<u>last time</u> $\quad G/_F$ split reductive, $\mathcal{B} \supseteq T$ Borel, Torus

(ψ Whittaker datum)

<u>Conj</u>: $\exists$ f.f. + exact $\mathfrak{Z}_G$-linear functor

$$R_G^\psi : \mathbb{D}^+(\mathrm{Rep}_{[T,1]} G) \longrightarrow \mathbb{D}^+_{\mathrm{QCoh}}([X_{\hat{G}}/\hat{G}])$$

s.th. $\quad R_G^\psi \circ i_{\bar{P}}^G \cong R\beta_{P,*}\, L\alpha_P^* \circ R_M^\psi \quad$ for $P \subseteq G$ parabolic ($\supseteq \mathcal{B}$)

$\to$ Small corrections

(i) $\quad G = GL_n$; this should recover the Langlands parametrization as follows:

given $(\varphi, N) \in X_{\hat{G}}(\mathbb{C})$, φ semisimple

$$\rightsquigarrow \quad LL(\varphi, N) \twoheadleftarrow LL^{mod}(\varphi, N) \quad \text{unique simple quot}$$

$$LL^{mod}(\varphi, N) \overset{def}{=} i_P^G(St(\lambda_1, r_1) \otimes \_ \otimes St(\lambda_s, r_s))$$

for appropriate ordering

where $\quad (\mathbb{C}^n, \varphi, N) \cong \bigoplus_{i=1}^{s} Sp(\lambda_i, r_i) \longleftarrow \begin{cases} \varphi_i = \mathrm{diag}(\lambda_i, q^{-1}\lambda_i, \dots, q^{-(r_i-1)}\lambda_i) \\ N_i = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix} \end{cases}$

and $\quad St(\lambda_i, r_i) = LL(Sp(\lambda_i, r_i)) = LL^{mod}(\lambda_i, r_i)$

corresponding generalized Steinberg

<u>expect</u>: $\quad R_{GL_n}(LL^{mod}(\varphi, N)) = \mathcal{O}_{X_{\varphi,N}}$

$$X_{\varphi,N} = \bigcup_{\substack{(\varphi, N_0) \in X_{\hat{G}} \\ N_0 \leq N}} \overline{\{(\varphi', N') \in X_{\hat{G}} \mid (\varphi'^{ss}, N') \text{ conj. to } (\varphi, N_0)\}}$$

closure relation in nilpotent cone

(ii) R_{GL_n} induces

$$K_0(\mathrm{Rep}^{fg}_{\rho[T,1]} GL_n(F)) \longrightarrow K_0(\mathrm{Coh}^{\hat{G}}(X_{\hat{G}}))$$

$$[\text{c-ind}_K^G \sigma_P] \longmapsto [\mathcal{O}_{X_P}] + [\mathcal{F}]$$

P Partition of $\{1,\dots,n\}$ $\hat{=}$ irred compo X_P

$\mathcal{F}$ supported on $\bigcup_{P' > P} X_{P'}$ (but not clear what precisely this is)

§2 Results

Thm 1 Conj. is true for GL_2

Thm 2 for $G = GL_n$ $\exists$ explicit candidate

$$R_{GL_n} : \pi \longmapsto \pi^I \otimes_{\mathcal{H}_G} \mathcal{M}_G$$

satisfying requirements of Conj. (except so far can't prove fully faithfulness) over $[X^{reg}_{\hat{G}}/\hat{G}] \subseteq [X_{\hat{G}}/\hat{G}]$

open subset of "regular elements" (see below)

Rem: Zhu, Ben-Zvi – Chen – Helm – Nadler announce full proof

Aim for rest of this talk:

what is $\mathcal{M}_G$, and why should R_{GL_n} satisfy the Conj.?

