A Support Preserving Hahn-Banach Property to Determine Helson-$S$ Sets

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Abstract. We imbed the space of pseudo-measures $\mathcal{A}'(E)$ supported by a closed totally disconnected set $E \subseteq R/2\pi Z$ into a space of distributions on an “imbedding” group. The basic technique is to find a sequence of measures $\mu_m$ on $E$ (non discrete measures generally) associated with each $T \in \mathcal{A}'(E)$, so that, with an additional arithmetic condition, $\{\mu_m\}$ converges in a weaker than weak $*$ topology to a measure $\mu_0$ and $\mu = T$. Using this framework we prove that a Helson set is a set of spectral synthesis if and only if certain of our distributions have a support preserving extension. We also introduce a uniqueness criterion, and show that the extension condition and uniqueness condition imply that $\mathcal{A}'(E)$ is the space of measures supported by $E$.

$E \subseteq \hat{\Gamma} = R/2\pi Z$ is a closed totally disconnected set. We introduce a method to associate a sequence of measures to a given pseudo-measure supported by $E$ (§5). We employ this approach to derive necessary and sufficient conditions that Helson sets be sets of spectral synthesis ($S$) (§8); the first set of conditions is in terms of a Hahn-Banach extension property with “boundary” constraints (viz., $(e_T)$ of §5). We also introduce a (Cantor) uniqueness set type of criterion (§6) on $E$, which, when coupled with $(e_T)$, implies $E$ is $S$ (Prop. 6.1). This is interesting since we show that the Cantor set (a non-Helson set) satisfies the uniqueness condition, and are able to conclude that if a Helson set is to be an $S$ set, then a property of Helson sets must be used to verify $(e_T)$. Körner has recently constructed independent Helson sets which are neither $S$ sets nor uniqueness sets.

§6 contains the necessary notation and some remarks about the general problem to associate measures with a given pseudo-measure. §§1–4 are necessary for §§5–6, §8, and are used in §7 to prove an analytic result on absolutely convergent Fourier series. In §§1–3 we construct a large group $\Gamma$ in which to imbed $E$, and define an $LB$ space of functions on $\Gamma$ which plays a key role in the sequel. The technique of §6 centers about the notions of §5 and the Dieudonné-Grothendieck theorem.

0. Notation and Remarks

$A(\hat{\Gamma})$ is the space of absolutely convergent Fourier series on $\hat{\Gamma}$, normed by $\| \phi \|_A = \sum |a_k|$, where $\phi(\gamma) = \sum a_k e^{i k \gamma}$. We define

$$k(E) = \{ \phi \in A(\hat{\Gamma}) : \phi = 0 \text{ on } E \}$$
$$j(E) = \{ \phi \in k(E) : \text{supp } \phi \cap E = \emptyset \};$$
and set $A(E) = A(\hat{\Gamma})/k(E), A_j(E) = A(\hat{\Gamma})/j(E)$ with corresponding quotient norms, $\| \cdot \|_{A(E)}, \| \cdot \|_j$. If $\psi \in A(\hat{\Gamma})$ we write $\tilde{\psi}(\gamma) = \psi(\gamma) + j(E)$. Also $m$ is Haar measure on $\Gamma$.

$C(E)$ is the space of continuous functions on $E$ with dual $M(E)$, the space of bounded Radon measures supported by $E$; the total variation norm on $M(E)$ is denoted by $\| \cdot \|_1$. The dual of $A(E)$ (resp., $A_j(E)$) is $A'_s(E)$ (resp., $A'_j(E)$). $A'(E)$ is the space of pseudo-measures with support in $E$. $A'_0(E) = \{ T \in A'(E): \lim_{|n| \to \infty} \hat{T}(n) = 0 \}$ and $M_0(E) = M(E) \cap A'_0(E)$.

$E$ is an S-set if $A'(E) = A'_s(E)$, an Helson set if $A'_s(E) = M(E)$, and a set of strong spectral resolution if $A'(E) = M(E)$. $E$ is a Kronecker set if for every $\varepsilon > 0$ and for every $\phi \in C(E), |\phi| = 1$, there is $n \in \mathbb{Z}$ such that

$$\sup_{\gamma \in E} |\phi(\gamma) - e^{i n \gamma}| < \varepsilon.$$ 

$E$ is a Cantor U-set (resp., weak U-set) if $A'_0(E) = \{ 0 \}$ (resp., $M_0(E) = \{ 0 \}$).

Finally, let $\mathcal{F}$ be the family of all open (in $E$) and closed subsets of $E$, and set $\mathcal{G} = \bigcup I_j$ where $I_j = (\lambda_j, \gamma_j)$ is an open interval, $m(I_j) = \varepsilon_j$, and $E \subseteq [0, 2\pi)$. For each $F \in \mathcal{F}, \phi \in C^\infty(\hat{\Gamma})$ denotes a function equal to 1 on a neighborhood of $F$ and equal to 0 on a neighborhood of $E - F$.

**Remark 1.** Let $T \in A'(E)$. If $m(E) = 0$ it is relatively straightforward [2] to construct a sequence of measures $\mu_n$ with finite support such that $\mu_n \to T$ on $C^1(\hat{\Gamma})$. The finite support is a mixed blessing since one cannot hope to conclude weak * convergence of such a sequence except in the simplest cases. Naturally, for a given Helson set $E$, it is desirable to find a weak * convergent sequence of measures for $T$ if $E$ is to be $S$. This is the motive for our approach in §5.

2. Remark 1 leads to the general problem to study those weaker than weak * topologies which preserve some of the important properties of weak * convergence. This is the motivation for [3], §2, and Prop. 5.1b.

### 1. The Imbedding Group $\Gamma$

Let $G$ be the additive group of real-valued elements in $A(\hat{\Gamma})$, taken with the discrete topology; and let $\Gamma$ be its compact dual group. $\Gamma$ is connected since $\{ 0 \}$ is the only compact subgroup of $G$, and thus $\Gamma$ is also a divisible group.

