrete description of their mod p points (Langlands-Rapoport, Kisin), canonical compactifications (Faltings-Chai, Lan, Madapusi Pera, ...), etc.
- $\lim_{K\to 1} H^*_{\text{ét}}(\mathrm{Sh}(G, X)_{K,\overline{E}}, \overline{\mathbf{Q}_\ell})$ has commuting $G(\mathbf{A}_f)$- and $\mathrm{Gal}(\overline{E}/E)$-actions. $\rightsquigarrow$ Construction of Galois representations associated with automorphic forms in many instances. (Huge industry; cf. recent work of Kret-Shin for interesting examples involving a Shimura variety *not* of Hodge type.)

Question: Is there a local analogue of Shimura varieties?

Local Shimura varieties

Fix a prime p and a finite extension $F/\mathbf{Q}_p$ with residue field $\mathbf{F}_q$. Set $\breve{F} = \widehat{F^{\mathrm{unr}}} \circlearrowleft \sigma =$ lift of $x \mapsto x^q$.

Definition (Rapoport-Viehmann)

A **local Shimura datum** *is a triple* $(G, \{\mu\}, b)$ *where* G/F *is a connected reductive group,* $\{\mu\}$ *is a conjugacy class of minuscule cocharacters* $\mathbf{G}_{m,\overline{F}} \to G_{\overline{F}}$*, and* $b \in G(\breve{F})$ *is an element such that* $b \in B(G, \mu)$*.*

(Can and do fix $\mu \in \{\mu\}$ defined over a minimal fin. extension $E = E(G, \{\mu\})/F$; ignore difference between μ and $\{\mu\}$. Set $\breve{E} = E.\breve{F}$.)

Example (Key example)

Take $F = \mathbf{Q}_p$, $G = \mathrm{GL}_n$, $\mu(z) = \mathrm{diag}(\underbrace{z, \ldots, z}_{d}, 1, \ldots, 1)$ *for some* $1 \leq d < n$, b *any element of* $\mathrm{GL}_n(\mathbf{Q}_p)$ *with characteristic polynomial* $X^n - p^d$.

Conjecture (Rapoport-Viehmann)

Given $(G, \{\mu\}, b)$ *as above, should have a natural tower of smooth rigid analytic spaces* $\{\mathcal{M}_K = \mathcal{M}(G, \{\mu\}, b)_K\}_K$ *over* $\breve{E}$*, indexed by open compact* $K \subset G(F)$*, equipped with commuting actions of* $G(F)$ *(on the whole tower) and* $G_b(F) := \{j \in G(\breve{F}) \mid \sigma(j) = bjb^{-1}\}$ *(on each* $\mathcal{M}_K$*). Moreover,...*

- $\mathcal{M}(G, \{\mu\}, b)_K$ *should often (but not always!) coincide with the generic fiber of a formal scheme (a "Rapoport-Zink space") parametrizing p-divisible groups w./w.o. extra structures. Works for* $G = \mathrm{GL}_n$, $\mathrm{GSp_{2n}}$, GSpin_n*... Doesn't work if e.g.* $G = \mathrm{PGL}_n$, E_7.
- *Should have a canonical (non-effective) descent datum to* $E \rightsquigarrow$ $\lim_{K\to 1} H^*_{c,\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbf{Q}_\ell})$ *has commuting* $G(F)$*-,* $G_b(F)$*-, and* W_E*-actions.*
- *(Many more.)*

Local Shimura varieties (cont'd)

Theorem (Scholze ∼late 2014, via Caraiani-Scholze, Fargues & Fargues-Fontaine, Kedlaya-Liu, Scholze-Weinstein)

Local Shimura varieties exist with all expected properties, as the solution to a natural moduli problem determined by the datum (G, μ, b). In fact, they exist in the larger category of diamonds for any μ, and even for more general input data $(G, \{\mu_i\}_{1 \leq i \leq n}, b)$, in which case the resulting spaces live over "n copies of $\mathrm{Spa}(\breve{E})$" in a precise sense.