§3 $\mathcal{M}_G$ as an $\mathcal{O}_{[X_{\hat{G}}/\hat{G}]}$-module (from now on: $G = GL_n$!)

if R_{GL_n} exists, then

$$\mathcal{H}_G = \mathrm{End}_G(\text{c-ind}_I^G \mathbb{1}) = R\mathrm{End}_G(\text{c-ind}_I^G \mathbb{1})$$

$$\cong R\mathrm{End}_{[X_{\hat{G}}/\hat{G}]}(R_{GL_n}(\underbrace{\text{c-ind}_I^G \mathbb{1}}_{= i_B^G(\text{c-ind}_{T_0}^T \mathbb{1})})) = R\mathrm{End}_{[X_{\hat{G}}/\hat{G}]}(\mathcal{M}_G)$$

where $\mathcal{M}_G = R_{GL_n}(i_{\bar B}^{G}(c\text{-ind}_{T^0}^{T}\mathbb{1})) = R\beta_{B,*}\, L\alpha_{B}^{*}\, \underbrace{R_T(c\text{-ind}_{T^0}^{T}\mathbb{1})}_{=\mathcal{O}_{[X_{\hat T}/\hat T]}}$

$$= R\beta_{B,*}\, \mathcal{O}_{[X_{\hat B}/\hat B]}$$

$\rightsquigarrow$ to prove Conj., need to identify $\mathcal{H}_G \cong R\mathrm{End}(\mathcal{M}_G)$

<u>Note</u> pullback $\tilde{\mathcal{M}}_G$ of $\mathcal{M}_G$ to $X_{\hat G}$ can be described as follows:

$$\tilde{\mathcal{M}}_G = R\tilde\beta_{B,*}\, \mathcal{O}_{\tilde X_{\hat B}}, \text{ with}$$

$\tilde X_{\hat B}$ = <u>derived</u> scheme parametrizing $(\varphi, N, F_\bullet)$

$N\varphi = q\varphi N$, $F_\bullet$ complete flag, stable under φ, N.

$$\begin{tikzcd} \tilde X_{\hat B} \arrow[r, hook, "i"] \arrow[d, "\tilde\beta_B"'] & Y = \left\{ (\varphi, N, F_\bullet) \;\middle|\; \begin{array}{l} F_\bullet \text{ stable under } \varphi, N, \\ \text{no relation between } \varphi, N \end{array} \right\} \arrow[d] \\ X_{\hat G} \arrow[r, hook] & \hat G \times \mathrm{Lie}\,\hat G \end{tikzcd}$$

$Ri_*\mathcal{O}_{\tilde X_{\hat B}}$ is a Koszul complex on Y.

<u>Conjecture 2</u> (i) $\tilde{\mathcal{M}}_G$ is concentrated in deg 0 and

$$R\mathcal{H}om_{X_{\hat G}}(\tilde{\mathcal{M}}_G, \omega_{X_{\hat G}}) \cong \tilde{\mathcal{M}}_G[\dim \hat G]$$

(in particular $\tilde{\mathcal{M}}_G$ is a max. Cohen-Macaulay module)

(ii) The canonical map

$$\begin{tikzcd} \tilde X_{\hat B} \arrow[r, "f"] \arrow[dr, "\tilde\beta_B"'] & X_{\hat G} \times_{\hat T/W} \hat T \arrow[d, "\pi"] \\ & X_{\hat G} \end{tikzcd}$$

induces a <u>surjection</u>

$$\mathcal{H}_T \otimes_{\mathfrak{Z}_G} \mathcal{O}_{X_{\hat G}} = \pi_*(\mathcal{O}_{X_{\hat G}\times_{\hat T/W}\hat T}) \twoheadrightarrow \tilde{\mathcal{M}}_G$$

$\mathcal{H}_T \cong \Gamma(\hat T, \mathcal{O}_{\hat T})$ Hecke alg of T.

about (i) Koszul complexes are self dual + Grothendieck duality

$\Rightarrow$ enough to show $\widetilde{\mathcal{M}}_G$ concentrated in deg ≤ 0

- true for GL_2, GL_3
- in general: would be implied by vanishing result for cohomology of certain line bdls on $\hat{G}/\hat{B}$ ($\underset{\text{Borel-Bott-Weil}}{\Leftarrow}$ combinatorial statement about roots)

about (ii) let $X_{\hat{G}}^{reg} \subseteq X_{\hat{G}}$ locus s.th

$$\begin{tikzcd} \widetilde{X}_{\hat{B}}^{reg} \arrow[d, "\text{finite}"'] \arrow[r, phantom, "\subseteq"] & \widetilde{X}_{\hat{B}} \arrow[d] \\ X_{\hat{G}}^{reg} \arrow[r, phantom, "\subseteq"] & X_{\hat{G}} \end{tikzcd}$$

