Affine Deligne-Lusztig varieties & Generalized affine Springer fibers

(ADLV) (GASF)

Notations:

F nonarch local field with valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$.

$\breve{F} = \widehat{F^{un}}$ with valuation ring $\mathcal{O}_{\breve{F}}$ and residue field $k = \overline{\mathbb{F}_q}$.

σ Frob morphism of $\breve{F}$ over F.

G conn reductive gp $/F$. $I \subset G(\breve{F})$ σ-stable Iwahori subgp.

$\tilde{W}$ Iwahori-Weyl gp.

Then
$$G(\breve{F}) = \coprod_{w \in \tilde{W}} IwI$$

If G is unramified, let $K \supset I$ be a hyperspecial parahoric subgp.

Then
$$G(\breve{F}) = \coprod_{\mu \text{ dominant}} K\mu K.$$

Affine Flag variety $\mathcal{Fl} = G(\breve{F})/I$

Affine Grassmannian $Gr = G(\breve{F})/K$.

Def (Rapoport) Let $b \in G(\breve{F})$, $w \in \tilde{W}$, and μ dominant (if G unramified)

ADLV

$$X_w(b) = \{gI \in \mathcal{Fl};\ g^{-1} b\sigma(g) \in IwI\}$$

$$X_\mu(b) = \{gK \in Gr;\ g^{-1} b\sigma(g) \in K\mu K\}$$

Rmk. ADLV plays an important role in the study of reduction of Shimura var.

Def (Kottwitz-Viehmann, Lusztig)

Let $\gamma \in G(\breve{F})$ regular semisimple, $w \in \breve{W}$, μ dominant.

GASF

$$Y_w(\gamma) = \{gI \in \mathcal{Fl};\ g^{-1}\gamma g \in IwI\}$$

$$Y_\mu(\gamma) = \{gK \in Gr;\ g^{-1}\gamma g \in K\mu K\}$$

Rmk. GASF encodes orbital integrals of spherical and Iwahori Hecke functions.

History

Problem	ADLV	GASF
Nonemptiness pattern	In Gr, Rapoport-Richartz '96, Kottwitz '03, Gashi '10 In Fl for basic b, Görtz-Haines-Kottwitz-Reuman '10 Görtz-H.-Nie '15	In Gr for split groups Kottwitz-Viehmann '12 Chi '19
Dimension formula	In Gr, Görtz-Haines-Kottwitz-Reuman & Viehmann '06 (Rapoport conjecture) Hamacher '15, X. Zhu '17. In Fl, Görtz-Haines-Kottwitz-Reuman '10 conjecture basic b, H. '14 & '16. nonbasic H. '20+	In Gr for split gps in equal char, Bouthier '15, Chi '19 via global argument
Irr comp	In Gr. Chen-Zhu conjecture Zhou-Y. Zhu '20 & Nie '19+ H.-Zhou-Zhu. Nie (ongoing)	In Gr, Chi conjecture proved for split gps in equal char for split γ.

My work in progress:

Motto: Information on ADLV + ASF (usual affine Springer fiber)

$\xrightarrow{\text{local argument}}$ Information on GASF.

First step: How to match $[b]$ with γ?

Let G be a residually split gp over F (so that σ acts trivially on $\tilde{W}$)

$$\begin{tikzcd}
{[b]} \in B(G)=G(\breve{F})/_\sigma G(\breve{F}) \arrow[rr, dashed] \arrow[dr, leftrightarrow, "\text{H.'14}"] \arrow[ddr, bend right, "\text{Kottwitz'85}"'] & & G(F)/G(F) \ni \{\gamma\} \arrow[dl, "\text{H.-Nie'20}"'] \arrow[ddl, bend left, "\text{Kottwitz-Viehmann'12}"] \\
& B(W)_{str}=(W/W)_{str} \arrow[d, "\text{H.-Nie'14}"] & \\
& \pi_1(G)_p \times X_*(T)^+_{\mathbb{Q}} &
\end{tikzcd}$$

Working in progress: Let G be a residually split group over F. Suppose that $[b] \leftrightarrow \{\nu\}$.

