On the derived category of the Iwahori–Hecke algebra, I

§1 Intro

fix F fin $/ \mathbb{Q}_p$ or $\mathbb{F}_p((t))$, $\supseteq \mathcal{O}_F$ integers, ϖ unif., k res. field

$q = \# k$

C (alg closed) field of char 0, $q^{1/2} \in C$.

<u>Recall</u> (i) local Langlands for $G = GL_n(F)$:

$$\left\{\begin{array}{l}\text{iso classes of irred.}\\ \text{smooth } G\text{-repn's}\\ \text{on } C\text{-vs}\end{array}\right\} \xrightarrow{1:1} \left\{\begin{array}{l}\text{iso classes of } n\text{-dim.}\\ \text{Frob. semi-simple}\\ \text{Weil-Deligne repn's}\end{array}\right\}$$

$\cup$ Simplest special case $\cup$

$$\left\{\begin{array}{l}\text{iso classes of those}\\ \pi \text{ s.th } \pi^I \neq 0\end{array}\right\} \xrightarrow{1:1} \left\{\begin{array}{l}\text{conj. classes of}\\ (\varphi, N) \in GL_n(C) \times \mathrm{Mat}_{n\times n}(C)\\ \varphi \text{ semi-simple} \quad N\varphi = q\varphi N\end{array}\right\}$$

where
$$I = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ \varpi * & & * \end{pmatrix}\right\} \subseteq K = GL_n(\mathcal{O}_F) \subseteq G \quad \text{std. Iwahori}$$

note: LHS $= \{$ those π that occur in $i_B^G \delta$, for some unram char. δ of $T \}$

$T \subseteq B \subseteq G$
split max torus   Borel

$\cong \{$ simple modules over $\mathcal{H}_G = \mathrm{End}_G(\text{c-ind}_I^G \mathbb{1}) \}$

Iwahori-Hecke alg.

(ii) more generally: $G/_F$ split reductive, I Iwahori

dual grp.

$$\left\{\begin{array}{l}\text{iso classes of sm.}\\ \text{irred } G\text{-repn's } \pi\\ \text{s.th } \pi^I \neq 0\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{conj. classes of}\\ (\varphi, N) \in \check{G}(C) \times \mathrm{Lie}\,\check{G}(C)\\ \varphi \text{ semi-simple, } \mathrm{Ad}(\varphi)N = q^{-1} N\end{array}\right\}$$

with fin. fibers param. by certain repn's of $\mathrm{Cent}(\varphi, N)$

<u>Aim</u>: lift this parametrization to the level of categories

related work by X. Zhu, Ben-Zvi – Chen – Helm – Nadler

§2 The Conjecture

$\mathbb{G}/_F$ split reductive, $\underset{\text{torus}}{T} \subseteq \underset{\text{Borel}}{B} \subseteq G = \mathbb{G}(F)$, $W =$ Weyl grp.

$\rightsquigarrow \check{G}, \check{B}, \check{T}$ dual grps $/\mathbb{C}$.

→ $\mathrm{Rep}\, G$ = cat. of smooth repn's on $\mathbb{C}$-vs

$$\begin{array}{ccl} & \cup\!\! \mathsf{I} & \\ \pi & \mathrm{Rep}_{[T,1]}\, G & \text{full sub cat of those } \pi \text{ s.th. the irred subquots of } \pi \\ \big\downarrow & \big\downarrow\cong & \text{occur in some } i_B^G \delta,\ \delta \text{ unram char.} \\ \pi^I & \mathcal{H}_G\text{-mod} & \end{array}$$

$\mathrm{Rep}_{[T,1]}\, G \circlearrowleft \mathfrak{Z}_G$ = center of $\mathrm{Rep}_{[T,1]}\, G \cong$ center of $\mathcal{H}_G$
$\cong \mathbb{C}[X_*(T)]^W = \Gamma(\check{T}/_W, \mathcal{O}_{\check{T}/_W})$

→ $X_{\check{G}} = \{ (\varphi, N) \in \check{G} \times \mathrm{Lie}\,\check{G} \mid \mathrm{Ad}(\varphi) N = q^{-1} N \}$ scheme of fin type $/\mathbb{C}$

$\rightsquigarrow$ $[X_{\check{G}}/\check{G}]$ stack quotient, $\mathbb{D}^+_{qcoh}([X_{\check{G}}/\check{G}])$ its derived cat of q coh. sheaves.

