What is a unipotent representation?

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Big unsolved problem (Gelfand)

Let G be a real reductive Lie group. Classify

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Conjecture (Arthur, Vogan, Adams, Barbasch,...)

The classification of $\mathrm{Irr}_u(G)$ should reduce to the classification of a small finite subset

$$\mathrm{Unip}(G) \subset \mathrm{Irr}_u(G)$$

RHS built out of LHS by *parabolic induction*. LHS indexed by *nilpotent orbits*.

Goals

1. Propose a definition of unipotent representations of a complex reductive group (geometric and case-free)
2. Describe key properties of unipotent representations (unitarity, restriction to K, maximality of annihilators, etc.)
3. Define an enhancement of Barbasch-Vogan duality
4. Give a classification of unipotent representations
5. Speculate about applications to real reductive groups?

Representations of Finite Groups of Lie Type

Let

$$\begin{aligned}
\mathbb{F}_q &= \text{finite field with } q \text{ elements} \\
\mathbf{G} &= \text{connected reductive algebraic group} \\
&\quad\ \text{defined and split over } \mathbb{F}_q \\
\mathbf{G}(\mathbb{F}_q) &= \mathbb{F}_q\text{-points of } \mathbf{G} \\
\mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q)) &= \text{irreducible finite-dim reps of } \mathbf{G}(\mathbb{F}_q)
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Problem

Determine $\mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q))$ for arbitrary $\mathbf{G}$.

Problem solved completely by Lusztig in early 1980s.

Construction of representations:

- Choose a Fr-stable maximal torus $\mathbf{T} \subset \mathbf{G}$ and a character $\theta : \mathbf{T}(\mathbb{F}_q) \to \mathbb{C}^\times$.
- Deligne-Lusztig define a virtual representation $R_{\mathbf{T}}(\theta)$ of $\mathbf{G}(\mathbb{F}_q)$ (obtained as the ℓ-adic cohomology of associated 'Deligne-Lusztig' variety)

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Theorem (Deligne-Lusztig, 1976)

Every irreducible representation of $\mathbf{G}(\mathbb{F}_q)$ appears in some $R_{\mathbf{T}}(\theta)$. Most $R_{\mathbf{T}}(\theta)$ are irreducible.

Unipotent Representations of Finite Groups of Lie Type

Definition (Deligne-Lusztig, 1976)

A unipotent representation of $\mathbf{G}(\mathbb{F}_q)$ is an irreducible representation appearing in some $R_{\mathbf{T}}(1)$. Write

$$\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q)) \subset \mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q))$$

for the set of unipotent representations.

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Theorem (Lusztig, 1984)

The classification of $\mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q))$ reduces to the classification of $\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q))$.

Classification of Unipotent Representations of Finite Groups of Lie Type

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Theorem (Lusztig, 1984)

There is a natural bijection between $\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q))$ and the set of triples

$$(\mathcal{O}^\vee = \text{special nilp},\ C = \text{conj class in } \overline{A},\ \xi = \text{irrep of } \overline{A}^C)$$

In particular, the classification is *independent of q*.

Recap

There is a finite set of irreducibles

$$\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q)) \subset \mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q))$$

such that

1. $\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q))$ is classified by certain geometric data related to nilpotent orbits, and
2. The classification of $\mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q))$ reduces to the classification of $\mathrm{Unip}(\mathbf{G}(\mathbb{F}_q))$ (analagous to the Jordan decomposition of matrices).

Replace

$$\begin{aligned} \mathbb{F}_q &\rightsquigarrow k \in \{\mathbb{R}, \mathbb{C}\} \\ \mathbf{G}(\mathbb{F}_q) &\rightsquigarrow \mathbf{G}(k) \\ \mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q)) &\rightsquigarrow \mathrm{Irr}_u(\mathbf{G}(k)) \end{aligned}$$

Unipotent Representations of Real and Complex Groups

Replace

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$$\mathrm{Irr}_{fd}(\mathbf{G}(\mathbb{F}_q)) \rightsquigarrow \mathrm{Irr}_u(\mathbf{G}(k))$$

Problem (Gelfand, 1930s)

Determine $\mathrm{Irr}_u(\mathbf{G}(k))$ for arbitrary $\mathbf{G}$.

