Support Preserving Measure Algebras and Spectral Synthesis

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In this paper we consider various subspaces of first order distributions (and, in particular, pseudo-measures) as algebras with a support preserving multiplication; that is, if supp $S$, supp $T \subseteq E$ then supp $ST \subseteq E$; $E \subseteq \mathbb{R}/2\pi \mathbb{Z}$ will always be a perfect set with Lebesgue measure $m(E)=0$. The main result says essentially that if the pseudo-measures supported by $E$ form a Banach algebra then the pseudo-measures are not only bounded but are quite close to being measures.

In Sec. 1 we define our various notation, operations, and algebras, and consider first order distributions as finitely additive set functions. Then in Sec. 2 we see that $M(E)$, the space of Radon measures with support in $E$, is a Banach algebra, and we calculate its maximal ideal space and see that $M(E)$ is symmetric (our multiplication is obviously not convolution). Sec. 3 is devoted to describing associated algebras that seem interesting in themselves and which are used to pinpoint the pseudo-measures on $E$ in Sec. 4.

1. Background, Notation, and Definition of Spaces

$A(\Gamma)$ is the space of absolutely convergent Fourier series $\phi \sim \sum a_n e^{in\gamma}$ with norm $\|\phi\|_{A} = \sum |a_n|$; $A'(\Gamma)$, the space of pseudo-measures, is the dual of $A(\Gamma)$ with canonical norm $\|\|_{A'}$; and $A'(E) \equiv \{ T \in A'(\Gamma) : \text{supp } T \subseteq E \}$. We designate the total variation norm on $M(E)$ by $\|\|$ and it is clear that $M(E) \subseteq A'(E)$.

Notationally, we set $\mathcal{E} = \bigcup_0^\infty I_j$ where $I_j = (\lambda_j, \gamma_j)$, $e_j = \gamma_j - \lambda_j$; and we refer to [4; 5] for preliminaries in pseudo-measures and Fourier analysis. Using the Hausdorff-Young theorem it is easy to see that if $T \in A'(E)$, $\hat{T}(0) = 0$, then $T = f_\gamma'$, distributionally, where

$$ f_\gamma = \sum_1^\infty k_j \chi_{I_j} \quad \text{a.e.} \quad (1.1) $$

and $f_\gamma \in L^p(\Gamma)$ for each $p \geq 1$. As such we let $D_\alpha(E)$ be the space of first order distributions $T$ where $T = f_\gamma'$, $f_\gamma \in L^p(\Gamma)$ for all $p \geq 1$, and $f_\gamma$ is given by (1.1). Without loss of generality we assume that if $T \in A'(E)$ (resp., $M(E)$) then $\hat{T}(0) = 0$; hence, $M(E) \subseteq A'(E) \subseteq D_\alpha(E)$ and $M(E)$, $A'(E)$ remain Banach spaces. Now, given $S, T \in D_\alpha(E)$ with corresponding $f_T = \sum k_j \chi_{I_j}$, $f_s = \sum h_j \chi_{I_j}$, we define

$$ ST \equiv (f_s f_T)' $$
noting that
\[ f_S f_T = \sum_1 k_j h_j \chi_{I_j} \quad \text{a.e.} \]  
(1.2)
Thus \( U = f'_U \),
\[ f_U = \sum_1 \chi_{I_j} \quad \text{a.e.,} \]
is a multiplicative identity in \( D_\infty(E) \).

If \( A'_S(E) \) consists of the elements \( T \) in \( A'(E) \) for which \( \langle T, \phi \rangle = 0 \) if \( \phi = 0 \) on \( E \), \( \phi \in A(G) \), then \( E \) is a spectral synthesis set if \( A'(E) = A'_S(E) \), and \( E \) is a Helson set if \( M(E) = A'_S(E) \). \( E \) is a set without true pseudo-measure (or strong spectral resolution set) if \( M(E) = A'(E) \). It is not known if every Helson set is a spectral synthesis set and so it is important to characterize sets of strong spectral resolution.