For each $\gamma \in \hat{\Gamma}$ we identify the element $f_\gamma \in \Gamma$ by

$$\forall \phi \in G, \quad (f_\gamma, \phi) = e^{i \phi(\gamma)}. \quad (1.1)$$

Thus we have the injection

$$u: \hat{\Gamma} \to \Gamma$$

$$\gamma \mapsto f_\gamma,$$
and by (1.1) and the definition of the topology on \( \Gamma \), \( u \) is continuous. In particular \( uE \) is compact in \( \Gamma \).

We could take \( G \) to have the further property that each \( \phi \in G \) equal zero at a fixed \( \lambda_0 \neq E \). This has the advantage of yielding an easy proof of the fact that

\[
\{ f \sim r_j, \gamma_j \in \Gamma : (f, \phi) = \prod e^{ir_j \phi(\gamma_j)}, \sum |r_j| < \infty, \gamma_j \in \tilde{\Gamma} \}
\]

is dense in \( \Gamma \) (e.g. [3]).

Define \( \mathcal{F} \) to be the set of trigonometric polynomials \( \Psi \) on \( \Gamma \), i.e.

\[
\Psi(f) = \sum_{j=1}^{n} a_j(f, \phi_j), \quad (1.2)
\]

some \( \phi_j \in G \). Then \( A(\Gamma) \), the space of absolutely convergent Fourier series on \( \Gamma \), is the completion of \( \mathcal{F} \) where \( \Psi \) (of (1.2)) is normed by

\[
\| \Psi \|_{A(\Gamma)} = \sum_{j=1}^{n} |a_j|.
\]

Finally, define \( k(uE) \) and \( j(uE) \) analogous to the way we defined \( k(E) \) and \( j(E) \).

2. An LB Function Space on \( \Gamma \)

For each \( \Psi \in A(\Gamma) \) define

\[
G_{\Psi} = \{ \phi \in G : \hat{\Psi}(\phi) \neq 0 \},
\]

and for each integer \( k > 0 \) let

\[
\mathcal{A}_k = \{ \Psi \in A(\Gamma) : \forall \phi \in G_{\Psi}, \| \phi \|_{A} \leq k \}.
\]

Clearly,

**Proposition 2.1.** \( \mathcal{A}_k \) is a closed vector subspace of \( A(\Gamma) \).

Thus we define the LB space

\[
\mathcal{A} = \bigcup \mathcal{A}_k
\]

noting that \( \mathcal{A} \) is bornological, barrelled, and non-metrizable, and that the dual \( \mathcal{A}' \) is Fréchet. Obviously, \( \mathcal{A} = A(\Gamma) \) since \( \mathcal{F} \subseteq \mathcal{A} \), the imbedding \( \mathcal{A} \to A(\Gamma) \) is continuous since \( \mathcal{A} \) is bornological, and consequently we have the natural imbedding \( A'(\Gamma) \subseteq \mathcal{A}' \). It is easy to check that \( A'(\Gamma) \) is \( \beta (\mathcal{A}', \mathcal{A}) \) dense in \( \mathcal{A}' \).
3. Canonical Maps

From the definition of $\mathcal{A}_k$,

**Proposition 3.1.** Let $\Psi \in \mathcal{A}$. Then

$$u' \Psi(\lambda) = \sum \hat{\Psi}(\phi) e^{i\phi(\lambda)}$$

(3.1)

is an element of $A(\mathcal{F})$.

$u'$ is thus a well-defined linear map $\mathcal{A} \to A(\mathcal{F})$ and so we can define the associated canonical linear map

$$u'_j: \mathcal{A} \to A_j(E).$$

**Proposition 3.2.** $u'_j \mathcal{A} = A_j(E)$.

**Proof.** Take $\psi(\gamma) = \sum a_k e^{ik\gamma} A(\mathcal{F})$ and $\psi_n(\gamma) = \sum_{|k| \leq n} a_k e^{ik\gamma}$.

Let $N$ be a neighborhood of $E$ with $m(N) < 2\pi$ and choose $\phi_k \in G$ such that $\phi_k(\gamma) = k\gamma$ on $N$.

Define $\Psi_n(f) = \sum_{|k| \leq n} a_k(f, \phi_k) \in \mathcal{A}$ noting that $u' \Psi_n = \psi_n$ on $N$ and $u' \Psi_n \in A(\mathcal{F})$.

Since $u' \Psi_n - \psi_n \in j(E)$ we have $u'_j \Psi_n = \tilde{\psi}_n$.

Therefore

$$||\tilde{\psi} - u'_j \Psi_n||_j = ||\tilde{\psi} - \tilde{\psi}_n||_j \leq \sum_{|k| > n} |a_k|.$$  \hspace{1cm} \text{qed.}

Since $\mathcal{A}$ is bornological and by properties of bounded sets in $\mathcal{A}$,

**Proposition 3.3.** $u'$ and $u'_j$ are continuous.

**Example 3.1.** Clearly, there are $\Psi \in A(\mathcal{F}) - \mathcal{A}$ such that $u'_j \Psi \in A_j(E)$. Take $||\phi_k||_A \to \infty$, $a_k = (\alpha/k^2) \exp \{-||\phi_k||_A\}$, and

$$\Psi(f) = \sum_{1}^{\infty} a_k(f, \phi_k).$$

Thus $||\Psi||_{A(\mathcal{F})}$ can be made as small as we like by choice of $\alpha$, and $\Psi \notin \mathcal{A}$ by choice of $\{\phi_k\}$. Now

$$||\sum_{1} a_k e^{i\phi_k}||_A \leq |\alpha| \sum_{1} \frac{1}{k^2}$$

so that $u'_j \Psi \in A_j(E)$.