This is a $\mathfrak{Z}_G$-lin. cat. via

$$X_{\check{G}} \xrightarrow{\text{proj.}} \check{G} \longrightarrow \check{G}/\!\!/\check{G} = \check{T}/_W = \mathrm{Spec}\, \mathfrak{Z}_G.$$

(in fact: $\check{T}/_W \cong X_{\check{G}} /\!\!/ \check{G}$ GIT quot)

<u>Conj:</u> for every (G, B, T) + choice of a Whitt. datum ψ:

there ex. a fully faithful $\mathfrak{Z}_G$-linear functor

$$R_G^\psi : \mathbb{D}^+(\mathrm{Rep}_{[T,1]}\, G) \longrightarrow \mathbb{D}^+_{qcoh}([X_{\check{G}}/\check{G}])$$

and for each $P \subseteq G$ parabolic with Levi M $(P \supseteq B)$

$\exists$ nat $\mathfrak{Z}_G$-lin. isom

$$\xi_P^G : R_G^{\psi} \circ i_{\bar{P}}^G \xrightarrow{\ \simeq\ } R\beta_{P,*} L\alpha_P^* R_M^{\psi_M}$$

where:
- ψ_M = Whitt datum for M induced by ψ
- $\alpha_P : [X_{\check{P}}/\check{P}] \longrightarrow [X_{\check{M}}/\check{M}]$

  $\beta_P : [X_{\check{P}}/\check{P}] \longrightarrow [X_{\check{G}}/\check{G}]$

  induced by $\check{P} \twoheadrightarrow \check{M}$ resp $\check{P} \hookrightarrow \check{G}$

These data satisfy:

(i) compatibility among the various ξ_P^G

(ii) if $G=T$ is a split torus, $T^{\circ} \subseteq T$ max compact, then R_T is induced by

$$\mathrm{Rep}_{[T,1]} T = \mathbb{C}[T/T^{\circ}]\text{-mod} = \mathbb{C}[X_*(T)]\text{-mod} = \mathrm{QCoh}(\check{T})$$

where $\check{T} = X_{\check{T}} \underset{\text{trivial}}{\circlearrowleft} \check{T}$

(iii) $$R_G^{\psi}\left((c\text{-}\mathrm{ind}_N^G \psi)_{[T,1]} \right) = \mathcal{O}_{[X_{\check{G}}/\check{G}]}$$

(here $\pi_{[T,1]}$ = image of π in Bernstein block $\mathrm{Rep}_{[T,1]} G \subseteq \mathrm{Rep}\, G$)

<u>Remark</u> (i) There is a variant for $\mathrm{Rep}\, G$ instead of $\mathrm{Rep}_{[T,1]} G$, replacing $[X_{\check{G}}/\check{G}]$ be the stack of all L-parameters for G

(ii) we have to work with derived categories, as we compare "flat structures with non-flat structures"

e.g.: $(c\text{-}\mathrm{ind}_N^G \psi)_{[T,1]}$ flat$/\mathfrak{Z}_G$, $X_{\check{G}} \longrightarrow \check{T}/W = \mathrm{Spec}\, \mathfrak{Z}_G$ <u>not</u> flat

($G = GL_n$: $\{\text{irred compos of } X_{\check{G}}\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{possible Jordan}\\ \text{can. forms of } N\end{matrix}\right\} = \check{G}\text{-orbits in } \mathcal{N}_{\check{G}}$

nilpot. cone

$X_{\check{G}} \longrightarrow \check{T}/W$ maps some irred compos to proper closed subschemes)

(iii) even worse: have to replace $X_{\check{P}}$ by its derived variant.