Since \( m(E) = 0 \), \( E \) is totally disconnected and we let \( F \) be the family of compact open sets in the topological space \( E \). \( F \) is a basis for the topology on \( E \) and an algebra of sets; and any distribution \( T \) with support in \( E \) is a finitely additive set function on \( F \) where
\[ T(F) = \langle T, \psi_F \rangle, \]
\( F \in F \) and \( \psi_F \in C^\infty(G) \) with \( \psi_F = 1 \) on a neighborhood of \( F \) in \( G \) and \( \psi_F = 0 \) on a neighborhood of \( E - F \) in \( G \). As such, we define
\[ \| T \|_v = \sup_{F \in F} |T(F)|, \]
and \( A'(E) = M(E) \) if and only if \( \| T \|_v < \infty \) for each \( T \in A'(E) \) (e.g., [2; 3] for related issues). We let \( F \subseteq F \) be the elements in \( F \) such that real-valued \( \psi_F \) can be found with the further properties that \( 0 \leq \psi_F \leq 1 \) and \( 0 < \psi_F < 1 \) on only finitely many \( I_j \). Then

**Proposition 1.1.** \( F = \subseteq F \).

**Proof.** Let \( F \in F \) and take \( \psi \in C^\infty(G) \) such that \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) on a neighborhood of \( F \) in \( G \), and \( \psi = 0 \) on a neighborhood of \( E - F \) in \( G \).

Let \( I_j \) have the property that \( 0 < \psi < 1 \) for some points of \( I_j \); and adjust \( \psi \) on \( I_j \) so that \( \psi = 0 \) on an open interval of \( I_j \) but so that it retains all its other properties. Do this for each \( j \) and hence
\[ \psi = \sum \psi_{F_j}, \]
where \( 0 < \psi_{F_j} < 1 \) on only two \( I_k, \psi_{F_j} = 1 \) on a neighborhood of \( F_j \in F \) in \( G \), and \( \psi_{F_j} = 0 \) on a neighborhood of \( E - F_j \) in \( G \).

Thus \( \{ F_j \} \) is an open cover of \( F \) so that \( F \) compact implies we can cover \( F \) by \( F_{n_1}, \ldots, F_{n_k} \); consequently, set \( \psi_F = \sum_{1}^{k} \psi_{F_{n_i}} \). Q.E.D.

We say that \( I_n \leq I_m \) if \( \lambda_n < \gamma_m \) and if we consider \( E \subseteq [0, 2\pi] \); also \( I_n \leq \cdots \leq I_m \) is a partition \( P \) of \( E \).
Proposition 1.2. The following are equivalent for \( T \in D_\omega(E) \), \( f_T = \sum k_j \chi_{I_j} \):

(a) \( T \in M(E) \).

(b) \( f_T(\pm \gamma) \) is defined on all of \( \Gamma \) (by taking limits) and \( f_T \) is of bounded variation.

(c) There is \( M > 0 \) for which

\[
\sup_p \sum_{j=1}^{k-1} |k_{n_{j+1}} - k_{n_j}| < M. \tag{1.3}
\]

Proof. (b) is equivalent to (a) by the Riesz representation theorem, and (b) implies (c) since \( f_T \) is of bounded variation. Assume (c) and let \( f_T \) be real-valued.

Set

\[
(Vf_T)(\gamma) = \sup_{\lambda} \left\{ \sum_{j=1}^{k-1} |k_{n_{j+1}} - k_{n_j}| : \lambda_{n_{j+1}} < \gamma, \lambda_{n_j} \in I_j \text{ for some } j \right\}.
\]

For \( \gamma \in \bigcup_0 I_j \) define

\[
f_1(\gamma) = \frac{1}{2}((Vf_T)(\gamma) + f_T(\gamma)),
\]

\[
f_2(\gamma) = \frac{1}{2}((Vf_T)(\gamma) - f_T(\gamma)).
\]

Clearly \( f_T = f_1 + f_2 \) and in the usual way we have that \( f_1 \) and \( f_2 \) are increasing functions on \( \bigcup_0 I_j \) considered as a subset of \([0, 2\pi]\).

Finally, for any \( \gamma \notin \bigcup_0 I_j \) set \( f_1(\gamma -) = \sup \{ f_1(\lambda) : \lambda \in \bigcup_0 I_j, \lambda < \gamma \} \), \( f_1(\gamma +) = \inf \{ f_1(\lambda) : \lambda \in \bigcup_0 I_j, \lambda > \gamma \} \), and similarly for \( f_2 \); because we are dealing with monotone functions these inf and sup exist and (b) follows. Q.E.D.

We set \( D_b(E) \) to be the space of those elements \( T \) in \( D_\omega(E) \) for which \( f_T \in L^\infty(\Gamma) \). Motivated by Proposition 1.2 and the properties of bounded variation functions we define the space \( \mathcal{B}(E) \) of \textit{generalized measures} to be those elements \( T \) of \( D_b(E) \) such that the corresponding \( f_T \) has the properties that \( f_T(\gamma \pm) \) exist for all \( \gamma \in \Gamma \) and \( f_T \) has at most countably many jump discontinuities. Also let \( A'_b(E) = A'(E) \cap D_b(E) \); this is the space of \textit{bounded pseudo-measures}.