This last construction is in parallel with the moduli spaces of local/global shtukas defined in equal characteristic p settings (Drinfeld, Varshavsky, Lafforgue, ...), which fiber over some finite self-product of copies of the base (which is either $\mathrm{Spec}\,\mathbf{F}_q((t))$ or a projective curve over $\mathbf{F}_q$).
$\rightsquigarrow$ Fargues-Scholze: Combining this with ideas of V. Lafforgue, get a construction associating a semisimple L-parameter with any irreducible smooth representation of any $G(F)$, giving a candidate for the local Langlands correspondence.

Rough idea of construction

Fix (G, μ, b) as before, with E the field of def. of μ. Let $C/\breve{E}$ be any complete algebraically closed extension. Let's unwind $\mathcal{M}_\infty(C) \overset{def}{=} \lim_{\underset{K}{\leftarrow}} \mathcal{M}_K(C)$.

Fargues-Fontaine: Any such $C \rightsquigarrow$ a certain "curve" $X = X_{C^\flat,F}$ (a connected reg. Noeth. 1-dim'l scheme over $\mathrm{Spec}\, F$) together with a distinguished closed point $\infty \in X$ such that $\kappa(\infty) \cong C$.

Fargues: Any $b \in G(\breve{F}) \rightsquigarrow$ a G-bundle $\mathcal{E}_b$ over X, with $\mathrm{Aut}(\mathcal{E}_b) \supset G_b(F)$ (and sometimes with equality; in particular, $\mathrm{Aut}(\mathcal{E}_{\mathbf{1}}) \cong G(F)$).

Idea

$\mathcal{M}_\infty(C) = \{$modifications $\mathcal{E}_{\mathbf{1}} \to \mathcal{E}_b$ supported at ∞, of type $\mu\}$, with the natural group actions on the r.h.s.

Can define a functor $\mathcal{M}_\infty$ representing essentially the same moduli problem on more general perfectoid spaces over $\breve{E}$; this turns out to be a diamond, and $\mathcal{M}_K \overset{def}{=} \mathcal{M}_\infty/K$ is a diamond too.

Some misc. remarks

- In the case of RZ spaces, Scholze-Weinstein had already reinterpreted their generic fibers in this language of modified vector bundles in 2012.
- For minuscule μ, not yet known in general whether $\mathcal{M}_\infty$ is representable by a perfectoid space. (True in situations related to RZ spaces, by Scholze-Weinstein, W. Kim,...)
- Local and global Shimura varieties are related by suitable *uniformization isomorphisms*: suitable pieces of global Shimura varieties should be uniformized by (quotients of) local Shimura varieties. Provably true in many cases (Rapoport-Zink, W. Kim, Howard-Pappas, Shen,...).

Expectations for cohomology

Fix (G, μ, b). Want to decompose the cohomology $R\Gamma_c(\mathcal{M}(G, \mu, b)_{\infty, \overline{E}}, \overline{\mathbf{Q}_\ell})$ representation-theoretically under the natural $G(F) \times G_b(F) \times W_E$-action.

First problem: This object is too big to be studied meaningfully. (Too big $=$ not admissible as a $G(F) \times G_b(F)$-representation.)

Solution: Given (G, μ, b), *fix* an admissible smooth $G_b(F)$-representation ρ, and consider

$$H^i_c(G, \mu, b)[\rho] = H^i(R\Gamma_c(\mathcal{M}_{\infty, \overline{E}}, \overline{\mathbf{Q}_\ell}) \otimes^{\mathbf{L}}_{\mathcal{H}(G_b(F))} \rho).$$

Still a $G(F) \times W_E$-representation.

Theorem (Fargues-Scholze)

If ρ is admissible, then $H^i_c(G, \mu, b)[\rho]$ is an admissible $G(F)$-representation, and $H^i_c(G, \mu, b)[\rho] = 0$ unless $0 \leq i \leq 2 \dim \mathcal{M}(G, \mu, b)_K$.

Can we describe these groups?

Expectations for cohomology, cont'd

Two key conjectures (stated roughly):

- Kottwitz: Suppose b is "basic" ($\leftrightsquigarrow G_b$ is an inner form of G). Let ρ be an irreducible smooth rep. of $G_b(F)$. Then

$$\rho \rightsquigarrow \mathcal{H}(G,\mu,b)[\rho] \overset{def}{=} \sum_{i\geq 0}(-1)^i H^i_c(G,\mu,b)[\rho]$$

  encodes the refined local Langlands correspondences for supercuspidal L-packets of $G(F)$ and $G_b(F)$.