- Problem remains *unsolved* in general.
- Answer known in special cases: connected compact groups (Weyl, 1920s), $SL_2(\mathbb{R})$ (Bargmann, 1947), $GL_n(k)$ (Vogan, 1986), complex classical groups (Barbasch, 1989), some low-rank groups...

One possible strategy for solving this problem is to try to *imititate* the approach for $\mathbf{G}(\mathbb{F}_q)$:

Conjecture (Vogan, 1987)

There is a finite subset $\mathrm{Unip}(\mathbf{G}(k)) \subset \mathrm{Irr}(\mathbf{G}(k))$ which completes the following analogy

$$\mathrm{Unip}(\mathbf{G}(k)) \quad \text{is to} \quad \mathrm{Irr}_u(\mathbf{G}(k))$$
$$\text{as}$$
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What does this mean?

Unipotent Representations of Real and Complex Groups

In light of Lusztig (1984), one expects:

1. $\mathrm{Unip}(\mathbf{G}(k))$ is classified by certain geometric objects related to nilpotent orbits, and
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5. Infinitesimal characters are 'as small as possible' in their translation families.

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$$\mathrm{Ann}_L(M) \subset U(\mathfrak{g}) \qquad \mathrm{Ann}_R(M) \subset U(\mathfrak{g})$$

  and associated variety

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$$V(M) = V(\mathrm{Ann}_L(M)) = V(\mathrm{Ann}_R(M)) \subset \mathcal{N}$$

- If M is irreducible, then $\mathrm{Ann}_L(M), \mathrm{Ann}_R(M)$ are primitive, and $V(M) = \overline{\mathcal{O}}$.

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- Dual orbit $\mathcal{O}^\vee$ determines infl char for $\mathfrak{g}$:

$$\mathcal{O}^\vee \mapsto (e^\vee, f^\vee, h^\vee) \mapsto \frac{1}{2} h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*$$

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- And hence a unique maximal ideal $J_{\frac{1}{2}h^\vee} \subset U(\mathfrak{g})$.
- $V(J_{\frac{1}{2}\mathcal{O}^\vee}) = \overline{d(\mathcal{O}^\vee)}$.

Special Unipotent Representations

Definition (Barbasch-Vogan, 1985)

The *special unipotent ideal* attached to $\mathcal{O}^\vee$ is the maximal ideal $J_{\frac{1}{2}h^\vee} \subset U(\mathfrak{g})$. A *special unipotent representation* attached to $\mathcal{O}^\vee$ is an irreducible HC bimodule M such that

$$\mathrm{Ann}_L(M) = \mathrm{Ann}_R(M) = J_{\frac{1}{2}h^\vee}$$

Write

$$\mathrm{Unip}^s_{\mathcal{O}^\vee}(G) \subset \mathrm{Irr}(G)$$

for the set of special unipotent representations attached to $\mathcal{O}^\vee$ and

$$\mathrm{Unip}^s(G) := \bigsqcup_{\mathcal{O}^\vee} \mathrm{Unip}^s_{\mathcal{O}^\vee}(G)$$

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- Annihilators are maximal.
- Unitary (Barbasch, Barbasch-Ciobotaru).
- Classified by geometric objects related to nilpotent orbits:

Theorem (Barbasch-Vogan, 1985)

There is a natural bijection

$$\mathrm{Unip}^{\mathrm{s}}_{\mathcal{O}^\vee}(G) \simeq \overline{A}(\mathcal{O}^\vee)^\wedge$$

Respects tensor products.

Special Unipotent Representations

Problem

Let $G = Sp(2n)$ and let $\mathcal{O} \subset \mathcal{N}$ be the minimal nilpotent orbit.

- Joseph ideal $J \subset U(\mathfrak{g})$. Maximal, completely prime, $V(J) = \overline{\mathcal{O}}$.
- Oscillator representations $M^{\pm}$. Unitary, irreducible, $\mathrm{Ann}_L(M^{\pm}) = \mathrm{Ann}_R(M^{\pm}) = J$.
- Since $\mathcal{O}$ is *rigid*, $M^{\pm}$ cannot be induced.
- Since $\mathcal{O}$ is not special, $M^{\pm} \notin \mathrm{Unip}^s(G)$.