Note that the mapping \( T \mapsto f_T \) for all our subspaces of \( D_\omega(E) \) is bijective.

2. The Support Preserving Banach Algebra \( M(E) \)

For each \( T \in M(E) \) define

\[
\| T \|_{1, \infty} = \| T \|_1 + \| f_T \|_\infty \tag{2.1}
\]

where \( T = f_T \), \( f_T = \sum k_j \chi_{I_j} \), and \( \| f_T \|_\infty = \sup_j |k_j| \). Clearly

\[
\| T \|_1 \leq \| T \|_{1, \infty} \leq 2 \| T \|_1.
\]

Generally, when dealing with Banach spaces which have a separately continuous multiplication and multiplicative unit \( U \), we employ the usual trick and identify

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the space with an algebra of operators that has a norm \( \| \cdot \| \) which satisfies
\[ \| ST \| \leq \| S \| \| T \| \text{ and } \| U \| = 1. \]

**Proposition 2.1.** \( M(E) \), with multiplication defined by (1.2) and with norm
given by (2.1), is a commutative Banach algebra with identity \( U \).

**Proof.** Given \( S, T \in M(E) \) with \( f_S = \sum \frac{k_j}{1} \chi_{I_j}, f_T = \sum \frac{h_j}{1} \chi_{I_j} \).
Letting \( I_{n_1} \leq \cdots \leq I_{n_m} \), we have
\[
\sum_{i=1}^{m-1} |k_{n+i, j, n+j} - k_{n, j, n}| \leq \sum_{i=1}^{m-1} \left| \left( k_{n+i, j, n} - k_{n, j, n} \right) \right| + \sum_{i=1}^{m-1} \left| h_{n+i, j, n} - h_{n, j, n} \right|.
\]
Thus
\[ \| ST \|_1 \leq \| f_T \|_{\infty} \| S \|_1 + \| f_S \|_{\infty} \| T \|_1 \leq \| T \|_{1, \infty} \| S \|_{1, \infty}. \quad \text{Q.E.D.} \]
We designate the Gelfand transform of \( T \in M(E) \) by \( T \).

**Proposition 2.2.** Let \( T \in D_\alpha(E), f_T = \sum \frac{k_j}{1} \chi_{I_j} \). The following are equivalent:
(a) \( T \in M(E) \).
(b) \( \| T \|_\alpha < \infty \).
(c) There is \( M > 0 \) such that if \( I_{n_1} \leq \cdots \leq I_{n_m} \) then
\[
\left| \sum_{j=1}^{m} (k_{n+j-1, j, n+j} - k_{n, j, n}) \right| < M. 
\]

**Proof.** The equivalence of (a) and (b) is given in [3] and the sum in (c) is
\[ \langle T, \psi \rangle \] for some \( \psi \). \quad \text{Q.E.D.} \]

We state Proposition 2.2 to observe the equivalence of (2.2) and (1.3).

Let's now describe the obvious elements of \( \mathcal{M}(M(E)) \), the maximal ideal space of \( M(E) \):
\[
X \equiv \{ F_n \in \mathcal{M}(M(E)) : F_n(T) \equiv k_n, f_T = \sum_1^m k_j \chi_{I_j}, n \geq 1 \}
\]
\[
X^+ \equiv \{ F_\gamma \in \mathcal{M}(M(E)) : F_\gamma(T) \equiv f_T(\gamma +), \gamma \in E \text{ inaccessible} \}
\]
\[
X^- \equiv \{ F_\gamma \in \mathcal{M}(M(E)) : F_\gamma(T) \equiv f_T(\gamma -), \gamma \in E \text{ inaccessible} \}
\]
\[
X^\lambda \equiv \{ F_\lambda \in \mathcal{M}(M(E)) : F_\lambda(T) \equiv f_T(\lambda +), \text{ some } n \}
\]
\[
X^\lambda \equiv \{ F_\lambda \in \mathcal{M}(M(E)) : F_\lambda(T) \equiv f_T(\lambda -), \text{ some } n \}
\]
Thus for \( X \equiv X_a \cup X^+_a \cup X^- \cup X^a \cup X_a \) we have \( X \subseteq \mathcal{M}(M(E)) \).