- Harris, Harris-Viehmann: Suppose b is *not* basic. Then

$$\mathcal{H}(G,\mu,b)[\rho] \simeq \oplus_{i\in I} \pm \mathrm{Ind}_{P_i(F)}^{G(F)} \mathcal{H}(M_i,\mu_i,b)[\rho]$$

  for some explicit finite list of local Shimura data (M_i,μ_i,b) assoc. with Levi subgroups of G. In particular, no supercuspidal $G(F)$-reps should appear.

Key example of the Kottwitz conjecture

Take $F = \mathbf{Q}_p$, $G = \mathrm{GL}_n$, $\mu(z) = \mathrm{diag}(\underbrace{z, \ldots, z}_{d}, 1, \ldots, 1)$ for some $1 \leq d < n$, b any element with characteristic polynomial $X^n - p^d$. Then $G_b(\mathbf{Q}_p) = A^\times$, $A/\mathbf{Q}_p$ central simple algebra of rank n^2 and Hasse invariant d/n. Jacquet-Langlands, Deligne-Kazhdan-Vignéras, Rogawski: Any supercuspidal representation π of $\mathrm{GL}_n(\mathbf{Q}_p)$ has a canonical transfer to a representation ρ_π of $A^\times$.

Conjecture (Kottwitz conjecture)

In this setup, should have $\mathcal{H}(G, \mu, b)[\rho_\pi] \simeq \pm \pi \boxtimes \mathrm{sym}^d(\varphi_\pi)$.

Here $\varphi_\pi : W_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$ is the L-parameter associated with π by local Langlands correspondence.
Proved for $d = 1$ by Harris-Taylor in their work on local Langlands; proved for arbitrary d by Fargues and Shin.
Today: First general results towards the Kottwitz conjecture.

Interlude: Local Langlands correspondence

Setup: Fix $F/\mathbf{Q}_p$ a finite extension, G/F a connected reductive group, C an algebraically closed field of characteristic zero.
Expectation: There should be a "natural" finite-to-one map

$$\mathrm{Irr}_C(G(F)) \to L - \text{parameters}\, \varphi : W_F \to {}^L G(C)$$

with many good properties. Write $\Pi_\varphi(G)$ for the fiber of this map over a given φ.
The precise structure of the fibers should be governed by the group $S_\varphi = \mathrm{Cent}_{\hat{G}}(\varphi)$ (or variants thereof).

Local Langlands correspondence cont'd

The group S_φ is a (possibly disconnected) reductive group, containing $Z(\hat{G})^{\Gamma_F}$ as a central subgroup.
Vague idea, redux: The structure of the packet $\Pi_\varphi(G)$ should be governed by the algebraic representations of S_φ.
Better idea (Vogan): Should try to parametrize **all** the packets $\Pi_\varphi(H)$ as H varies over inner forms of G.
Our next goal is to make this precise.

Isocrystal local Langlands correspondence

Set $B(G) = G(\breve{F})/(b \sim gb\sigma(g)^{-1})$. Kottwitz's set of isocrystals with G-structure.

This comes with a natural subset $B(G)_{\mathrm{bas}}$ of basic elements, and there is a natural bijection

$$\kappa : B(G)_{\mathrm{bas}} \to X^*(Z(\hat{G})^{\Gamma_F}).$$

For any $b \in B(G)_{\mathrm{bas}}$, get an inner form G_b as before.

Refined local Langlands conjecture, "isocrystal form" (Kottwitz, Kaletha). If G is quasisplit and φ is **supercuspidal**, then for every $b \in B(G)_{\mathrm{bas}}$, should have a natural bijection

$$\iota : \Pi_\varphi(G_b) \to \mathrm{Irr}(S_\varphi, \kappa(b)).$$

Here $\mathrm{Irr}(S_\varphi, \kappa(b))$ denotes the set of irreducible **algebraic** representations of S_φ whose restriction to $Z(\hat{G})^{\Gamma_F}$ is $\kappa(b)$-isotypic.