Using this example, the Baire category theorem yields

**Proposition 3.4.** a) $u'_j \mathcal{A}_k$ in nowhere dense in $A_j(E)$.

b) $A_j(E) = \bigcup \mathcal{A}_k$.

c) $\mathcal{A}$ and $\mathcal{B} \mathcal{A}$ are dense in $A_j(E)$.
We now define the canonical transpose
\[ u_j: A'(E) \rightarrow \mathcal{A}', \]
noting that \( u_j \) is a continuous 1-1 linear map since \( u_j \) is continuous and \( u_j^\prime \mathcal{A} = A_j(E) \). By definition, if \( T \in A'(E) \), \( t = u_j T \) is given by
\[ \forall \Psi \in \mathcal{A}, \quad \langle t, \Psi \rangle = \langle T, u_j^\prime \Psi \rangle. \]

It follows that

**Proposition 3.5.** Let \( T \in A'(E) \), \( u_j T = t \), and \( \Psi \in \mathcal{A} \). If \( \supp \Psi \cap u E = \emptyset \) then \( \langle t, \Psi \rangle = 0 \).

4. **Kronecker Sets in \( \Gamma \)**

Let \( F \in C(E) \), \( |F| = 1 \). Because \( F \) is uniformly continuous and \( E \) is totally disconnected there is \( \phi \in C(E) \), real-valued, such that
\[ \forall \gamma \in E, \quad F(\gamma) = e^{i\phi(\gamma)}. \]

Obviously, this statement can be strengthened considerably. The point for us is that by using the Stone-Weierstrass theorem we can show

**Proposition 4.1.** \( uE \) is Kronecker (and therefore independent).

Varopoulos [5] showed that Kronecker sets are \( S \)-sets in \( \Gamma \), and the proof is readily extended to arbitrary compact abelian groups in which the Kronecker set is 0-dimensional (e.g. [2, Chapter 2]).

Let us show that \( \dim uE = 0 \), i.e., that \( uE \) is totally disconnected. Take \( \lambda \in E \) and let \( \bigcap F_\lambda = \{ \lambda \} \), \( F_\lambda \in \mathcal{F} \), noting that each \( F_\lambda \) is open and compact. \( u \) continuous implies \( uF_\lambda \) compact, so that we will have the total disconnectedness of \( uE \) by Šura-Bura's theorem once we prove \( uF \) is open in \( uE(F \in \mathcal{F}) \). Let \( \phi \in G \) be equal to 1 on a neighborhood of \( F \in \mathcal{F} \) and equal to 0 on a neighborhood of \( E - F \). Take \( 0 < \varepsilon < |e^i - 1| \) and define the following open sets of \( uE \):
\[ \forall \lambda \in F, \quad V_\phi(\lambda, \varepsilon) \cap uE = \left\{ f_\gamma: |1 - e^{i(\phi(\lambda) - \phi(\gamma))}| < \varepsilon \right\}. \]

Clearly, if \( f_\gamma \in V_\phi(\lambda, \varepsilon) \cap uE \) then \( \gamma \in F \) for otherwise we get \( |e^i - 1| < \varepsilon \) by our choice of \( \phi \). Thus
\[ uF = \bigcup_{\lambda \in F} [V_\phi(\lambda, \varepsilon) \cap uE] \]
and so \( uF \) is open. Consequently, from Prop. 4.1, the fact that Kronecker sets are Helson, and the Varopoulos theorem generalized to \( \Gamma \) —

**Proposition 4.2.** \( A'(uE) = M(uE) \).

Actually, Saeki has extended Varopoulos' result by dropping the 0-dimensionality hypothesis. For our purposes of dealing with \( uE \) it is easier to proceed as we did than to invoke Saeki's technique.
5. Pseudo-Measures and Associated Measures

$T \in A'(E)$ is measure approximable (resp., synthesis measure approximable) if there is a sequence of subspaces $X_k \subseteq A_j(E)$ and measures $\mu_k \in M(E)$ such that

a) $\bigcup X_k = A_j(E)$ (resp., $\bigcup \bar{X}_k = A_j(E)$)
b) $\mu_k = T$ on $X_k$.

We shall observe (Prop. 5.3) that under a certain extension condition $(e_T)$, below) on $T \in A'(E)$, $T$ is measure approximable. Clearly, if $T \in A'(E)$ and $E$ is synthesis measure approximable then $T \in M(E)$ for $E$ Helson; however, in light of Prop. 3.4, this approach is not promising.

Given $t \in \mathcal{A}'$ we know that $t \in \mathcal{A}'_k$ for each $k$ and so there is $s_k \in A'(\Gamma)$ such that $s_k = t$ on $\mathcal{A}_k$. Thus, because of the annihilating properties of $t = u_j T$, $T \in A'(E)$ (e.g., Prop. 3.5, Example 5.2 below), the following condition is meaningful:

Given $t = u_j T$, $T \in A'(E)$, $\exists$ infinitely many $k$ for which $(e_T)$

$$\exists s_k \in A'(uE), \quad s_k = t \quad \text{on} \quad \mathcal{A}_k.$$ 

We can not expect $(e_T)$ to hold generally for all $T \in A'(E)$ (e.g. Example 6.2).

Example 5.1. Note that if $u_j T = t \in A'(\Gamma)$ then $T \in M(E)$. In fact, if there is $K > 0$ such that for all $\phi \in G$, $F \in \mathcal{F}$,

$$|\langle T, e^{i\phi_r}\rangle| = |\hat{t}(\phi_f)| < K$$

then $T \in M(E)$ (e.g. [3]). Further, from Prop. 5.2, $t \in M(uE)$.

Implicit in $(e_T)$ is the hypothesis that $\mathcal{A}$ contains a large number of elements from $j(uE)$ (modifications of this observation will appear in forthcoming work).

On the other hand, there are many elements $\Psi \in \mathcal{A}$ which vanish on $uE$ and such that $\langle t, \Psi \rangle = 0$ for $t = u_j T$; and so it is reasonable to consider "supp $t$" $\subseteq uE$. In fact —

Example 5.2. Take $U \subseteq V$, two compact neighborhoods of $E$, and let $\phi \in G$ be $2\pi$ on $U$ and 0 on $\partial V$. Thus, $\Psi(f) = 1 - (f, \phi) \in \mathcal{A}$ and if $t = u_j T$ then $\langle t, \Psi \rangle = 0$ since supp $T \subseteq E$. Assume without loss of generality that $U - N + N = V$, $N$ a neighborhood of origin. Then $\Psi \in \mathcal{A}_k$ where

$$m(U - N)/m(N) < k^2$$

by a standard approximate identity argument.