**Proposition 2.3.** (a) The elements of \( X \) are identified with monotone convergent sequences \( \{ \lambda_{m_n} \} \), \( \lambda_{m_n} \) accessible in \( E \).
(b) \( F \in X \) if and only if there is a subsequence \( \{ I_{m_n} \} \) such that for all \( T \in M(E) \),
\[
f_T = \sum_1^m k_j \chi_{I_j}, \quad F(T) = \lim_n k_{m_n}. \quad \text{ (2.3)}
\]

**Proof.** (a) Given \( \{ \lambda_{m_n} \} \) a monotone (decreasing, say) convergent sequence in \([0, 2\pi]\) and let \( \lambda_{m_n} \to \gamma; \gamma \in E \) since \( E \) is closed.
For $T \in M(E)$, $f_T = \sum_{I_j} k_j \chi_{I_j}$, $\lim k_{m_n}$ exists.

We set $F_\gamma(T) = \lim_{n} k_{m_n}$. $F_\gamma$ is clearly a homomorphism, and, by monotonicity, $F_\gamma(T) = f_T(\gamma +)$. Conversely, if $F_\gamma \in X$ then without loss of generality we have $F_\gamma(T) = f_T(\gamma +)$ for all $T \in M(E)$.

Since the accessible points are dense in $E$ we choose $\lambda_{m_n} \to \gamma$, and, since in this case we are dealing with right hand limit points, we take $\lambda_{m_n}$ monotone decreasing to $\gamma$.

(b) For $F_\gamma \in X$ we take a monotone sequence as in (a) and the corresponding intervals $\{I_{m_n}\}$.

For any $T \in M(E)$, since $F_\gamma(T) = f_T(\gamma +)$, say, and

$$\lim_{n} k_{m_n} = f_T(\gamma +), \quad f_T = \sum_{I_j} k_j \chi_{I_j},$$

we have (2.3).

Assume (2.3); that is, let $\{\lambda_{m_n}\}$ have the property that for all $T \in M(E)$,

$$f_T = \sum_{I_j} k_j \chi_{I_j}, \quad \lim_{n} k_{m_n}$$

exists—we designate this limit by $F(T)$.

It is easy to see that we can choose a monotone subsequence of $\{\lambda_{m_n}\}$, call it $\{\lambda_{m_n}\}$ again, and hence apply (a). Q.E.D.

Obviously in the correspondence of Proposition 2.3 there are many monotone subsequences for any $F \in X$. Also, in the second part of the argument of Proposition 2.3b the existence of $\lim k_{m_n}$ for all $T$ implies the existence of a limit $\gamma$ of $\{\lambda_{m_n}\}$ with the property that $\lambda_{m_n} \geq \gamma$ (or $\gamma \geq \lambda_{m_n}$) for all but a finite number of the $\lambda_{m_n}$.

**Theorem 2.1.** (a) $X = M(M(E))$.

(b) $M(E)$ is a symmetric, semi-simple algebra with $\bar{X} = M(M(E))$.

**Proof.** (a) Clearly $X \subseteq M(M(E))$.

For $T \in M(E)$, $f_T = \sum_{I_j} k_j \chi_{I_j}$, we define $f_T = \sum \tilde{k}_j \chi_{I_j}$ and note that for any partition $I_{n_1} \leq \cdots \leq I_{n_m}$,

$$\sum_{j=1}^{m-1} |\tilde{k}_{n_{j+1}} - \tilde{k}_{n_j}| = \sum_{j=1}^{m-1} |k_{n_{j+1}} - k_{n_j}|;$$

thus $f_T = \bar{T} \in M(E)$.

Also, if $F_\gamma \in X$ we define

$$M_{F_\gamma} = \{T \in M(E): F_\gamma(T) = 0\};$$

since $F_\gamma(T) = f_T(\gamma +)$, say, it is easy to see that $M_{F_\gamma}$ is a maximal (and hence closed) ideal in $M(E)$.

Taking any proper ideal $I \subseteq M(E)$ we show $I \subseteq M_{F_\gamma}$ for some $F_\gamma \in X$, and this proves that $X$ consists of all maximal ideals.

If $I \not\subseteq M_{F_\gamma}$ for some $F_\gamma \in X$ then for all $F \in X$ there is $T_F \in I$ such that $F(T_F) \neq 0$; we get a contradiction to this assumption.
Define \( S_{T_F} \equiv g_{T_F}^* \) where

\[
g_{T_F} \equiv f_{T_F} \cdot f_{T_F},
\]

so that since \( I \) is an ideal we have \( g_{T_F} \in I \) and for all \( G \in X \) and all \( T_F \)

\[
G(S_{T_F}) \geq 0;
\]

(2.5) follows by definition of \( X \), from (2.4), and because \( G \in X \).

We now show that \( X \) is closed in \( \mathcal{M}(M(E)) \), where, of course, we have the weak * topology from \( M(E) \).