($\Rightarrow \widetilde{X}_{\hat{B}}^{reg}$ is a classical scheme)

, then - $\widetilde{\mathcal{M}}_G|_{X_{\hat{G}}^{reg}}$ conc. in deg 0

- $\widetilde{X}_{\hat{B}}^{reg} \hookrightarrow X_{\hat{G}}^{reg} \times_{\hat{T}/W} \hat{T}$ closed imm.

$\Rightarrow$ (ii) ok over $X_{\hat{G}}^{reg}$.

Reformulation of (ii): let $Y_{\hat{G}} =$ closure of $\widetilde{X}_{\hat{B}}^{reg} \subseteq X_{\hat{G}} \times_{\hat{T}/W} \hat{T}$.

then $Rf_* \mathcal{O}_{\widetilde{X}_{\hat{B}}} = \mathcal{O}_{Y_{\hat{G}}}$

in fact: can prove (ii) over $X_{\hat{G}}^{reg} \subseteq \underbrace{\mathcal{U}}_{\text{open, compliment of codim} \geq 2} \subseteq X_{\hat{G}}$

§4 Constructing an $\mathcal{H}_G$-action on $\widetilde{\mathcal{M}}_G$

Idea / Construction (Emerton-Helm / Helm)

$\exists!$ family V_G of G-rep's on $X_{\hat{G}}$ s.th.

$$(c\text{-}\mathrm{ind}_N^G \psi)_{[T,1]} \otimes_{\mathfrak{Z}_G} \mathcal{O}_{X_{\hat{G}}} \twoheadrightarrow V_G$$

$$V_G \otimes k(x) = LL(\varphi_x, N_x) \text{ for } x = (\varphi_x, N_x) \text{ generic point of } X_{\hat{G}}.$$

Conj 3 (E.-H.) for arbitrary $x = (\varphi_x, N_x)$:

$$V_G \otimes k(x) = LL^{mod}((\varphi_x^{ss}, N_x)^\vee)^\vee$$

Thm There is a comm diag

$$\begin{array}{ccc} (c\text{-ind}_N^G \psi)^I \otimes_Z \mathcal{O}_{X_{\hat{G}}} & \twoheadrightarrow & V_G^I \\ \wr\| & & \wr\| \\ \mathcal{H}_T \otimes_Z \mathcal{O}_{X_{\hat{G}}} & \twoheadrightarrow & \pi_* \mathcal{O}_{Y_{\hat{G}}} \end{array}$$

comm diag of $\mathcal{H}_T \otimes_Z \mathcal{O}_{X_{\hat{G}}}$ - modules.

In particular: get action of $\mathcal{H}_G \otimes_Z \mathcal{O}_{X_{\hat{G}}}$ on $\pi_* \mathcal{O}_{Y_{\hat{G}}}$.

$\rightsquigarrow$ assuming Conj. 2 (i.e. $\tilde{\mathcal{M}}_G \cong \pi_* \mathcal{O}_{Y_{\hat{G}}} \cong V_G^I$)

compatibility with parabolic induction for $B \subseteq P \subseteq G$

comes down to:

- $(c\text{-ind}_N^G \psi)^I = \mathcal{H}_G e_{St,G} = \mathcal{H}_T e_{St,G}$ free of rk $1/\mathcal{H}_G$

  idempot elt def by $c\text{-ind}_K^G St_G \subseteq c\text{-ind}_I^G \mathbb{1}$

  fin. dim Steinberg of K

- $\mathcal{H}_M e_{M,St} = \mathcal{H}_T e_{M,St} \xrightarrow{\cong} \mathcal{H}_T e_{G,St} = \mathcal{H}_G e_{G,St}$

  is an <u>isom of $\mathcal{H}_M$</u> - modules