Example 5.3. There are other extension conditions to guarantee that $T \in P'(E)$ be measure approximable and hence to deduce that if $E$ is Helson and satisfies one of these conditions then $T \in M(E)$. For example, noting that for all $k$ there is $s_k \in A'(\Gamma)$ such that $s_k = t$ on $\mathcal{A}_k$, we assume
$s_k = u_j S_k$, $S_k \in A'(E)$. Thus $S_k \in M(E)$ and $s_k \in M(uE)$ by Example 5.1, and so $T$ is measure approximable.

The following is clear from Prop. 4.2 and the definition of inductive limit topologies.

**Proposition 5.1.** Assume that $(e_T)$ is satisfied. Then there is $\{v_{k_n}\} \subseteq M(uE)$ such that

a) $\forall k_n > k_m$, $t = v_{k_n} = v_{k_m}$ on $\mathcal{A}_{k_m}$.

b) $v_{k_n} \rightarrow t$ in $\beta(\mathcal{A}', \mathcal{A})$ where $\beta(\mathcal{A}', \mathcal{A})$ is the strong topology on $\mathcal{A}'$.

By standard measure theory —

**Proposition 5.2.** $u_j: M(E) \rightarrow M(uE)$ is a bijective continuous linear map where $M(E)$ has the induced topology from $A'(E)$ and $M(uE)$ has the induced topology from $\mathcal{A}'$.

Prop. 5.2 is true for $M(X)$ and $M(uX)$, with any closed $X \subseteq \widehat{\Gamma}$.

Summing up the previous observations and letting $X_k = u_j A_k$ we have —

**Proposition 5.3.** Assume $(e_T)$ is satisfied. Then $T \in A'(E)$ is measure approximable; further, the subspaces $X_k$ can be taken to be increasing and $\mu_m = \mu_k$ on $X_k$ if $m \geq k$.

### 6. Conditions for Strong Spectral Resolution

We begin with the following definition. $E$ has the weak uniqueness property $U_N$ if there is $N$ satisfying the following condition:

$$\forall I_a, I_b \subseteq [0, 2\pi), \quad \text{for which} \quad I_k \subseteq (\lambda_a, \gamma_b)$$

for infinitely many $k$, $\exists n I_{k_j} \subseteq (\lambda_a, \gamma_b)$ such that

$$d(I_{k_{j-1}}, I_{k_j}) \leq N \min(e_{k_{j-1}}, e_{k_j}) \quad \text{for} \quad j = 1, \ldots, n,$$

where $k_0 = a$ and $k_{n+1} = b$

(and where $d(I, J)$ is the distance between $I$ and $J$).

**Remark 1.** We refer to $U_N$ as a weak uniqueness property because the classical necessary conditions for weak uniqueness due to Bary and Civin-Chrestenson [1, Chapter 14, §13]; their result is only stated for $U$ sets but it is shown that $M_0(E) \neq \{0\}$ in the proof. In this regard, consider the following condition, $\check{U}_1$, which is weaker than $U_1$:

For every interval $J$, which contains infinitely many $I_k$, $\exists I_{k_1}, I_{k_2} \subseteq J$ such that

$$d(I_{k_1}, I_{k_2}) \leq \min(e_{k_1}, e_{k_2}).$$
Thus if \( E \) is not \( \hat{U}_1 \) and if we build \( E \) by throwing out the largest \( I_j \) between any given \( I_p, I_q \) (which have already been thrown out), we have
\[
\varepsilon_j/d(I_p, I_q) < 1/3
\]
(6.1)
since \( \varepsilon_j \leq \min(\varepsilon_p, \varepsilon_q) \). One of Bary’s necessary conditions for weak uniqueness is that ratios such as (6.1) tend to 0, and for a long time it was thought that such convergence was all that was necessary.

2. Since we are interested in whether certain Helson sets are \( S \) sets, and since Helson sets are weak \( U \)-sets [2, Chapter 7], it is not unexpected to have uniqueness conditions for strong spectral resolution (e.g. Prop. 6.1 below).

**Example 6.1.** Clearly every countable \( E \) is \( U_1 \). On the other hand \( E = \{0, 1/n; n = 1, \ldots\} \) is not \( U_n \) for any \( N \). In fact, given \( I_a = [-\pi, 0) \) and \( I_b = I_n = (1/(n+1), 1/n) \), if \( m > n \) with \( I_m = (1/(m+1), 1/m) \)
\[
1/m = d(I_a, I_m) > 1/m(m+1) = \min(e_a, e_m);
\]
and if \( n \) is chosen large enough, \( 1/m > N/m(m+1) \) for \( m > n \). Note that \( \{0, 1/n; n = 1, \ldots\} \) is not Helson. Now take \( E = \{0, 1/3^n; n = 1, \ldots\} \) (resp., \( \{0, 1/2^n; n = 1, \ldots\} \)) which is Helson. Let \( I_a \) be as above and let \( I_b = I_n = (1/3^{n+1}, 1/3^n) \) (resp., \( I_b = I_n = (1/2^{n+1}, 1/2^n) \)). Then for \( m > n \)
\[
1/3^{m+1} \text{ (resp., } 1/2^{m+1}) = d(I_a, I_m) \leq 2/3^{m+1} \text{ (resp., } 1/2^{m+1}) = \min(e_a, e_m),
\]
and hence these Helson sets are \( U_1 \). We mention both the \( 1/2^n \) and \( 1/3^n \) cases since \( \{0, 1/2^n\} \) is close to the “boundary” of being a Helson set, a fact which is illustrated by observing that the above inequality is not as strong for the \( 1/2^n \) case as for the \( 1/3^n \) case.

**Example 6.2.** The Cantor ternary set \( E \) is \( U_1 \) (and, of course, not Helson). Again, this is an easy calculation. In light of Prop. 6.1 and the fact that \( A_S(E) + M(E), (e_\tau) \) is not satisfied for some \( T \in A'(E) \).

