Modular Forms Handout

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Structure & Goals

1. Show why one should care about modular forms from classical number theory.
2. Give a definition of modular forms and see a few examples (e.g., $G_{2k}(\tau)$).
3. Describe the structure of modular forms to obtain the dimension formula and a basis.
4. Count the ways a natural number can be written as a sum of 4 squares:
   (a) Show $\theta^4(\tau)$ is a modular form of weight 2 for a subgroup $\Gamma_0(4)$ of $\mathrm{SL}_2(\mathbb{Z})$.
   (b) Recall from Euler that the q-expansion of $\theta^k(\tau)$ has $r_k(n)$ as coefficients.
   (c) Use that the space of modular forms of weight 2 for $\Gamma_0(4)$ is two-dimensional with basis in terms of modified Eisenstein series to write $\theta^4(\tau)$ in another way.
   (d) Equate coefficients to get a formula for $r_4(n)$.
5. For those with background, present the modular curve perspective. List modern applications.

Notational Conventions

- The Greek letter τ will represent a member of the upper half plane $\mathcal{H} := \{z \in \mathbb{C} \colon \mathrm{Im} z \geq 0\}$.
- We always have $q \in \mathbb{C}$ with $|q| < 1$. In particular, $q = \exp(2\pi i \tau)$ where $\tau \in \mathcal{H}$.
- We denote by S and T the generators of $\mathrm{SL}_2(\mathbb{Z})$: $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- The number of ways to write $n \in \mathbb{N}$ as a sum of k squares is denoted $r_k(n)$.
- The space of k-modular forms for $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ is $M_k(\Gamma)$. The cusp forms are $S_k(\Gamma)$.

The Definition

A **modular form of weight** k **for** $\mathrm{SL}_2(\mathbb{Z})$ is a *holomorphic* function $f : \mathcal{H} \to \mathbb{C}$ such that $f(\tau) = f(\tau + 1)$ and $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$ for all $\tau \in \mathcal{H}$. Moreover, we require $\lim_{\mathrm{Im}\tau \to \infty} f(\tau) < \infty$.

A **cusp form** is a modular form with no constant term in the q-expansion. (See: Key Facts.) The **principal congruence subgroup of** $\mathrm{SL}_2(\mathbb{Z})$ **of level** N is

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a **congruence subgroup** if $\Gamma(N) \subset \Gamma$.

The definition of a modular form for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is similar.

Main Examples

- The **Eisenstein series** $G_k(\tau) := \displaystyle\sum_{\substack{(m,n) \neq (0,0) \\ (m,n) \in \mathbb{Z}^2}} (m\tau + n)^{-k}$ and the weighted versions $E_k := G_k / 2\zeta(k)$ are modular forms of weight k for all even $k \geq 4$.
- The **modular discriminant** $\Delta(\tau) := \dfrac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q \displaystyle\prod_{n=1}^{\infty} (1 - q^n)^{24}$ is in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
- The **theta function** $\theta(q) := \displaystyle\sum_{n \in \mathbb{Z}} q^{n^2} = \sum_{n=0}^{\infty} r_k(n) q^n$ is a modular form of "half-integral weight" for some congruence subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$.

Key Facts

1. Every modular form has a **Fourier expansion** $f(\tau) = g(q(\tau)) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n \left(e^{2\pi i \tau}\right)^n$.
2. The vector space $M_k(\Gamma)$ is finite-dimensional over $\mathbb{C}$.
   For the Γ we need, it has a **basis in Eisenstein series**.

Applications

1. Operator theory on modular forms.
2. Count the number of partitions of an integer, $p(n)$.
3. Provide natural functions on elliptic curves through lattice view.
4. The Modularity Theorem and its classical partial converse: For any rational newform $f \in S_2(\Gamma_0(N))$, there is an elliptic curve $E/\mathbb{Q}$ such that the L-function of f is the L-function of E.
5. Monstrous moonshine (Borcherds, et al.).
6. Sphere packing in 8D and 24D (Viazovska, et al.).
7. Physics: quantum and statistical mechanics, CFT, string theory.
8. Properties of ζ and irrationality proofs ($\zeta(2k)$, $\zeta(3)$, etc.).
9. "Explain why" $e^{\pi\sqrt{163}} \approx 262537412640768743.99999999999992$.
10. "LMFDB universe" of automorphic forms, Galois representations, L-functions, motives.

Further Reading

1. 2021 AWS: *A friendly introduction to the theory of modular forms* (Alex Barrios). *An introduction to modular groups* (Lori Watson).
2. *A Course in Arithmetic.* J.P. Serre.
3. *The 1-2-3 of Modular Forms.* Don Zagier, et al.
4. *Introduction to Modular Forms.* Keith Conrad. CTNT 2016.
5. *A First Course in Modular Forms.* Fred Diamond & Jerry Shurman.
6. *Elliptic Curves, Modular Forms, and Their L-Functions.* Álvaro Lozano-Robledo.
7. *Modular Forms: A Computational Approach.* William Stein.
8. *Tutorial on modular forms.* Sam Marks. Harvard, Summer 2020.
9. *Modular Forms.* Richard Borcherds. YouTube.
10. *The Geometry of SL(2,Z).* Kristaps Balodis ("K-Theory"). YouTube.
11. *A Beautiful Group, SL2(Z).* Roy Williams. https://roywilliams.github.io/play/js/sl2z/.

The slides for this talk will also be uploaded at https://www.math.umd.edu/ jheavner/.