First let \( F \in X \) and say that for all \( T \in M(E) \), \( F(T) = f_T(\gamma^+) \), some \( \gamma \in E \). A subbasic neighborhood of \( F \) is

\[
N \equiv \{ H \in \mathcal{M}(M(E)) : |H(T) - F(T)| < \varepsilon \},
\]

and if \( f_T = \sum k_j \chi_{I_j} \) we have \( k_{n_j} \to F(T) \) where \( \lambda_{n_j} \) is monotone convergent (in \([0, 2\pi])\); thus if \( F_{n_j} \in X_a \) corresponds to \( I_{n_j} \) we have \( F_{n_j} \to F \).

Therefore \( \bar{X}_a = X \) where \( X \) has the induced weak * topology.

Thus \( \bar{X}_a = \bar{X} \) where \( \bar{X} \) is the weak * closure in \( \mathcal{M}(M(E)) \) of \( X \).

Hence, for \( F \in \bar{X} \) there is \( \{ F_{n_j} \} \subseteq X_a \) such that \( F_{n_j} \to F \) — that is, if \( f_T = \sum k_j \chi_{I_j}, \)

\( k_{n_j} \to F(T) \); consequently, we apply Proposition 2.3 and so \( F \in X \) and \( X \) is closed in \( \mathcal{M}(M(E)) \).

Because \( X \) is weak * closed in \( \mathcal{M}(M(E)) \) and \( \mathcal{M}(M(E)) \) is weak * compact we have \( X \) weak * compact in \( \mathcal{M}(M(E)) \).

Without loss of generality take \( F(T_F) = 1 \) and hence \( F(S_{T_F}) = 1 \).

Now, for all \( F \in X \) let \( N_F \subseteq \mathcal{M}(M(E)) \) be a weak * neighborhood of \( F \) such that \( |\tilde{S}_{T_F}| > \frac{1}{2} \) on \( N_F \); there is no problem about doing this since \( \tilde{S}_{T_F} \) is continuous and \( \tilde{S}_{T_F}(F) \equiv F(S_{T_F}) = 1 \).

Further, \( \tilde{S}_{T_F} \), \( \frac{1}{2} \) on \( X \cap N_F \), and since \( X \) is weak * compact, \( X \subseteq N_{F_1}, \ldots, N_{F_k}, \)

\( F_j \in X \), and

\[
\bar{S} \equiv \sum_{j=1}^{k} \tilde{S}_{T_{F_j}} > \frac{1}{2} \quad \text{on } X.
\]

Therefore, \( I \) an ideal implies \( S \in I \). Thus, if \( f_S = \sum h_j \chi_{I_j} \) there is \( f_{S^{-1}} \equiv \sum \frac{1}{h_j} \chi_{I_j} \) with \( S^{-1} \in M(E) \) because \( X_a \subseteq X \) and \( \bar{S} > \frac{1}{2} \) on \( X \).

Consequently, \( U = SS^{-1} \in I \) and hence \( I = M(E) \), a contradiction.

(b) We showed \( \bar{X}_a = X \) in (a) so that since \( X = \mathcal{M}(M(E)) \) we have \( \bar{X}_a = \mathcal{M}(M(E)) \).

For the symmetry, recall from (a) that if \( T \in M(E) \) then

\[
\bar{T} \equiv f_T \in M(E).
\]

Hence, \( \bar{T} = \tilde{T} \) on \( X_a \) which does it.

For the semi-simplicity let \( \tilde{T} \equiv 0 \) on \( X \), \( T \in M(E) \). Then if \( f_T = \sum k_j \chi_{I_j} \) we have each \( k_j = 0 \) since \( \tilde{T}(F_j) = 0 \) and \( \tilde{T}(F_j) = F_j(T) = k_j \). Q.E.D.
3. The Algebras $D_\omega(E)$ and $\mathcal{G}(E)$

For $D_\omega(E)$ we define the natural metric topology given by the countable family of norms

$$\|T\|_p \equiv \left( \frac{1}{2\pi} \int_0^{2\pi} |f_T(y)|^p \, d\gamma \right)^{1/p}, \quad T \in D_\omega(E), \quad p = 1, 2, \ldots.$$  

**Proposition 3.1.** $D_\omega(E)$ is a Fréchet space and a continuous topological algebra with unit.

**Proof.** Note that the metric space $D_\omega(E)$ is complete; in fact, if $\{T_n\} \subseteq D_\omega(E)$ is Cauchy we have $f_{T_n} \in L^p(\Gamma)$ such that $\|f_{T_n} - f_{T_m}\|_p \to 0$ for all $p \geq 1$. In particular $f_{T_n} \to f$ in measure and so there is a subsequence (call it $\{f_{T_{n_k}}\}$ again) which converges to $f$ a.e.