**Example 6.3.** There is a standard technique to construct perfect Helson sets \( E \) due to Carleson, Kahane-Salem, and Rudin [2, Chapter 5]. We shall show that such an \( E \) is \( U_N \).

Take \( \lambda_1 \in (0, \pi/2) \) such that \( \{\pi, \lambda_1\} \) is independent (over the rationals), choose \( \gamma_1 \in (3\pi/2, 2\pi) \), and form the interval \( I_0 = (\gamma_1, \lambda_1) \subseteq [-\pi, \pi) \) of length \( e_0 \). We next take \( \lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 \) such that \( e_0 \geq \gamma_1 - \lambda_1 = \gamma_2 - \lambda_2 = L_1 \), \( \{\pi, \lambda_1, \lambda_2\} \) is independent, and \( e^0 = \lambda_2 - \gamma_1 \geq L_1 \); and write \( \lambda_1 < \gamma_1 \), \( E_1 = S_1 \cup S_2 \), \( I_0 = (\gamma_1, \lambda_2) \). For the inductive step we first recall from Kronecker’s theorem [2, Chapter 5] that if \( \{\pi, \lambda_1, \ldots, \lambda_{2^n}\} \) is independent and a measure \( \mu \) is supported by \( \{\lambda_1, \ldots, \lambda_{2^n}\} \) then there is \( N_n \) for which
\[
\sup_{0 < m < N_n} |\hat{\mu}(m)| > \frac{1}{2} \|\mu\|_1;
\]
next we note that if \( \{\pi, \lambda^n_1, \ldots, \lambda^n_{2^n}\} \) is independent and \( V_1, \ldots, V_{2^n} \) are intervals disjoint from the \( \lambda^*_j, \pi \) then there are \( \lambda_j \in V_j \) such that
\[
\{\pi, \lambda^n_1, \ldots, \lambda^n_{2^n}, \lambda_1, \ldots, \lambda_{2^n}\}
\]
is independent [2, Chapter 4]. We form \( E = \bigcap E^n \) where \( E^n = S^n_1 \cup \cdots \cup S^n_{2^n}, \)
\( m(S^n_j) = I^n, S^n_{2^n-1}, S^n_{2^n} \subseteq S^n_{j} \), and \( \{\pi, \lambda^n_1, \ldots, \lambda^n_{2^n}\} \) is independent. Thus at the \( n \)-th stage we throw out the open intervals \( I^n_j \) of length \( e^n_{j-1} \), \( j = 1, \ldots, 2^n \), subject to the conditions
\[
L^n_j \leq e^n_{j-1}, \quad j = 1, \ldots, 2^n - 1,
\]
and
\[
\lim_{n} N_n 2^n L^n_j = 0.
\]
We need these two conditions to show \( A'(E) = M(E) \), and the inequality obviously tells us that \( E \) is \( U_1 \).

**Proposition 6.1.** Let \( E \) be a \( U_N \) set. If \( (e_T) \) is satisfied for \( T \in A'(E) \) then \( T \in M(E) \).

**Proof.** Take \( X_k = u_j \cdot A_k \) and \( \mu_k \) as in the definition of measure approximable and Prop. 5.3.

Without loss of generality assume \( N = 1 \).

Let \( V \subseteq E \) be open (in the relative topology obviously).

We shall show that \( \lim \mu_k(V) \) exists so that by the Grothendieck-Dieudonné theorem, \( \mu_k \) converges in the weak topology on \( M(E) \) to a measure \( \mu \); consequently, \( \mu_k \to \mu \) in the weak* topology.

For \( \bar{\phi} \in \bigcup X_k \), with \( \bar{\phi} \in X_m \), say, we have
\[
\langle \mu, \bar{\phi} \rangle = \lim \langle \mu_k, \bar{\phi} \rangle = \langle \mu_m, \bar{\phi} \rangle = \langle T, \bar{\phi} \rangle;
\]
thus \( T = \mu \) on a dense subset of \( A_j(E) \) and so \( T = \mu \).

Now, since \( V \) is open, we write
\[
V = \bigcup F_n,
\]
where \( F_n \subseteq V, F_n \in \mathcal{F}, \text{ and } F_n \subseteq F_{n+1} \).

Recall that the de la Vallée-Poussin kernel \( \phi_{\varepsilon, \gamma} \) about \( \gamma \in I^n \) is non-negative, equals 1 on an \( \varepsilon \)-neighborhood of \( \gamma \), equals 0 outside a \( 2\varepsilon \)-neighborhood of \( \gamma \), and has the property that \( \| \phi_{\varepsilon, \gamma} \|_A \leq 3 [2, \text{ Chapter 1}]. \)

For each \( F_n \) we observe that there is a finite number of de la Vallée-Poisson kernels \( \phi_{n,1}, \ldots, \phi_{n,k_n} \) (for various \( \varepsilon \)'s and \( \gamma \)'s in \( F_n \)), such that
\[
\text{supp } \phi_{n,j} \cap \text{supp } \phi_{n,k} = \emptyset,
\]
\[
\forall \gamma \in F_n, \exists j \text{ such that } \phi_{n,j}(\gamma) = 1,
\]
\[
\forall \gamma \in E - F_n, \forall j, \phi_{n,j}(\gamma) = 0.
\]
To see this we first note that since $V$ is open, $V = E \cap (\cup J_m)$ where $J_m$ is an open interval; and a straightforward compactness argument shows that $F \in \mathcal{F}$ is the intersection of $E$ with a finite number of intervals each having their endpoints in the $I$'s.

By the previous observation $F \in \mathcal{F}$, $F_n \cap J_m$ determines a finite number of $I_{a_j}$, $I_{b_j}$ depending on $n$ and $m$, and for each of these we use the $U_1$ property. Let us label all the closed intervals in $J_m$ contiguous to the $I$'s (obtained from $U_1$) by $H_1, \ldots, H_p$.