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I will propose a natural definition of $\mathrm{Unip}(G)$ which generalizes BV.

Nilpotent Covers

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- Choose $e \in \mathcal{O}$ and $x \in \widetilde{\mathcal{O}}$ over e. Then

$$\pi_1^G(\mathcal{O}) \simeq G_e/G_e^o$$

and

$$\pi_1^G(\widetilde{\mathcal{O}}) \simeq G_x/G_x^o \subseteq G_e/G_e^o \simeq \pi_1^G(\mathcal{O})$$

This defines a Galois correspondence between nilpotent covers of $\mathcal{O}$ (up to isomorphism) and subgroups of $\pi_1^G(\mathcal{O})$ (up to conjugacy).

Quantizations of Nilpotent Covers

- There is a G-eqvt symplectic form $\omega \in \Omega^2(\mathcal{O})$ (Kostant), inducing a G-eqvt symplectic form $p^*\omega \in \Omega^2(\widetilde{\mathcal{O}})$.

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- There is a G-eqvt symplectic form $\omega \in \Omega^2(\mathcal{O})$ (Kostant), inducing a G-eqvt symplectic form $p^*\omega \in \Omega^2(\widetilde{\mathcal{O}})$.
- Symplectic form induces graded G eqvt Poisson bracket on $\mathbb{C}[\widetilde{\mathcal{O}}]$

Definition

A *quantization* of $\widetilde{\mathcal{O}}$ is a pair $(\mathcal{A}, \theta)$ consisting of a filtered algebra $\mathcal{A}$ such that

$$[\mathcal{A}_{\leq m}, \mathcal{A}_{\leq n}] \subseteq \mathcal{A}_{\leq m+n-1}$$

and an isomorphism of graded Poisson algebras

$$\theta : \mathrm{gr}(\mathcal{A}) \simeq \mathbb{C}[\widetilde{\mathcal{O}}]$$

Quantizations of Nilpotent Covers

Every $\widetilde{\mathbb{O}}$ defines

1. A Levi subgroup $L \subset G$.
2. A finite group W acting on $\mathfrak{z}(\mathfrak{l})$ by reflections.

Theorem (Loseu, Loseu-MB-Matvieievskyi)

There is a canonical bijection

$$\{\text{quantizations of } \widetilde{\mathbb{O}}\} \simeq \mathfrak{z}(\mathfrak{l})/W$$

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Quantizations of Nilpotent Covers

Example

Let $\mathcal{O} \subset \mathcal{N}$ be the principal nilpotent orbit. Then

$$L = T \qquad W = W(G)$$

Conclusion:

$$\mathfrak{t}^*/W(G) \simeq \{\text{quantizations of } \mathcal{N}\}$$

In this case, bijection is very easy to describe.

$$\lambda \longmapsto U(\mathfrak{g})/\langle \ker \chi_\lambda \rangle$$

Canonical Quantizations

Observation

If $\widetilde{\mathcal{O}}$ is an eqvt nilpotent cover, there is a distinguished quantization corresponding to $0 \in \mathfrak{z}(\mathfrak{l})/W$. We call this quantization the *canonical quantization* and denote it by $\mathcal{A}_0$.

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- $\mathcal{A}_0$ has a large automorphism group (all Poisson automorphisms of $\mathbb{C}[\widetilde{\mathbb{O}}]$ lift to filtered automorphisms of $\mathcal{A}_0$).
- $\mathcal{A}_0$ is G-eqvt (i.e. G-action on $\mathbb{C}[\widetilde{\mathbb{O}}]$ lifts to $\mathcal{A}_0$).

Unipotent Ideals

The G-action on $\mathcal{A}_0$ is *Hamiltonian*, i.e. there is a *(quantum) co-moment map*

$$\Phi : U(\mathfrak{g}) \to \mathcal{A}_0$$

lifting the *(classical) co-moment map*

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Definition (Loseu-MB-Matvieievskyi)

The *unipotent ideal* attached to $\widetilde{\mathcal{O}}$ is the primitive ideal

$$J(\widetilde{\mathcal{O}}) := \ker\left(\Phi : U(\mathfrak{g}) \to \mathcal{A}_0\right) \subset U(\mathfrak{g})$$

Unipotent Ideals

Two important examples:

Example

Let $\mathcal{O} = \{0\}$. Then $\mathcal{A}_0 = \mathbb{C}$, $\Phi : U(\mathfrak{g}) \to \mathbb{C}$ is the augmentation map, and $J(\mathcal{O})$ is the augmentation ideal. In particular $\lambda(\mathcal{O}) = \rho$.