(otherwise: compatibility of various ξ_P^G can't work)

$$\begin{tikzcd} X_{\check{P}} \arrow[r, hook] \arrow[d, "\Gamma"] & \check{P} \times \mathrm{Lie}\,\check{P} \arrow[d] & (\varphi, N) \arrow[d, mapsto] \\ \{0\} \arrow[r, hook] & \mathrm{Lie}\,\check{P} & \mathrm{Ad}\varphi\, N - q^{-1} N \end{tikzcd}$$

derived fib. prod

This is not a classical scheme in general

(main reason: $\check{P} \curvearrowright \mathrm{Lie}\,\check{P} \cap \mathcal{N}_{\check{G}}$ not via fin. many orbits)

§3 Results

Thm 1 Conj is true for GL_2

Thm 2 $G = GL_n$, $\exists$ explicit candidate

$$R_G : \pi \longmapsto \pi^I \otimes_{\mathcal{H}_G} \mathcal{M}_G$$

satisfying the requirements (except maybe fully faithfulness)

over $[X^{reg}_{\check{G}} / \check{G}] \subseteq [X_{\check{G}} / \check{G}]$

open subset of "regular" (φ, N) ($\rightsquigarrow$ next time)

Aim for the remaining time:

what is $\mathcal{M}_G$ and why should R_G satisfy Conj?

we restrict to $G = GL_n$ ($\rightarrow$ choice of ψ unique up to isom.)

1) $\mathcal{M}_G$ as an $\mathcal{O}_{[X_{\check{G}}/\check{G}]}$ - module

if $\exists$ fully faithful functor R_G, then:

$$\mathcal{H}_G = \mathrm{End}_G(\text{c-ind}_I^G \mathbb{1}) = R\mathrm{End}_G(\text{c-ind}_I^G \mathbb{1})$$

$$= R\mathrm{End}_{[X_{\check{G}}/\check{G}]}\big(R_G(\underbrace{\text{c-ind}_I^G \mathbb{1}}_{= \, i_{\check{B}}^G (\text{c-ind}_{T^\circ}^T \mathbb{1})})\big)$$

$$= R\mathrm{End}_{[X_{\check{G}}/\check{G}]}(\mathcal{M}_G)$$

where $\mathcal{M}_G = R_G(\iota_{\bar{B}}^G \text{ c-ind}_{T^\circ}^T \mathbb{1})$

$$= R\beta_{B,*} L\alpha_B^* \underbrace{R_T(\text{c-ind}_{T^\circ}^T \mathbb{1})}_{= \mathcal{O}_{[X_{\check{T}}/\check{T}]}}$$

$$= R\beta_{B,*} \mathcal{O}_{[X_{\check{B}}/\check{B}]}$$

$\rightsquigarrow$ to prove the Conj: need to identify $\mathcal{H}_G \cong R\mathrm{End}_{[X_{\check{G}}/\check{G}]}(\mathcal{M}_G)$

2) how to construct an $\mathcal{H}_G$-module structure on $\mathcal{M}_G$?

(in fact: $\mathcal{H}_G \otimes_{\mathfrak{Z}_G} \mathcal{O}_{[X_{\check{G}}/\check{G}]}$ - module str.)

next time: construct an explicit family $\mathcal{V}_G$ of G-repn's on $[X_{\check{G}}/\check{G}]$ ("Emerton-Helm family")

s.th. $\mathcal{V}_G$ "interpolates (modified) local Langlands"

and (conjecturally) $\mathcal{V}_G^I = \mathcal{M}_G$

(true over $[X_{\check{G}}^{reg}/\check{G}] \subseteq [X_{\check{G}}/\check{G}]$)