Consequently we choose $\phi_{n,j}$ about $\gamma \in H_k$ so that it is equal to 1 on $H_k$, and we have (6.2).

Now, for a given $F_n$, define $\tilde{\phi}_n \in A_j(E)$ by

$$\phi_n(\gamma) = \frac{1}{1 - e^j} \left[ k_n - \sum_{j=1}^{k_n} e^{j\phi_{n,j}(\gamma)} \right].$$

Clearly, $\tilde{\phi}_n \in X_3$ since $\|\phi_{n,j}\|_A \leq 3$.

Because of (6.2) we note that if $\gamma \in F_n$,

$$\phi_n(\gamma) = \frac{1}{1 - e^j} \left[ k_n - (k_n - 1) - e^j \right] = 1,$$

since $\phi_{n,j}(\gamma) = 1$ for precisely one $j$ and $\phi_{n,j}(\gamma) = 0$ for the remaining $j$; similarly, if $\gamma \in E - F_n$, $\phi_{n,j}(\gamma) = 0$ for each $j$ and so $\phi_n(\gamma) = 0$.

Thus $\lim \phi_n = \chi_V$ pointwise; and from our definitions

$$\sup \{ |\phi_n(\gamma)| : \gamma \in E, n \} < \infty.$$

Therefore, we can use the Riesz representation theorem and see that for each $k \geq 3$, $\mu_k(V)$ is well-defined by

$$\lim_n \langle \mu_k, \tilde{\phi}_n \rangle = \mu_k(V).$$

Now,

$$\lim_{k \geq 3} \mu_k(V) = \lim_{k \to \infty} \lim_n \langle \mu_k, \tilde{\phi}_n \rangle = \lim_{k \to \infty} \langle T, \tilde{\phi}_n \rangle = \lim_n \langle T, \tilde{\phi}_n \rangle,$$

where the right hand side exists since $\lim_n \langle T, \tilde{\phi}_n \rangle = \lim_n \langle \mu_k, \tilde{\phi}_n \rangle$.

Thus, $\lim_{k \to \infty} \mu_k(V)$ exists. \textit{qed}.

We can't replace $U_1$ by $\bar{U}_1$ in the proof since there is no guarantee that $V$ would be covered by the above procedure.

\textit{Example 6.4.} Let us look at the relation between $U_N$ and Helson sets. For convenience of explication suppose that $E$ is $\bar{U}_1$ (the point of this example being the same for $U_N$). Assume $E$ is a perfect Helson set. If $E$ is not $\bar{U}_1$ then there is an admissible $J$ (as in the definition of $\bar{U}_1$) such that for all $I_{k_1}$, $I_{k_2} \subseteq J$, $d(I_{k_1}, I_{k_2}) > \min(e_{k_1}, e_{k_2})$. Without loss of generality we suppose that $J$ has endpoints in $I_p$, $I_q$, each with length greater than or
equal to the length of any \( I \) between them. We build \( J \cap E \) by throwing out \( I_1^1 \), the largest \( I \) between \( I_p \) and \( I_q \); then at the second step we throw out \( I_2^1, I_2^2 \) the largest \( I \)'s between \( I_p \) and \( I_1^1 \), and \( I_1^1 \) and \( I_q \), respectively; etc. In this way because of estimates like (6.1), it is reasonable to try to find situations where there are perfect \( P, Q \) for which

\[
P + Q \subseteq J \cap E \subseteq E
\]

(since at each stage of building \( J \cap E \) there is a certain symmetry and we have much of \( J \) left over).

We can prove that \( E \) Helson and (6.3) give a contradiction so that in this setting we'd have "Helson implies \( \bar{U}_1 \)." To get the contradiction we construct the continuous Cantor-Lebesgue positive measures \( \mu_P, \mu_Q \) supported by \( P, Q \), respectively, and have

\[
\mu_P * \mu_Q (E) > 0
\]

since \( P + Q \subseteq E \). This contradicts the recent Salinger-Varopoulos theorem [2, Chapter 7]: if \( E \) is Helson and \( \mu, \nu \) are positive continuous measures in \( R/2 \pi \mathbb{Z} \) then \( \mu * \nu (E) = 0 \). This latter result, by the way, is based on the Kahane-Salem theorem which estimates the number points of "general" arithmetric progressions that lie in a Helson set.

Remark. In light of the Bary theorem mentioned in Remark 1 of this section and the importance of the non-existence of inclusions like (6.3) for Helson sets, we would now like to define a more general notion than \( U_N \). The point is to find such a notion so that Helson sets are included and the analogue of Prop. 6.1 is true; and to investigate the cases where (6.3) holds for perfect non-\( U_N \) sets.

7. A Property of \( A(\mathbb{F}) \)

Consider the following property on \( A(\mathbb{F}) \) and \( E \): \( \exists K \) such that \( \forall n \in \mathbb{Z}, \exists \phi_n \in G \) and \( \exists N_n \), a finite disjoint union of closed intervals covering \( E \), for which

\[
\| \phi_n \|_A < K, \\
\forall \gamma \in N_n, \quad e^{i \gamma} = e^{i \phi_n (\gamma)}.
\]

(P)

Note that for Helson sets there is \( K > 0 \) such that \( \| \phi_n \|_{A(E)} < K \) where \( \phi_n (\gamma) = e^{i \gamma} \) on \( E \). Also observe that our requirements for \( N_n \) are weaker than stipulating that \( N_n \) be a neighborhood of \( E \).

The following result (and proof) is amusing for one-point sets, \( E \).

**Proposition 7.1.** Property P never holds.

**Proof.** Let \( \psi (\gamma) = \sum a_k e^{i k \gamma} \in A(\mathbb{F}) \) and define

\[
\Psi_n (f) = \sum_{|n| \leq N} \sum a_n (f, \phi_n)
\]
where the $\phi_n$ are chosen by property $P$. Thus there is a $k$ such that for all $N$, $\mathcal{F}_N \in \mathcal{A}_k$.

