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Fourier Analysis of Riemann Distributions and Explicit Formulas

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A *Riemann distribution* R on the real line $\mathbb{R}$ has the form

$$R = \sum_{\gamma \in \Gamma} \delta_\gamma ,$$

where $\Gamma \subsetneqq \mathbb{R} \setminus \{0\}$ is countable, δ_γ is the Dirac measure at $\gamma \in \Gamma$, and $\sum_{\Gamma} 1/\gamma^2 < \infty$. Riemann (1859) first studied the primitive $W_{(1)}$ of the Fourier transform of R at the end of his fundamental memoir on the Riemann zeta function, $\zeta(s)$. For Riemann, Γ was the set, $\Gamma(\zeta) = \{\gamma : \zeta(\frac{1}{2} + i\gamma) = 0\}$, and the study of $W_{(1)}$ was carried on by von Mangoldt [16, pp. 292ff.], cf., Landau [12]. The notation, $W_{(1)}$, is used because

$$W_{(1)}(y) = 2 \sum_{\substack{\gamma \in \Gamma(\zeta) \\ \gamma > 0}} \frac{\sin \gamma y}{\gamma}$$

is the first primitive of *Weil's distribution*, W, defined as

$$\langle W, F \rangle = \sum_{\gamma \in \Gamma(\zeta)} \int_{-\infty}^{\infty} F(y)\, e^{i\gamma y} dy ,$$

e.g., Sect. 4; W was introduced by Weil [24] to formulate general explicit formulas, cf., Besenfelder [4] and Besenfelder and Palm [5]. The Fourier analysis of R is a natural project for $\Gamma = \Gamma(\zeta)$, and some interesting contributions have been made by Hardy and Littlewood [8], Cramer [7], Ingham [10], Rademacher [12], Rubel and Straus [13], and Montgomery [17], as well as by A. Wintner in a series of papers in the American Journal of Mathematics on Gibbs phenomenon and the distribution function of the remainder term of the prime number theorem (PNT). Another family of number theoretic Schwartz distributions is studied by Benedetto [3, Examples 5.2–5.4].

The basic problem considered herein is to estimate

$$\sum_{\substack{\gamma \in \Gamma \\ 0 < \gamma < g(y)}} \frac{\sin \gamma y}{\gamma} \tag{1}$$

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for large y where g is given and $\lim_y g(y)=\infty$. The results are limited by the fact that only interesting functions g are considered. The series $\sum_{\gamma\in\Gamma}(\sin\gamma y)/\gamma$ is a Besicovich almost periodic function whose frequencies are proportional to the inverse of corresponding amplitudes. The musical interpretation of such series has been indicated by Rubel in his lecture, "Harpsichords, pianos, and the Riemann hypothesis". In this paper the asymptotic behavior of the series (1), for $y\to\infty$, is studied by means of a decomposition of (1) into a Fourier series on $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ and a Fourier transform on $\mathbb{R}$; and the Fourier series can be viewed as a linear approximation to (1) on short intervals. This decomposition, including estimates on the Fourier series and Fourier transform constructed by the decomposition, is the point of view of the paper.

The problem of asymptotic estimates for (1) is motivated by an important open question in analytic number theory: is it true that

$$\psi(x)=x+o(x^{1/2}\log^2 x), \qquad x\to\infty, \tag{2}$$

where the Riemann hypothesis (RH) is assumed and where $\psi(x)$ is the Tchebychev function, e.g., Sect. 4? In 1901, von Koch verified that $\psi(x)=x+O(x^{1/2}\log^2 x)$, $x\to\infty$, when one assumes RH. P. X. Gallagher and J. Mueller have proved (2) assuming both RH and Montgomery's conjecture [17].

In Sect. 1 the above-mentioned decomposition is described and various norm estimates are proved for (1). Section 2 provides a Fourier analysis of the Fourier series in the decomposition. It should be noted that these Fourier series can be studied quite effectively, e.g., Sect. 5; and so precise local information can be obtained from the decomposition if the period size for the Fourier series is made smaller. This topic is not developed in the paper. Section 3 gives the corresponding Fourier analysis for the Fourier transform of the decomposition. This section establishes the need for a stationary phase analysis of various distribution functions in order to estimate (1) in a more satisfactory way, e.g., Proposition 3.7 and Example 3.1.