Thus if $\gamma, \lambda \in I$, and $f_{T_n}(\gamma), f_{T_n}(\lambda)$ converge to $f(\gamma), f(\lambda)$, respectively, we have $f(\gamma) = f(\lambda)$ since $f_{T_n}(\gamma) = f_{T_n}(\lambda)$. Thus $f_T \in D_\omega(E)$.

Now, for $S, T \in D_\omega(E)$ and $q \geq 2$ we note that $(f_S f_T)^q \in L^1(\Gamma)$.

In fact, if $s \geq 1$, $f_S^q, f_T^q \in L^1(\Gamma)$; and so if $\frac{1}{p} + \frac{1}{q} = 1$ we have $f_S^q \in L^p(\Gamma), f_T^q \in L^p(\Gamma)$ so that $(f_S f_T)^q \in L^1(\Gamma)$ by Hölder’s inequality.

Hence $ST \in D_\omega(E)$, and, again by Hölder, $(ST)\to ST$ is continuous. Q.E.D.

Notationally we set $\mathcal{M}(D_\omega(E)) = \{F \in (D_\omega(E))' : F \not\equiv 0, \quad F(ST) = F(S)F(T)\}$. Also if $M(E) \subseteq B \subseteq D_\omega(E)$ is any Banach algebra define $\mathcal{M}(B) = \{F \in B' : F \not\equiv 0, \quad F(ST) = F(S)F(T)\}$; and for each $T \in B$, $\hat{T}$ is the Gelfand transform of $T$. For example, $D_b(E)$, when normed by $\|T\|_\infty = \|f_T\|_\infty, T \in D_b(E)$, is a Banach algebra.

**Proposition 3.2.** (a) $\overline{M(E)} = D_\omega(E)$.

(b) $X_\omega = \mathcal{M}(D_\omega(E))$ and so $\mathcal{M}(D_\omega(E))$ is dense in $\mathcal{M}(M(E))$.

**Proof.** (a) Let $T \in D_\omega(E)$, $f_T = \sum_{j=1}^n k_j \chi_{I_j}$, and set $f_{T_n} = \sum_{j=1}^n k_j \chi_{I_j}$.

Letting $p \geq 1$

$$2\pi \|T - T_n\|_p = \left( \int_0^{2\pi} \left| \sum_{j=1}^n k_j \chi_{I_j}(\gamma) \right|^p \, d\gamma \right)^{\frac{1}{p}} = \sum_{j=1}^n |k_j|^p \varepsilon_j;$$

but

$$2\pi \|T\|_p^p = \sum_{j=1}^\infty |k_j|^p \varepsilon_j$$

and so

$$\lim_{n} \|T - T_n\|_p = 0.$$

(b) Let $F_n \in X_\omega$ and let $T_m \to 0$ in $D_\omega(E), T_m \in D_\omega(E)$. If $f_{T_m} = \sum_{j=1}^\infty k_{m,j} \chi_{I_j}$ and $p \geq 1$ we have $F_m(T_m) = k_{m,n}$ and

$$\frac{1}{2\pi} \varepsilon_n |k_{m,n}|^p \leq \frac{1}{2\pi} \sum_{j=1}^\infty |k_{m,j}|^p \varepsilon_j = \|T_m\|_p^p.$$

Thus, with $n$ and $p$ fixed, $k_{m,n} \to 0$ as $m \to \infty$ since $\|T_m\|_p \to 0$; consequently, $F_n \in \mathcal{M}(D_\omega(E)).$

Now if $F \in \mathcal{M}(D_\omega(E))$ let $T \in D_\omega(E)$ be such that $F(T) = 0$. 

Setting \( f_T \equiv \sum_{i} k_j \chi_{I_j} \) let \( f_{T_N} \equiv \sum_{i} k_j \chi_{I_j} \) have the property that \( F(T_N) \neq 0 \) by (a).

Therefore, if \( n > N \) and \( S \equiv \chi'_{I_n} \)

\[
F(S) F(T_N) = F(ST_N) = 0
\]

so that \( F(S) = 0 \).

By linearity of \( F \) there is \( 1 \leq n \leq N \) such that \( F(P) \neq 0 \), \( P = \chi'_{I_n} \); if \( R = \chi'_{I_k} \), \( k \neq n \) and \( 1 \leq k \leq N \), then

\[
F(R) F(P) = F(RP) = 0
\]

so that \( F(R) = 0 \).

Also \( F(P) = 1 \) since \( F(P) = F(PP) \).

Hence, by applying (a) again, we have \( F(T) = k_n \) and so \( F \equiv F_n \in X_a \). Q.E.D.