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Example

Let $\mathcal{O} = \mathcal{O}_{\text{prin}}$. Then $\mathcal{A}_0 = U(\mathfrak{g})/\langle \ker \gamma_0 \rangle$, $\Phi : U(\mathfrak{g}) \to \mathcal{A}_0$ is the quotient map, and $J(\mathcal{O}) = \langle \ker \gamma_0 \rangle$. In particular, $\lambda(\mathcal{O}) = 0$.

Unipotent Ideals

Example

Let $G = SL(2)$ and let $\widetilde{\mathcal{O}} \to \mathcal{O}_{\mathrm{prin}}$ be the 2-fold G-eqvt cover.

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \hookrightarrow & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathcal{O} & \hookrightarrow & \mathcal{N} \end{array}$$

The Weyl algebra $\mathbb{C}[x, \partial x]$ is the *unique* quantization of $\mathbb{C}^2$. Has a $\mathbb{Z}_2$ action (by negation). There is a surjection $\Phi : U(\mathfrak{g}) \twoheadrightarrow \mathbb{C}[x, \partial x]^{\mathbb{Z}_2}$

$$e \mapsto \frac{1}{2}x^2 \quad f \mapsto -\frac{1}{2}\partial x^2 \quad h \mapsto x\partial x + \frac{1}{2}$$

with kernel $\Omega + \frac{3}{4}$. Recall $\gamma_\lambda(\Omega) = \lambda^2 - 2\lambda$. Hence, $\lambda(\widetilde{\mathcal{O}}) = \frac{1}{2}$.

Properties of Unipotent Ideals

Theorem (Loseu-MB-Matvieievskyi)

The following are true:

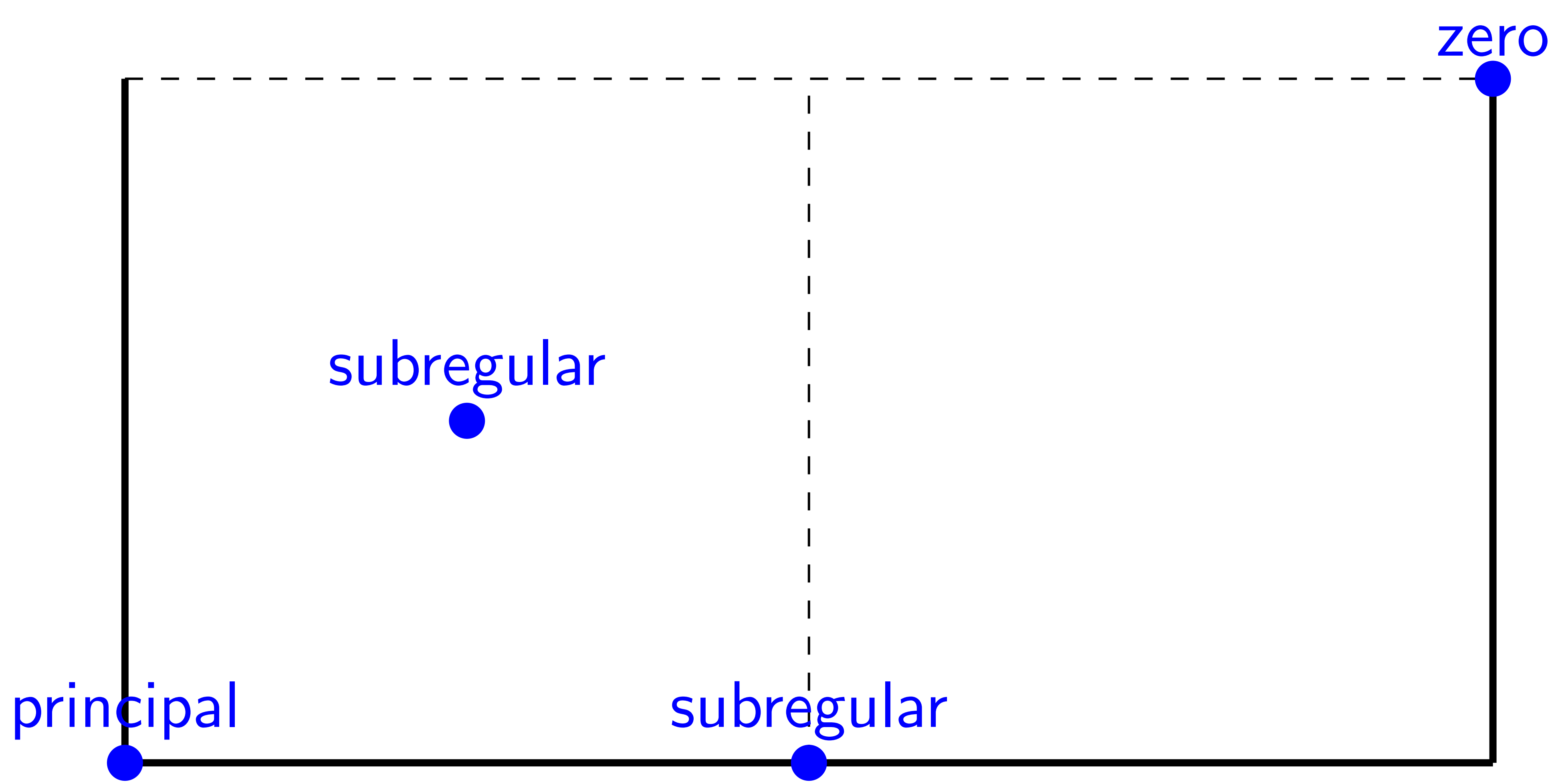
(i) $V(J(\widetilde{\mathcal{O}})) = \overline{\mathcal{O}}$