The remainder of the paper deals with problem (2). Section 4 is devoted to the number theoretic background necessary to discuss (2). This material includes Littlewood's and Weil's explicit formulas, as well as the distributional setup for Sect. 6. RH is assumed in Sects. 5 and 6. In Sect. 5 the asymptotic behavior is determined for the Fourier series arising in the decomposition of (1) applied to the case (2). This behavior is much better than the desired "o" of (2), but, of course, it is only applicable to the linear approximation provided by the Fourier series. In Sect. 6, Weil's explicit formula is viewed as a distributional equation on $\mathbb{R}$; and bounds are obtained on the second primitive f of a certain distribution in the equation. The results of Sect. 6 appear technical but they establish the limitation of explicit formulas for dealing with (2) in the following way. A number theoretic sequence $\{a_n\}$ is defined in conjunction with f, and Theorem 6.1 establishes the relation

$$f(\log m)=\sum_1^m a_n\log(m/n)\ll\log m;$$

then, Theorem 6.2 recalls that the estimate,

$$\psi(x)=x+O(x^{1/2}\log x), \qquad x\to\infty,$$

is equivalent to the estimate,

$$\sum_1^m a_n \ll \log m. \tag{3}$$

The left hand side of (3), which can be thought of in terms of the sum $W_{(1)}$, is the distributional *derivative* of f at $\log m$; and so Theorems 6.1 and 6.2 show that results such as (2) are not derived from explicit formulas, manifested in these remarks by f, but from local conditions, manifested by f'. These local conditions can only follow from a finer knowledge of $\{a_n\}$ or, equivalently, from more precise knowledge of $\Gamma(\zeta)$, cf., the previous remark about Montgomery's conjecture.

The usual notation from distribution theory and harmonic analysis is employed, e.g., Benedetto [2].

1. Norm Estimates and the Series $\sum \sin\gamma y/\gamma$

Let $\{\gamma_j\}\subseteq[1,\infty)$ increase to infinity and define

$$\forall m\geqq 1, \qquad d(m)=\operatorname{card}\{\gamma_j : m\leqq\gamma_j<m+1\}.$$

Assume throughout that for any $h>0$ there are positive constants $A(h)$, $B(h)$, and T_h such that

$$\forall T\geqq T_h, \qquad A(h)\log T\leqq\operatorname{card}\{\gamma_j : T\leqq\gamma_j<T+h\}\leqq B(h)\log T. \tag{1.1}$$

In particular,

$$\log m \ll d(m) \ll \log m. \tag{1.2}$$

Let $M_y=[e^{y/2}]-1$ and define the functions

$$f_m=\sum_{m\leqq\gamma_j<m+1}\frac{1}{\gamma_j}\chi_{[m,\gamma_j)},$$

$$f_y=\sum_{m=1}^{M_y} f_m, \qquad f=\sum_{m=1}^{\infty} f_m,$$

and

$$\hat{f}_y(\lambda)=\int_{-\infty}^{\infty} f_y(x)\cos x\lambda\,dx,$$

noting that $\hat{f}_y$ is even on $\mathbb{R}$. The following facts are obvious: $0\leqq f(m)\leqq d(m)/m$, $\lim_{x\to\infty} f(x)=0$, $f, f_y\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, and $\lim_{y\to\infty}\|f-f_y\|_\infty=0$. It is also true that

$\lim_{y\to\infty} \|f-f_y\|_2=0$ since

$$\left\|\sum_{m=M}^{\infty} f_m\right\|_2^2 \leqq \sum_{m=M}^{\infty} \left\|\sum_{m\leqq\gamma_j,\gamma_n<m+1} \frac{1}{\gamma_j\gamma_n}\chi_{[m,\gamma_j)}\chi_{[m,\gamma_n)}\right\|_1$$
$$\leqq \sum_{m=M}^{\infty}\left(\frac{d(m)}{m}\right)^2,$$

noting that there are $d(m)^2$ terms within $\| \ \|_1$.

Define a function $c(y)\geqq C>0$ and then define $C(y)=yc(y)$. $c(y)$ and $C(y)$ must be defined for all large y. Next, for a given m, let $\gamma_{n_m}\geqq m$ be the first element of $\{\gamma_j\}$ greater than or equal to m. (If $\gamma_j=\gamma_{j+1}$ we choose either one.) When there is no confusion one writes

$$n+1=n_m \quad\text{and}\quad \gamma_n=m.$$

Thus, $\gamma_n=m\leqq\gamma_{n+1}\leqq\gamma_{n+2}\leqq \dots \leqq\gamma_{n+d(m)}<m+1$. Finally, define the "triangular sum"

$$c_m=\sum_{j=0}^{d(m)-1}((\gamma_{n+d(m)-j}-m)+(\gamma_{n+d(m)-(j+1)}-m)+\dots+(\gamma_{n+1}-m)).$$

The Fourier series in Proposition 1.1 represents the straight line approximation to f on each $[m,m+1)$.