Setting $\psi_N(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma}$,

$$||\tilde{\psi} - u_j^* \mathcal{F}_N||_j \leq ||\tilde{\psi} - \tilde{\psi}_N||_j + ||\tilde{\psi}_N - u_j^* \mathcal{F}_N||_j.$$  

Clearly, $||\tilde{\psi} - \tilde{\psi}_N||_j \to 0$ as $N \to \infty$.

Also, since $u^* \mathcal{F}_N = \psi_N$ on an $N_n$ we have

$$||\tilde{\psi}_N - u_j^* \mathcal{F}_N||_j = 0,$$

because $N_n$ is an $S$-set.

Consequently, $\tilde{\psi} \in \overline{u_j^* \mathcal{A}_k}$, and hence $\bigcup_k \overline{u_j^* \mathcal{A}_k} = A_j(E)$, contradicting Prop. 3.4. $\text{q.e.d.}$

Remark. 1. E Helson implies $m(E) = 0$ [2, Chapter 7].
2. If $m(E) = 0$ then

$$\forall D > 0, \; \forall M > 0, \; \forall n, \; \text{and} \; \forall 1 \leq p < \infty$$

there is a neighborhood $N_{n,p}$ of $E$ of the form $\bigcup_{\ell} I_j, I_j$ open disjoint intervals, and a function $\phi$, sup $\{ |\phi(\gamma)| : \gamma \in N_{n,p} \} < D$ such that

$$\phi(\gamma) = n\gamma + 2\pi k_j \quad \text{on} \quad I_j$$

and

$$\left( \sum_{N_{n,p}} |\phi'(x)|^2 \right)^{\frac{1}{2}} < M.$$

8. Extension and Helson Set Conditions for Spectral Synthesis

Let $\mathcal{A}(uE)$ be the quotient space of restrictions of the elements of $\mathcal{A}$ to $uE$; and recall from the proof of Prop. 4.1 that $u^* \mathcal{F} \in C(E)$ when $\mathcal{F} \in C(uE)$.

**Proposition 8.1.** E is Helson if and only if $\mathcal{A}(uE) = C(uE)$.

**Proof.** Take $\mathcal{F} \in C(uE)$. Since E is Helson there is

$$\phi(\gamma) = \sum_{n} a_n e^{in\gamma} \in A(I^\infty)$$

such that $\phi = u^* \mathcal{F}$ on $E$.

For each $n$, consider a finite decomposition $\{F_1, \ldots, F_{m_n}\} \subseteq \mathcal{F}$ of $E$ and $\psi_n \in G$ such that $\|\psi_n\|_\infty \leq 31$ and

$$\psi_n(\gamma) = n\gamma + 2\pi k_j$$

on $F_j, j = 1, \ldots, m_n$.

Let $K_E > 0$ be the Helson constant: $||A(E)|| \leq K_E ||,\infty, E.$
Thus, there is \( k > 31 K_E \) and \( \{ \phi_n \} \subseteq G \) such that \( \| \phi_n \|_A \leq k \) and \( \phi_n = \psi_n \) on \( E \); the fact that we can take \( \phi_n \) real follows since \( (\phi_n + \phi_n)/2 = \psi_n \) on \( E \) and \( \| (\phi_n + \bar{\phi}_n)/2 \|_A \leq \| \phi_n \|_A \).

Consequently, \( \Phi(f) = \sum a_n(f, \phi_n) \in \mathcal{A}_k \) and \( \Phi = \Psi \) on \( uE \). \( \text{qed.} \)

**Proposition 8.2.** Let \( E \) be Helson.

a) If \( (e_T) \) is satisfied then \( T \in M(E) \).

b) \( E \) is S if and only if \( (e_T) \) is satisfied for all \( T \in A'(E) \).

**Proof.** a) Because of Prop. 5.1 b, \( \lim \langle v_k, \Psi \rangle \) exists for all \( \Psi \in \mathcal{A} \) (where we preserve the notation of § 5).

Since \( v_k \in M(uE) \), if \( \Psi \in \mathcal{A}(uE) \) and \( \Psi_e \in \mathcal{A} \) is any extension, then \( \langle v_k, \Psi \rangle = \langle v_k, \Psi_e \rangle \).

Thus, \( \lim \langle v_k, \Psi \rangle \) exists for all \( \Psi \in \mathcal{A}(uE) \); and so, from Prop. 8.1, \( \{ v_k \} \) converges in the weak * topology to a measure \( \nu \in M(uE) \).

Consequently \( \nu = \lambda \) on \( \mathcal{A} \) and so \( \lambda \in M(uE) \) since \( \mathcal{A} = C(\bar{\Gamma}) \) (because \( \mathcal{T} \subseteq \mathcal{A}, \Gamma \) is compact, \( G \) separates points on \( \Gamma \), and by the Stone-Weierstrass theorem).

We are done by Example 5.1 or Prop. 5.2.

b) We need only show that \( M(E) = A'(E) \) implies \( (e_T) \). This is obvious since if \( T \in M(E) \) then \( \lambda = u_j T \in M(uE) \). \( \text{qed.} \)

In the same way,

**Proposition 8.3.** Let \( E \) be a Helson set. \( E \) is an S set if and only if \( t = u_j T \) is well-defined on \( \mathcal{A}(uE) \) for each \( T \in A'(E) \).

In fact, for \( E \) Helson and \( T \in A'(E) \) we need only show: For each \( F \in \mathcal{F} \), there is \( \psi = \phi_F \) with \( \| \psi \|_A \leq K_E \) (since \( E \) is Helson such \( \psi \) exist) such that \( \hat{T}(\psi) = \hat{T}(\phi_F) \); then \( T \in M(E) \).