Suppose that the residue char of F is large.

(1) For any $w \in \tilde{W}$, if $Y_w(\nu) \neq \emptyset$, then $X_w(b) \neq \emptyset$.

(2) For almost all $w \in \tilde{W}$.

$$\dim Y_w(\nu) = \dim X_w(b) + \dim \text{ of ASF associated to } \nu.$$

(3) There exists a natural bijection

$$Z^{\circ}_{G(\breve{F})}(\nu) \backslash \mathrm{Irr}^{top}\, Y_\mu(\nu) \xleftrightarrow{1-1} J_b \backslash \mathrm{Irr}^{top}\, X_\mu(b).$$

Rmk. ① In the general case, the definition of ASF associated to ν involves the straight elements of $\tilde{W}$ and is a bit technical.

② If $\nu \in I$, we may take $b=1$ and in this case, ASF associated to ν is just Fl^{ν}, the fixed pt set.

And $\dim Fl^{\nu}$ is known due to Kazhdan-Lusztig, Bezrukavnikov.

Consequence: Under a mild assumption on the residue char (might be removable)

(1) $Y_\mu(\gamma) \neq \phi$ iff the Mazur's inequality is satisfied, i.e.

$\kappa(\mu) = \kappa(\gamma)$ and $\nu_\gamma \leq \mu^\diamond$. Here κ is the Kottwitz map and ν_γ is the Newton pt.

In this case, $\dim Y_\mu(\gamma) = \langle \mu, \rho \rangle + \frac{1}{2}(d(\gamma) - c(\gamma))$,

where $d(\gamma)$ is the discriminant valuation of γ, ie val of $\det(\mathrm{Id} - \mathrm{ad}_\gamma : \mathfrak{g}(F)/\mathfrak{g}_\gamma(F)^\partial)$.

and $c(\gamma) = \mathrm{rank}(G) - \mathrm{rank}_F(Z_G(\gamma))$

(2) Let $w = x t^{\lambda_w} y$ with x, y in finite Weyl gp. and $t^{\lambda_w} y$ is in dom chamber.

If $\lambda_w \geq \nu_\gamma + 2\rho^\vee$, then $Y_w(\gamma) \neq \phi$ iff $\kappa(w) = \kappa(\gamma)$, $yx \in W_0$ has full support.

In this case, $\dim Y_w(\gamma) = \frac{1}{2}(\ell(w) + \ell(yx) + d(\gamma) - c(\gamma))$

(3) If $Y_\mu(\gamma) \neq \phi$, then $\exists$ natural bijections

$$Z_G(\gamma)^\circ \backslash \mathrm{Irr}^{top}\, Y_\mu(\gamma) \simeq MV_\mu(\lambda_G(\gamma)) \simeq \mathbb{B}_\mu(\lambda_G(\gamma))$$

Mirkovic-Vilonen cycle canonical basis

Here $\lambda_G(\gamma)$ is the "best integral approximation" of ν_γ.

Strategy:

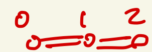
① H. '14 DL reduction method on $X_w(b)$

from arbitrary w to min length w

Such reduction step is still valid for $Y_w(\gamma)$. And the reduction procedure is encoded in the so-called class polynomial of affine Hecke alg.

② For w of min length, $X_w(b)$ is easy to understand.

But $Y_w(\gamma)$ is rather difficult.

/ Eg. 

w may be in

$W_0 = W_{\{1,2\}}$ ← able to handle

or $W_{\{0,2\}}$ ← hard to handle.

- Full strength of Chen-Zhu conj on the orbifold $J_b \backslash \mathrm{Irr}^{top} X_\mu(b))$

  $\rightsquigarrow$ study of $Y_w(\gamma)$ for $w \in W_0$!!!

- Kazhdan-Lusztig, Bezrukavnikov's work on the regular locus of Fl^γ.

- Lusztig's work on classical generalized Springer fiber

  (precursor of the theory of character sheaves)