Proposition 1.1. *For each* $y\geqq 1$ *and* $\lambda\in\mathbb{R}\backslash\{0\}$,

$$\frac{1}{\lambda y}\sum_{0<\gamma_j<e^{y/2}}\frac{\sin\gamma_j\lambda}{\gamma_j}=\frac{1}{y}\hat{f}_y(\lambda)+\frac{1}{\lambda y}\sum_{m=1}^{M_y}\frac{d(m)}{m}\sin m\lambda+\varepsilon(\lambda,y),$$

where $|\varepsilon(\lambda,y)|\leqq K/(y|\lambda|)$.

Proof.

$$\frac{1}{y}\int_{-\infty}^{\infty} f_y(x)\cos x\lambda dx$$
$$=\frac{1}{y}\sum_{m=1}^{M_y}\sum_{m\leqq\gamma_j<m+1}\frac{1}{\gamma_j}\int_m^{\gamma_j}\cos x\lambda dx$$
$$=\frac{1}{\lambda y}\sum_{m=1}^{M_y}\sum_{m\leqq\gamma_j<m+1}\frac{1}{\gamma_j}(\sin\gamma_j\lambda-\sin m\lambda)$$
$$=\frac{1}{\lambda y}\sum_{0<\gamma_j<M_y+1}\frac{\sin\gamma_j\lambda}{\gamma_j}-\frac{1}{\lambda y}\sum_{m=1}^{M_y}\sin m\lambda\left(\sum_{m\leqq\gamma_j<m+1}\frac{1}{\gamma_j}\right)$$
$$=\frac{1}{\lambda y}\sum_{0<\gamma_j<e^{y/2}}\frac{\sin\gamma_j\lambda}{\gamma_j}-\frac{1}{\lambda y}\sum_{m=1}^{M_y}\frac{d(m)\sin m\lambda}{m}$$
$$+\frac{1}{\lambda y}\sum_{m=1}^{M_y}\sin m\lambda\left(\frac{d(m)}{m}-\sum_{m\leqq\gamma_j<m+1}\frac{1}{\gamma_j}\right)-\frac{1}{\lambda y}\sum_{M_y+1\leqq\gamma_j<e^{y/2}}\frac{\sin\gamma_j\lambda}{\gamma_j}.$$

The bound on $\varepsilon(\lambda,y)$ follows from (1.2). qed.

Proposition 1.2. a) *For each* $m \geqq 1$,

$$\frac{1}{(m+1)^2} c_m \leqq \|f_m\|_2^2 \leqq \frac{2}{m^2} c_m . \tag{1.3}$$

b)

$$\left(\sum_{m=1}^{M_y} c_m/(m+1)^2\right)^{1/2} \leqq \|f_y\|_2 \leqq \sqrt{2}\left(\sum_{m=1}^{M_y} c_m/m^2\right)^{1/2}, \tag{1.4}$$

and, in fact,

$$\sum_{m=1}^{M_y}\left(\frac{\log^2 m}{m^2}\right)^{1/2} \ll \|f_y\|_2 \ll \left(\sum_{m=1}^{M_y}\frac{\log^2 m}{m^2}\right)^{1/2}. \tag{1.5}$$

Proof. a) By definition,

$$\begin{aligned}\|f_m\|_2^2 &= (\gamma_{n+1}-m)\left(\frac{1}{\gamma_{n+1}}+\frac{1}{\gamma_{n+2}}+\dots+\frac{1}{\gamma_{n+d(m)}}\right)^2\\ &\quad+(\gamma_{n+2}-\gamma_{n+1})\left(\frac{1}{\gamma_{n+2}}+\frac{1}{\gamma_{n+3}}+\dots+\frac{1}{\gamma_{n+d(m)}}\right)^2+\dots\\ &\quad+(\gamma_{n+d(m)-1}-\gamma_{n+d(m)-2})\left(\frac{1}{\gamma_{n+d(m)-1}}+\frac{1}{\gamma_{n+d(m)}}\right)^2\\ &\quad+(\gamma_{n+d(m)}-\gamma_{n+d(m)-1})\frac{1}{\gamma_{n+d(m)}^2}.\end{aligned} \tag{1.6}$$