(ii) $J(\widetilde{\mathcal{O}})$ is completely prime

(iii) $J(\widetilde{\mathcal{O}})$ is maximal.

(iv) $m_{\overline{\mathcal{O}}}(J(\widetilde{\mathcal{O}})) = 1$ if and only if $\widetilde{\mathcal{O}} \to \mathcal{O}$ is Galois.

Unipotent Ideals for $G = Sp(4)$



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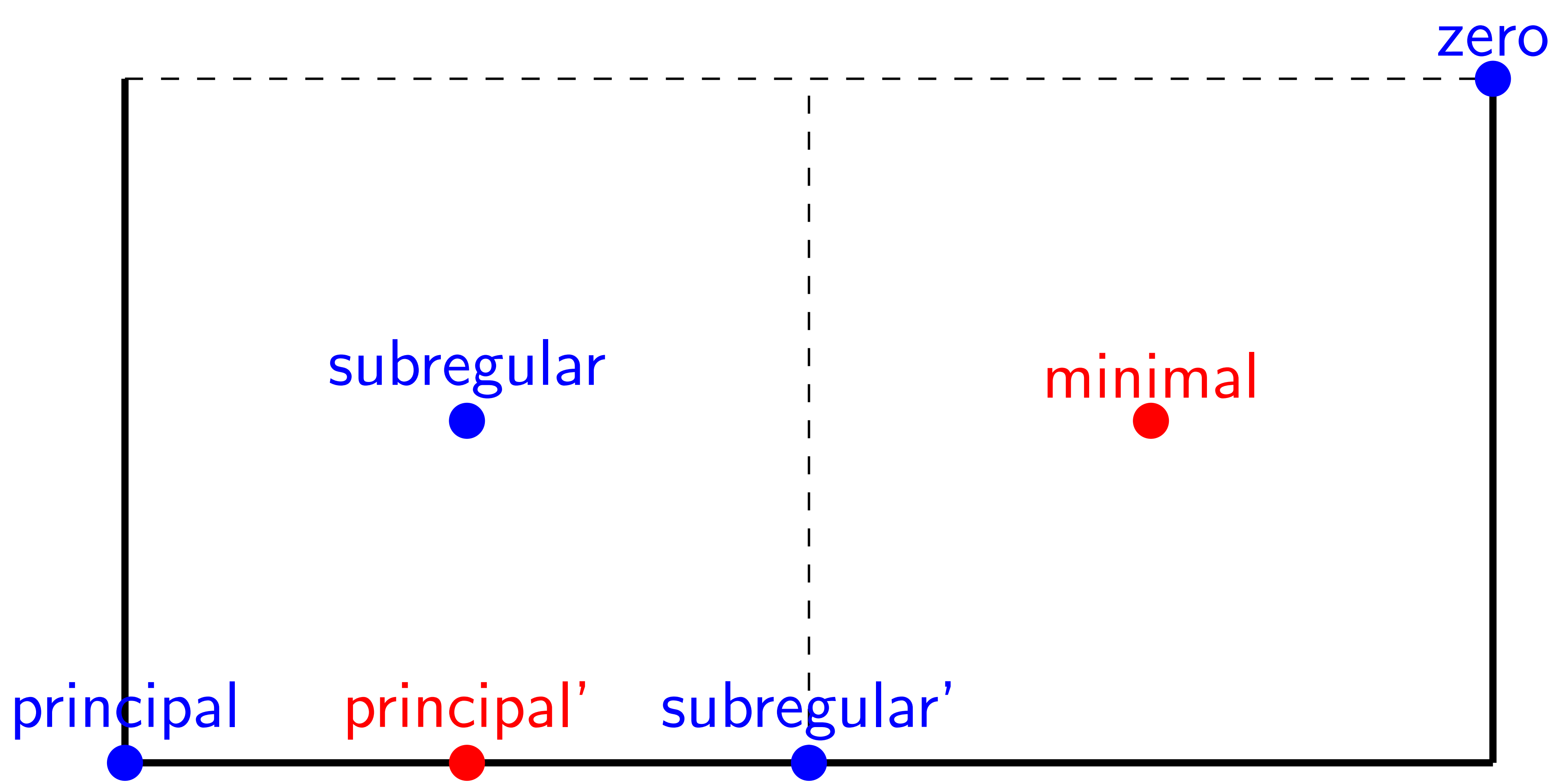

Unipotent Ideals for $G = Sp(8)$

The unipotent ideals $J(\widetilde{\mathcal{O}})$ are the maximal ideals with the following infl chars:

$\tilde{\mathcal{O}}$	$\lambda(\tilde{\mathcal{O}})$	$\tilde{\mathcal{O}}$	$\lambda(\tilde{\mathcal{O}})$	$\tilde{\mathcal{O}}$	$\lambda(\tilde{\mathcal{O}})$
(8)	$(0,0,0,0)$	$(42^2)_2$	$(1,1,\frac{1}{2},0)$	$(3^22)_2$	$(\frac{3}{2},1,\frac{1}{2},\frac{1}{2})$
$(8)_2$	$(\frac{1}{2},0,0,0)$	$(42^2)_2$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	(3^21^2)	$(2,1,\frac{1}{2},\frac{1}{2})$
(62)	$(\frac{1}{2},\frac{1}{2},0,0)$	$(42^2)_2$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},0)$	(2^4)	$(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_2$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$	$(42^2)_4$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(2^4)_2$	$(2,1,1,0)$
$(62)_2$	$(1,0,0,0)$	(421^2)	$(2,1,0,0)$	(2^31^2)	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_2$	$(1,\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,0,0)$	$(2^31^2)_2$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_4$	$(1,\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,0,0)$	(2^21^4)	$(3,2,1,0)$
(61^2)	$(\frac{3}{2},\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,\frac{1}{2},0)$	$(2^21^4)_2$	$(3,2,1,0)$
$(61^2)_2$	$(\frac{3}{2},\frac{1}{2},0,0)$	$(421^2)_4$	$(2,1,\frac{1}{2},0)$	(21^6)	$(\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$
(4^2)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	(41^4)	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},0)$	$(21^6)_2$	$(\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$
$(4^2)_2$	$(1,\frac{1}{2},\frac{1}{2},0)$	$(41^4)_2$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},0)$	(1^8)	$(4,3,2,1)$
(42^2)	$(1,1,0,0)$	(3^22)	$(1,1,1,0)$		

Blue = special unipotent. Note: all such appear.