**Remark.** 1. Since \( D_\omega(E) \) is not locally \( m \)-convex, a fact which is clear by the properties of \( L^p \)-spaces, we expect [7, p. 355] that there is a non-invertible \( T \in D_\omega(E) \) such that for all \( F \in \mathcal{M}(D_\omega(E)) \), \( F(T) \neq 0 \); and this is obviously the case.

2. It is also easy to see that \( \mathcal{M}(D_\omega(E)) \) is not weak \( \ast \) compact; for if it were, \( X_a \) would be weak \( \ast \) compact in \( \mathcal{M}(M(E)) \), by the continuity of the natural injection (by Proposition 3.2a) of \( \mathcal{M}(D_\omega(E)) \) into \( \mathcal{M}(M(E)) \), and this contradicts Theorem 2.1b.

The following is easy to prove from the properties of \( D_\omega(E) \), and we refer to [1; 7] for general and related results.

**Proposition 3.3.** \( \mathcal{M}(D_\omega(E)) \) is the space of closed maximal ideals in \( D_\omega(E) \).

It is also clear (e.g., Theorem 4.1) that—

**Proposition 3.4.**

(a) \( \mathcal{B}(E) \) is a closed subalgebra of \( D_\omega(E) \).

(b) \( \mathcal{M}(\mathcal{B}(E)) = \mathcal{M}(M(E)) \).

(c) The space \( C(\mathcal{M}(\mathcal{B}(E))) \) of continuous functions on \( \mathcal{M}(\mathcal{B}(E)) \) is precisely \( \{\tilde{T} : T \in \mathcal{B}(E)\} \).

4. Subalgebras of \( \mathcal{B}(E) \)

**Theorem 4.1.** Let \( M(E) \subseteq B \subseteq D_\omega(E) \), \( B \) a Banach algebra with

\[
\mathcal{M}(B) \subseteq \mathcal{M}(M(E)).
\]  

(4.1)

Then

(a) \( \mathcal{M}(B) = \mathcal{M}(M(E)) \), as sets and topologically.

(b) \( B \subseteq \mathcal{B}(E) \).

Proof. Since \( M(E) = D_\omega(E) \) we have \( \bar{B} = D_\omega(E) \) and hence the canonical adjoint \( D_\omega(E) \to B' \) is injective; thus

\[
\mathcal{M}(D_\omega(E)) \subseteq \mathcal{M}(B).
\]

From Theorem 2.16, Proposition 3.26, and (4.1) we have

\[
\bar{\mathcal{M}(B)} = \mathcal{M}(M(E)).
\]  

(4.2)
It is easy to check that the natural injection $\mathcal{M}(B) \to \mathcal{M}(M(E))$ is continuous, where both domain and range have their respective weak $\ast$ topologies.

By this continuity and (4.2) we have $\mathcal{M}(B) = \mathcal{M}(M(E))$ as sets and (a) follows by properties of compact spaces.

Let $T \in B$; then there is $\{T_n\} \subseteq M(E)$ such that $\hat{T}_n \to \hat{T}$ in the sup norm topology of $C(\mathcal{M}(M(E)))$ since $M(E)$ is symmetric.

Because $X_a \subseteq \mathcal{M}(B)$ and $\hat{T}_n(F_j) = k_{n,j}$, for $f_{T_n} = \sum_{j=1}^{\infty} k_{n,j} \lambda_{I_j}$ and $f_j \in X_a$, we have $f_{T_n} \to f_T$ uniformly on $\bigcup I_j$.

Similarly, if $F_j \in X - X_a$ assume, without loss of generality, that $\bar{S}(F_j) = F_j(S) = f_S(\gamma +)$, $\gamma$ an inaccessible point of $E$ and $S \in M(E)$.

Let $\{\lambda_{m_j}\}$ be monotone decreasing (as a subset of $[0, 2\pi]$) and with the property that for all $S \in M(E)$, $f_S(\gamma +) = \lim_{j} h_{m_j}$ where $f_S = \sum_{j} h_j \chi_{I_j}$; we can do this from the results of Sec. 2.