Note that $(\ )_j^2=\left(\frac{1}{\gamma_{n+j}}+\dots+\frac{1}{\gamma_{n+d(m)}}\right)^2$ has $(d(m)-j+1)^2$ terms with numerator 1. Thus the terms of $(\ )_j^2$ include those of $(\ )_{j+1}^2$. The cardinality of the difference is $1+2(d(m)-j)$ and

$$(\ )_j^2-(\ )_{j+1}^2=\frac{1}{\gamma_{n+j}^2}+\frac{2}{\gamma_{n+j}}\left(\frac{1}{\gamma_{n+j+1}}+\dots+\frac{1}{\gamma_{n+d(m)}}\right).$$

Therefore, rearranging terms in (1.6) yields

$$\begin{aligned}\|f_m\|_2^2 &= \frac{1}{\gamma_{n+d(m)}}-m\left(\frac{1}{\gamma_{n+1}}+\dots+\frac{1}{\gamma_{n+d(m)}}\right)^2\\ &\quad+\sum_{j=1}^{d(m)-1}\left(\frac{1}{\gamma_{n+j}}+\frac{2}{\gamma_{n+j+1}}+\dots+\frac{2}{\gamma_{n+d(m)}}\right).\end{aligned} \tag{1.7}$$

In (1.7) the sum $1/\gamma_{n+d(m)}+\sum_1^{d(m)-1}(\ )$ has $d(m)^2$ terms with numerator 1. In fact,

$$\begin{aligned}1+\sum_1^{d(m)-1}(2(d(m)-j)+1) &= 1+d(m)-1\\ &\quad+2d(m)(d(m)-1)-2((d(m)-1)d(m)/2)\\ &= d(m)(d(m)-1)+d(m).\end{aligned}$$

Also, $-m(\ )^2$ in (1.7) has $d(m)^2$ terms of the form $-m/(\gamma_a\gamma_b)$.

Thus, regrouping (1.7) into $d(m)^2$ differences, it follows that the part of (1.7) with the factor $1/\gamma_{n+d(m)}$ has the form

$$\frac{1}{\gamma_{n+d(m)}}+\frac{2(d(m)-1)}{\gamma_{n+d(m)}}-\frac{m}{\gamma_{n+d(m)}^2}-2m\left(\frac{1}{\gamma_{n+1}}+\ldots+\frac{1}{\gamma_{m+d(m)-1}}\right)$$

$$=\frac{1}{\gamma_{n+d(m)}}\left(1-\frac{m}{\gamma_{n+d(m)}}\right)+\frac{2}{\gamma_{n+d(m)}}$$

$$\cdot\left((d(m)-1)-\left(\frac{m}{\gamma_{n+1}}+\ldots+\frac{m}{\gamma_{n+d(m)-1}}\right)\right)$$

$$=\frac{1}{\gamma_{n+d(m)}}\left(1-\frac{m}{\gamma_{n+d(m)}}\right)+\frac{2}{\gamma_{n+d(m)}}$$

$$\cdot\left(\left(1-\frac{m}{\gamma_{n+d(m)-1}}\right)+\ldots+\left(1-\frac{m}{\gamma_{n+1}}\right)\right).$$

Proceeding in the same way with the other factors yields

$$\|f_m\|_2^2=\frac{1}{\gamma_{n+d(m)}}\left(1-\frac{m}{\gamma_{n+d(m)}}\right)$$
$$+\frac{2}{\gamma_{n+d(m)}}\left(\left(1-\frac{m}{\gamma_{n+d(m)-1}}\right)+\ldots+\left(1-\frac{m}{\gamma_{n+1}}\right)\right)$$
$$+\frac{1}{\gamma_{n+d(m)-1}}\left(1+\frac{m}{\gamma_{n+d(m)-1}}\right)$$
$$+\frac{2}{\gamma_{n+d(m)-1}}\left(\left(1-\frac{m}{\gamma_{n+d(m)-2}}\right)+\ldots+\left(1-\frac{m}{\gamma_{n+1}}\right)\right)$$
$$+\ldots+\frac{1}{\gamma_{n+2}}\left(1-\frac{m}{\gamma_{n+2}}\right)+\frac{2}{\gamma_{n+2}}\left(1-\frac{m}{\gamma_{n+1}}\right)+\frac{1}{\gamma_{n+1}}\left(1-\frac{m}{\gamma_{n+1}}\right). \tag{1.8}$$