Classification of Unipotent Ideals

Definition (Loseu-MB-Matvieievskyi)

Suppose $\widetilde{\mathcal{O}}^1$ and $\widetilde{\mathcal{O}}^2$ are eqvt covers of $\mathcal{O}$. We say $\widetilde{\mathcal{O}}^1$ and $\widetilde{\mathcal{O}}^2$ are equivalent if the affine varieties $\mathrm{Spec}([\widetilde{\mathcal{O}}^1])$ and $\mathrm{Spec}([\widetilde{\mathcal{O}}^2])$ have the same codimension 2 singularities. Denote the equivalence classes by $[\widetilde{\mathcal{O}}]$.

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Theorem (Loseu-MB-Matvieievskyi)

The map $\widetilde{\mathcal{O}} \mapsto J(\widetilde{\mathcal{O}})$ defines a bijection

$$\{\text{equivalence class of covers}\} \simeq \{\text{unipotent ideals}\}$$

Classification of Unipotent Ideals

Example

Let $G = SL(2)$.

$\{0\}$	$\mathcal{O}$	$\widetilde{\mathcal{O}}$
J_1	J_0	$J_{\frac{1}{2}}$

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Example

Let $\mathcal{O}$ be the minimal orbit (for any G). Unless $\dim(\mathcal{O}) = 2$, $\mathrm{Spec}(\mathbb{C}[\mathcal{O}])$ has *no* codim 2 singularities. Hence, all covers of $\mathcal{O}$ are equivalent. Corresponding ideal is Joseph ideal.

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Want to show that all special unipotent ideals are unipotent.

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There is an injective map

$$\tilde{d} : \{\mathcal{O}^\vee\} \hookrightarrow \{[\widetilde{\mathcal{O}}]\}$$

with the following properties:

(i) $\tilde{d}(\mathcal{O}^\vee)$ covers $d(\mathcal{O}^\vee)$

(ii) $\frac{1}{2}h^\vee =$ infl char of $J(\tilde{d}(\mathcal{O}^\vee))$.

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Since unipotent ideals are maximal, this shows that special unipotent $\implies$ unipotent.

New Definition of Unipotent Representations

Definition (Loseu-Mason-Brown-Matvievskyi)

A *unipotent representation* attached to $\widetilde{\mathcal{O}}$ is an irreducible HC bimodule M such that

$$\mathrm{Ann}_L(M) = \mathrm{Ann}_R(M) = I_0(\widetilde{\mathcal{O}})$$

Write

$$\mathrm{Unip}_{\widetilde{\mathcal{O}}}(G) \subset \mathrm{Irr}(G)$$

for the set of unipotent representations attached to $\widetilde{\mathcal{O}}$ and

$$\mathrm{Unip}(G) := \bigsqcup_{\widetilde{\mathcal{O}}} \mathrm{Unip}_{\widetilde{\mathcal{O}}}(G)$$

Properties of Unipotent Representations

Theorem (Loseu-MB-Matvieievskyi)

If G is linear classical all representations in $\mathrm{Unip}(G)$ are unitary.

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If G is linear classical all representations in $\mathrm{Unip}(G)$ are unitary.

Let G and $\widetilde{\mathcal{O}}$ be arbitrary. We prove a conjecture of Vogan:

Theorem (Loseu-MB-Matvieievskyi)

Let $V \in \mathrm{Unip}_{\widetilde{\mathcal{O}}}(G)$. Then there is a distinguished good filtration on V giving rise to an isomorphism of G-representations

$$V \simeq_G \Gamma(\mathcal{O}, \mathrm{gr}(V))$$

Classification of Unipotent Representations

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$$\Gamma(\widetilde{\mathcal{O}}) := \mathrm{Gal}(\widetilde{\mathcal{O}}/\mathcal{O})$$

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Theorem (Loseu-MB-Matvieievskyi)

There is a natural bijection

$$\mathrm{Unip}_{\widetilde{\mathcal{O}}}(G) \simeq \Gamma(\widetilde{\mathcal{O}})^{\wedge}$$