By hypothesis, $\lim_{n} \hat{T}_n(F_j) = \hat{T}(F_j)$ exists, and we show that
\[
\lim_{j} k_{m_j} = \hat{T}(F_j), \quad f_T = \sum_{j} k_j \chi_{I_j}. \tag{4.3}
\]

Now given $\varepsilon > 0$, for any $j$,
\[
|\hat{T}(F_j) - k_{m_j}| \leq |\hat{T}(F_j) - \hat{T}_n(F_j)| + |\hat{T}_n(F_j) - k_{m_j}|,
\]
and
\[
|\hat{T}_n(F_j) - k_{m_j}| \leq |f_{T_n}(\gamma +) - k_{n,m_j}| + |k_{n,m_j} - k_{m_j}|.
\]

There is $N$ such that for all $n \geq N$ and for all $j$,
\[
|k_{n,m_j} - k_{m_j}| < \varepsilon/4 \quad \text{and} \quad |\hat{T}_n(F_j) - \hat{T}_n(F_j)| < \varepsilon/2.
\]

For this $N$ there is $J_N$ such that for all $j \geq J_N$, $|f_{T_n}(\gamma +) - k_{n,m_j}| < \varepsilon/4$.

Thus, for $\varepsilon > 0$ we've found $J \equiv J_N$ so that if $j \geq J$, $|\hat{T}(F_j) - k_{m_j}| < \varepsilon$ and hence (4.3) holds.

Finally, to show $B \subseteq \mathcal{G}(E)$ we must prove that $f_T(\gamma +) \neq f_T(\gamma -)$ for at most countably many $\gamma \in E$.

Given $k > 0$. There is $N > 0$ such that for all $n \geq N$ and for all $\gamma \in E$
\[
|f_T(\gamma \pm) - f_{T_n}(\gamma \pm)| < 1/4k.
\]

For any fixed $n \geq N$ there are at most countably many $\gamma$ for which
\[
|f_{T_n}(\gamma +) - f_{T_n}(\gamma -)| > 1/k;
\]
thus for any $\lambda$, not one of these $\gamma$,
\[
|f_T(\lambda+) - f_T(\lambda-)| \leq |f_T(\lambda+) - f_{T_n}(\lambda+) + f_{T_n}(\lambda-) - f_T(\lambda-) + f_T(\lambda-) - f_T(\lambda-)| \leq 1/k.
\]

Therefore for a given $k$ there are at most countably many $\gamma$ for which $|f_T(\gamma +) - f_T(\gamma -)| > 1/k$. Q.E.D.
Corollary 4.1.1. Let \( M(E) \subseteq B \subseteq D_\omega(E) \), \( B \) a Banach algebra, and assume \( M(E) \) is weakly dense (and hence norm dense) or dense in the spectral norm in \( B \). Then

(a) \( \mathcal{M}(B) = \mathcal{M}(M(E)) \), as sets and topologically.

(b) \( B \subseteq \mathcal{G}(E) \).

Proof. By the weak denseness or spectral denseness (4.1) holds, and we apply the theorem. Q.E.D.

Remark. If \( E \) is Helson and a spectral synthesis set then \( \mathcal{A}'(E) \) is the Banach algebra \( M(E) \). If \( E \) is not Helson but \( \mathcal{A}'(E) \) is a Banach algebra (containing \( M(E) \)) then \( \mathcal{A}'(E) \subseteq \mathcal{G}(E) \) if \( E \) satisfies either of the denseness conditions of Corollary 4.1.1, or, more generally, if (4.1) is satisfied.

Theorem 4.2. Let \( M(E) \subseteq B \subseteq D_\omega(E) \), \( B \) a Banach algebra, and assume that the identity homomorphism \( M(E) \rightarrow M(E) \) extends to a homomorphism \( j: B \rightarrow M(E) \). Then \( B \subseteq \mathcal{G}(E) \).

Proof. Since \( j \) is surjective \( j(B) \) is dense in \( M(E) \) with the spectral norm. Thus \( \mathcal{M}(M(E)) \rightarrow \mathcal{M}(B) \) homeomorphically.

For \( T \in B \), \( \hat{T} \) is the restriction of \( \hat{T} \) to \( \mathcal{M}(M(E)) \).

Since \( \hat{T} \in C(\mathcal{M}(M(E))) \) and \( M(E) \) is symmetric \( \hat{T} \) is the uniform limit of \( \hat{T}_n \), \( T_n \in M(E) \).

By the calculation at the end of Theorem 4.1, \( B \subseteq \mathcal{G}(E) \). Q.E.D.

Remark. Generally, when one wishes to show \( M(E) = Y \) for some subspace \( Y \) of \( \mathcal{A}'(E) \), it is natural to extend the identity map \( M(E) \rightarrow M(E) \) to a linear transformation \( j: Y \rightarrow M(E) \), \( Y \) a Banach space [2; 6]; this process has built into it that \( j \) is injective. In Theorem 4.2 we need more initial structure on the space (viz., \( B \) must be a Banach algebra not just a Banach space) and on the map (viz., \( j \) must be a homomorphism) but we do not make any requirements concerning injectiveness.

Bibliography


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