(1.3) follows from (1.8).

b) (1.4) is immediate from part a) and the fact that

$$\|f_y\|_2^2=\sum_{m=1}^{M_y}\|f_m\|_2^2.$$

For the second inequality of (1.5) it is sufficient to verify that $c_m \ll d(m)^2$ and this follows since

$$0\leqq c_m\leqq\sum_{j=0}^{d(m)-1}(d(m)-j)=d(m)^2-d(m)(d(m)-1)/2.$$

For the first inequality of (1.5), (1.1) is used in the following way.

Take $h=1/4$ and $A=A(1/4)$ and define $I_m=[m+\frac{1}{2}, m+\frac{3}{4})$ and $J_m=[m+\frac{3}{4}, m+1)$.

If m is large then (1.1) establishes more than $A\log m$ elements γ in each of the intervals I_m and J_m.

Write $c_m = \sum_{j=0}^{d(m)-1} (\ )_j$ and choose any j for which $\gamma_{n+d(m)-j} \in J_m$. Then $(\ )_j > (A/2)\log m$ since there are more than $A\log m$ elements γ in I_m.

Thus, $\sum(\ )_j > (A^2/2)\log^2 m$ if the sum is taken over all j for which $\gamma_{n+d(m)-j} \in J_m$. The first inequality of (1.5) is established. qed.

Example 1.1. It will be desirable later to compare estimates of $|\hat{f}_y(y)|$ by $c(y)$ with estimates of the remaining terms of Proposition 1.1 by $C(y)$. If, instead of $|\hat{f}_y(x)| \leqq c(y)$, one considers the inequality $\|f_y(x)\|_p \|\cos xy\|_q \leqq c(y)$, for some $p \in (1, \infty]$, then $c(y)$ has exponential growth.

Proposition 1.3. a) $\sum_{0<\gamma_j<e^{y/2}} \sin\gamma_j y/\gamma_j \ll y^2$, *cf.*, *Propositions 4.1 and 4.2.*

b) $\sum_{m=1}^{M_y} \frac{d(m)}{m} \sin my \ll y^2$.

c) $\hat{f}_y(y) \ll y$.

Proof. a) follows from (1.2) by summing $\sum 1/\gamma_j$ in terms of $\sum d(m)/m$; b) follows from (1.2) and the integral test; and c) follows from Proposition 1.1 and parts a) and b). qed.

Proposition 1.4. $y^2 \ll \|f_y\|_1 = \hat{f}_y(0) \ll y^2$.

Proof. The equality is clear, and a direct calculation yields

$$\|f_m\|_1 = d(m) - m\left(\frac{1}{\gamma_{n+1}} + \dots + \frac{1}{\gamma_{n+d(m)}}\right)$$

and

$$\frac{1}{m+1} \sum_{j=0}^{d(m)-1} (\gamma_{n+(j+1)} - m) \leqq \|f_m\|_1 \leqq \frac{1}{m} \sum_{j=0}^{d(m)-1} (\gamma_{n+(j+1)} - m).$$

Thus, the second inequality to be proved follows from the integral test since $\|f_m\|_1 \leqq d(m)/m$ and $\|f_y\|_1 = \sum_1^{M_y} \|f_m\|_1$.

The first inequality follows from the same argument as given in part b) of Proposition 1.2. qed.

2. The Fourier Series $\sum(d(m)/m)\sin mx$

Define the Fourier series

$$g_{d,y}(x) \sim \sum_1^{M_y} (d(m)/m)\sin mx, \qquad g_d(x) \sim \sum_1^{\infty} (d(m)/m)\sin mx,$$

and

$$g(x) \sim \sum_1^{\infty} (\log m/m)\sin mx.$$

Proposition 2.1. $g, g_d \in (\cap L^p(\mathbb{T})) \backslash L^{\infty}(\mathbb{T})$.

Proof. $g, g_d \in \cap L^p(\mathbb{T})$ since $\sum |\log m/m|^q < \infty$ for each $q>1$ and by the Hausdorff-Young theorem. By general properties of Fourier sine series f with decreasing coefficients $\{a_n\}$, $f \in L^1(\mathbb{T})$ if and only if $\sum a_n/n < \infty$.