**Example 8.1.** Since Helson sets \( E \) have \( m(E) = 0 \) a natural technique to prove that \( t = u_j T \) is well-defined on \( \mathcal{A}(uE) \) (i.e., Prop. 8.3) is to utilize the representation

\[
T = \sum k_j (\delta_{\lambda_j} - \delta_{\gamma_j}) \quad \text{on} \quad C^1(\hat{\Gamma}),
\]

\[
\sum e^{r|k_j|} e_j < \infty, \quad \text{some r}
\]

for \( \hat{T}(0) = 0 \) [2, Chapter 2]. In fact, if we chose \( G \) to be the real continuously differentiable elements of \( A(\hat{\Gamma}) \) and defined \( \mathcal{A}_k \) by the restriction that \( \| \phi' \|_\infty \leq k \) for \( \phi \in G_F \) then \( \lambda = u_j T \) is well-defined. On the other hand we do not get (8.1) with these spaces and so cannot conclude that \( T \in M(E) \) as in Prop. 8.3.

**Example 8.2.** Along the same lines of Prop. 8.3 and Example 8.1 we see that if \( E \) is Helson and \( G = G_E \subseteq A(\hat{\Gamma}) \) exists such that

\[
\forall F \in \mathcal{F} \quad \exists \phi \in G \quad \text{for which} \quad \phi = 1 \text{ on a neighborhood of } F
\]

\[
\text{and} \quad \phi = 0 \text{ on a neighborhood of } E - F
\]
and
\[ \exists M > 0 \quad \text{for which} \quad \|\phi'\|_\infty \leq M \|\phi\|_A \]
then \( E \) is an \( S \)-set. Verifying such conditions is of course related to Remark 1, §6.

We say \( uE \) in \( \mathcal{A} - S \) if for all \( t \sim T \in A'(E) \) and for all \( \Psi \in \mathcal{A} \) vanishing on in \( E \), \( \langle t, \Psi \rangle = 0 \). The above results show that if \( E \) is Helson then \( E \) is \( S \) if and only if \( uE \) if \( \mathcal{A} - S \). It would be interesting to weaken the Helson set hypothesis here. Neither Malliavin’s non-\( S \) criteria or Körner’s example provide any help. On the other hand we can easily check that –

**Proposition 8.4.** Assume that for each \( t \sim T \in A'(E) \), \( \{\phi \}_n \leq G \) with \( \sup \|\phi\|_A < \infty \), and \( \{s_n\} \leq R \) increasing to infinity, there is \( \mu \in M(uE) \) such that for all \( n \), \( |\hat{\mu}(\phi_n) - \hat{\phi}(\phi_n)| < s_n \). Then \( uE \) is \( \mathcal{A} - S \).

The hypothesis of Prop. 8.4 characterizes \( S \) sets when we deal only with pseudo-measures (as a simple Hahn-Banach argument shows).

Using the technique of Prop. 8.1 we find

**Proposition 8.5.** Given real \( \phi \in C^\infty(\hat{\mathcal{F}}) \) and \( E \) Helson. There is real \( \psi_n \in C^\infty(\hat{\mathcal{F}}) \) such that
\[ \|\psi_n\|_A \leq 31 K_E + 1 \]
and for each \( \gamma \in \mathcal{S}_n = \{\lambda_1, \gamma_1, \ldots, \lambda_n, \gamma_n\}, \]
\[ e^{i\psi_n(\gamma)} = e^{i\phi(\gamma)}. \]

This leads us to define the following set of numbers \( \{\phi(n, F): F \in \mathcal{S}, \quad n = 1, 2, \ldots\} \) which we call the Helson indicator for a given Helson set \( E \). From Prop. 8.5 and for a given \( n \) and \( \phi = \phi_F \) choose \( q_n \in k(S_n) \) and \( g_n \in j(S_n) \) such that \( \|\phi + q_n\|_A \leq 31 K_E + 1 \) and \( \|q_n - g_n\|_A < 1 \); let \( \varepsilon = \varepsilon(n, F) \) have the property that
\[ \|g_n * p_\varepsilon - g_n\|_A < 1 \]
where \( p_\varepsilon \) is the Friedrich mollifier of support \([-\varepsilon, \varepsilon]\).

For convenience in Prop. 8.6 we consider the bounded pseudo-measures \( A'_b(E) \) (e.g. [3]); this amounts to considering those \( T \in A'(E) \) with \( \{k_j\} \) (of Example 8.1) bounded. The result is easily reformulated for \( A'(E) \).

**Proposition 8.6.** Given \( E \) Helson and \( T \in A'_b(E) \). If there is \( K \) such that for all \( F \in \mathcal{F} \) we can choose \( n \) for which
\[ \sum_{j=n+1} \varepsilon_j < K \varepsilon(n, F), \]
then \( T \in M(E) \).

**Proof.** Set \( \phi = \phi_F \), \( F \in \mathcal{F} \), and \( f_n = e^{i\phi} - e^{i\psi_n} \) from Prop. 8.5. From the structure of \( \phi \) there is \( N_F \) such that \( \phi' = 0 \) on \( (\lambda_j, \gamma_j) \) if \( j \geq N_F \).
Taking $n > N_F$ we have

$$|\langle T, f_n \rangle| = \left| \sum_{n+1}^{\infty} k_j \int_{\lambda_j}^{\gamma_j} \psi_n'(\gamma) e^{i\psi_n'(\gamma)} d\gamma \right|,$$

and since $\psi_n' = \phi' + g_n * p'_\varepsilon$ we compute

$$|\langle T, f_n \rangle| \leq C \sum_{n+1}^{\infty} k_j \int_{0}^{2\pi} |p'_\varepsilon| = \frac{C}{\varepsilon(n, F)} \sum_{n+1}^{\infty} \varepsilon_j.$$

Given the hypothesis and the norm bound on $\|\psi_n\|_A$ we have $|\langle T, e^{i\phi_F} \rangle|$ uniformly bounded. Hence $T \in M(E)$. \textit{qed}.

Because of Prop. 8.6 we consider $E$ with the property: there is $x_E > 0$ such that for each $N$ we can find $n > N$ for which

$$\min d(t_j, t_k) \geq x_E \sum_{n+1}^{\infty} \varepsilon_j,$$

where $j, k \leq n$ and $t_j$ is $\lambda_j$ or $\gamma_j$. The Cantor set is not (P), which is encouraging, whereas some calculations show that there is no reason to expect a pseudo-measure on an Helson-(P) set to be a measure.

References

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