Example one: $G = \mathrm{GL}_n$

Let's take $G = \mathrm{GL}_n$. Then $\hat{G} = \mathrm{GL}_n$, and φ is supercuspidal iff it is irreducible. In this situation, get $S_\varphi = Z(\hat{G})^{\Gamma_F} = \mathbf{G}_m$. Hence

$$B(G)_{\mathrm{bas}} \cong X^*(Z(\hat{G})^{\Gamma_F}) \cong \mathbf{Z}.$$

Concretely, if $\kappa(b) = d$, then $G_b(F) = A^\times$, A/F the central simple algebra of rank n^2 and Hasse invariant d/n mod 1. (So G_b depends only on d mod n.)
Refined LLC then boils down to the expectation that each $\Pi_\varphi(G_b)$ is a singleton. This is known; moreover, the elements of these packets are "Jacquet-Langlands transfers" of each other (Jacquet-Langlands, Deligne-Kazhdan-Vignéras, Rogawski).

Example two: $G = \mathrm{GSp}_4$

Now take $G = \mathrm{GSp}_4$. Then $\hat{G} = \mathrm{GSp}_4$ (accidentally!) and $Z(\hat{G})^{\Gamma_F} = \mathbf{G}_m$, so again

$$B(G)_{\mathrm{bas}} \cong X^*(Z(\hat{G})^{\Gamma_F}) \cong \mathbf{Z}.$$

Concretely, if $\kappa(b) = d$, then G_b depends only on the parity of d: get GSp_4 if d is even, and $J = \mathrm{GU}_2(D)$ (unique inner form!) if d is odd. But now it gets interesting: if φ is supercuspidal, can look at $\mathrm{std} \circ \varphi : W_F \to \mathrm{GL}_4$. It turns out that one of two things can happen:

- $\mathrm{std} \circ \varphi$ is irreducible. Then $S_\varphi = \mathbf{G}_m$, and $\Pi_\varphi(G)$ and $\Pi_\varphi(J)$ are singletons.
- $\mathrm{std} \circ \varphi \simeq \varphi_1 \oplus \varphi_2$ with φ_i two-dimensional, distinct and irreducible (and with equal determinants). Then S_φ is disconnected with neutral component $\mathbf{G}_m$ of index two. Both $\Pi_\varphi(G)$ and $\Pi_\varphi(J)$ contain **two** elements.

Kottwitz conjecture: setup

Fix a local Shimura datum (G, μ, b) as before, with b **basic**. For any $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$, consider the virtual $G(F) \times W_E$ representation

$$\mathcal{H}(G, \mu, b)[\rho] \stackrel{def}{=} \sum_{i \geq 0} (-1)^i H^i_c(G, \mu, b)[\rho].$$

How to describe this explicitly?
First observation: From μ, get an algebraic representation $r_\mu : {}^L G \to \mathrm{GL}_m$. For any L-parameter φ, the composition $r_\mu \circ \varphi|_{W_E}$ is a representation of $W_E \times S_\varphi$ (think about def. of S_φ).
Second observation: From refined LLC, given any $\pi \in \Pi_\varphi(G)$ and $\rho \in \Pi_\varphi(G_b)$, can extract an algebraic representation $\delta_{\pi,\rho}$ of S_φ which measures the "relative positions" of π and ρ. For G quasisplit this is given by $\iota(\pi) \otimes \iota(\rho)^\vee$, but there is a general recipe.

Kottwitz conjecture: statement

Conjecture (Kottwitz)

Fix a basic local Shimura datum (G, μ, b) and a supercuspidal L-parameter $\varphi : W_F \to {}^L G(\overline{\mathbf{Q}_\ell})$. Let $\rho \in \Pi_\varphi(G_b)$ be any element. Then

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu \circ \varphi|_{W_E})$$

as virtual $G(F) \times W_E$ representations.

Strictly speaking, I am ignoring a Tate twist.

Weakened Kottwitz conjecture: statement

If we ignore the Weil group action on the cohomology, the statement becomes simpler.

Conjecture (Kottwitz)

Fix a basic local Shimura datum (G, μ, b) and a supercuspidal L-parameter $\varphi : W_F \to {}^L G(\overline{\mathbf{Q}_\ell})$. Let $\rho \in \Pi_\varphi(G_b)$ be any element. Then

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu)]\pi$$

as virtual $G(F)$ representations.

The full Kottwitz conjecture is only known in a handful of situations:

- The Lubin-Tate/Drinfeld towers (Harris-Taylor)
- Local Shimura varieties of "unramified EL" type (Fargues, Shin)
- Some unitary local Shimura varieties of "unramified PEL" type (Bertoloni-Meli–Nguyen)

All of these works rely crucially on **global** methods (comparison with cohomology of global Shimura varieties). For the weakened Kottwitz conjecture, purely **local** methods can be brought to bear.

Main theorem

Theorem (H.-Kaletha-Weinstein)

Fix a basic local Shimura datum (G, μ, b) and a supercuspidal L-parameter $\varphi : W_F \to {}^L G(\overline{\mathbf{Q}_\ell})$. Let $\rho \in \Pi_\varphi(G_b)$ be any element. Then

$$\mathcal{H}(G, \mu, b)[\rho] = (-1)^{\langle 2\rho, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu)]\pi + \mathrm{err}$$

as virtual $G(F)$ representations, where err is a **non-elliptic** *virtual representation.*

Recall that a regular semisimple element $g \in G(F)$ is elliptic if the torus $\mathrm{Cent}_G(g)/Z(G)$ is anisotropic. A virtual representation is non-elliptic if its Harish-Chandra character vanishes on all elliptic elements.
Here we assume a certain refined form of LLC due to Kaletha. **Assuming** a certain compatibility between this and the Fargues-Scholze construction of L-parameters, we can also show that $\mathrm{err} = 0$ as expected.

Commentary

- We actually prove a more general result, for general moduli of local shtukas (no restriction on μ). Here one essentially needs to take **intersection cohomology** in the definition of $\mathcal{H}(G, \mu, b)[\rho]$.
- We also prove a result for more general **discrete** L-parameters. Here the error term can definitely be nonzero!
- The key idea, of using the Lefschetz trace formula in some form, goes back to a (literal) dream of Harris in the early 90s.
- There are previous results of Faltings (for GL_2), Strauch (for GL_n), and Mieda (for GSp_4), also using trace formula methods. However, our implementation of this idea is totally different, and uses the full force of "modern" p-adic geometry a la Scholze.

Proof sketch

Key idea: Define an **explicit** operator $T_{b,\mu}^{G_b \to G} : C(G_b(F)_{\mathrm{ell}}//G_b(F), \overline{\mathbf{Q}_\ell}) \to C(G(F)_{\mathrm{ell}}//G(F), \overline{\mathbf{Q}_\ell})$, and then prove two separate claims.

Claim 1: There is an equality

$$T_{b,\mu}^{G_b \to G}(\Theta_\rho)(g) = (-1)^{\langle 2\rho, \mu \rangle} \sum_{\pi \in \Pi_\varphi(G)} [\dim \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_\mu)] \Theta_\pi(g)$$

for all elliptic g.

Claim 2: There is an equality

$$T_{b,\mu}^{G_b \to G}(\Theta_\rho)(g) = \Theta_{\mathcal{H}(G,\mu,b)[\rho]}(g)$$

for all elliptic g and **any** $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$.

These two claims immediately imply the theorem.

Proof sketch cont'd

Claim 1 follows from a direct (though subtle) analysis of the operator $T_{b,\mu}^{G_b \to G}$, using the **endoscopic character identities** in the refined LLC. Claim 2 follows from an application of the Lefschetz trace formula, together with a subtle continuity argument to reduce to the case where ρ admits a $\overline{\mathbf{Z}}_\ell$-lattice. Some key inputs:

- Lu-Zheng's new point of view on the Lefschetz trace formula, via the **symmetric monoidal 2-category of cohomological correspondences**.
- The monumental work of Fargues-Scholze: $\mathcal{H}(G, \mu, b)[\rho]$ in terms of Hecke operators on the stack Bun_G, ULA sheaves in p-adic geometry, geometric Satake for the B_{dR}-affine Grassmannian,...
- Recent work of Varshavsky on local terms in the Lefschetz trace formula.

Thank you for listening!