Thus, $f \sim \sum (1/\log^2 n) \sin nx \in L^1(\mathbb{T})$ and so $g, g_d \notin L^\infty(\mathbb{T})$ since the duality, $\langle f, g\rangle = \sum (1/\log^2 m)(\log m/m)$, diverges. The fact, $g, g_d \notin L^\infty(\mathbb{T})$, also follows since the L^∞-norms of the Fejèr partial sums are unbounded. qed.

Remark. g and g_d cannot be expected to have bounded mean oscillation since the $A(\mathbb{T})$-norms over lacunary blocks are unbounded:

$$\sum_{e^k}^{e^{k+1}} \log m/m \geqq C((k+1)^2 - k^2).$$

Example 2.1. Writing $g_{d,y}$ as a convolution one obtains

$$\|g_{d,y}\|_{L^\infty(\mathbb{T})} \ll \left\| \sum_1^{M_y} d(m) \cos mx \right\|_1,$$

an estimate of limited value since, by a direct arc length computation, the right hand side is bounded below by $\sum_1^{M_y} d(m)/m$.

Remark. Using the Hausdorff-Young theorem and a standard L^p-estimate of the Dirichlet kernel it follows that

$$\forall p \in (1,2], \quad \|g_{d,y}\|_{L^\infty(\mathbb{T})} \ll \left(\frac{p}{p-1}\right)^{1/p} \exp\left\{\frac{y}{2}\left(1-\frac{1}{p}\right)\right\} \left(\sum_{m=1}^{M_y} \left(\frac{\log m}{m}\right)^p\right)^{1/p}, \tag{2.1}$$

where the constants are independent of y and p. Estimates of the right hand side of (2.1) with p as a function of y do not produce bounds of $\|g_{d,y}\|_{L^\infty(\mathbb{T})}$ better than y^2.

Proposition 2.2. *Let $y = x_y + 2\pi n_y$ where $x_y \in [0, 2\pi)$ and n_y is an integer. Then $g_{d,y}(y) = 0$ if $x_y = 0$ and*

$$g_{d,y}(y) \ll y + \frac{1}{\sin(x_y/2)} \sum_1^{M_y} |d(m+1) - d(m)|/m, \tag{2.2}$$

otherwise.

Proof. The result follows by partial summation using the fact that $|\varepsilon_m(x)| \leqq 2/((m+1)\sin x/2)$, where

$$\sum_1^m \frac{\sin jx}{j} + \varepsilon_m(x) = \frac{\pi - x}{2},$$

$$\varepsilon_m(x) = -\tfrac{\pi}{2} - \frac{\sin mx}{2m} + \tfrac{1}{2}\int_x^{2\pi} \sin mt \cot\frac{t}{2}\, dt,$$

and $x \in (0, 2\pi)$. qed.

Proposition 2.3. a) *If* $\{d(m)\}$ *is eventually increasing then*

$$\|g_{d,y}\|_{L^\infty(\mathbb{T})} \ll y.$$

b) *If* $\{d(m)/m\}$ *has bounded variation then the Fourier series* g_d *converges uniformly on each interval* $[\varepsilon, 2\pi-\varepsilon]$, $\varepsilon>0$.

c) *If there is* $M>0$ *such that*

$$\forall m \geqq M \quad and \quad \forall \delta>0, \qquad \left(1+\frac{1}{m}\right)^\delta \geqq \max\left(\frac{d(m)}{d(m+1)}, \frac{d(m+1)}{d(m)}\right)$$

then

$$g_d(x) \sim \tfrac{1}{2}\pi \log\frac{1}{x}, \qquad x \to 0+.$$

d) *The Fourier series* g_d *converges everywhere except possibly on a set* $E \subsetneqq [0, 2\pi)$ *whose capacity of order* α, $\operatorname{cap}_\alpha E$, *is* 0 *for each* $\alpha \in (0,1)$.

The proof of Proposition 2.3 follows from well-known results in Fourier series. Part a) uses partial summation; parts b) and c) can be found in Zygmund [25, Chap. 1 and p. 188, resp.]; and part d) is in Kahane and Salem [11]. Because of (1.2), the conclusion of part d) can be refined to the statement, $\operatorname{cap}_{\log(1/r)} E = 0$.

3. The Computation of $\lim (1/y)\hat{f}_y(y)$

Assume throughout that the sequence $\{\gamma_j\}$ has the property that

$$\lim_{y\to\infty} \sum_{0<\gamma_j<M_y} \frac{\sin \gamma_j \lambda}{\gamma_j} \tag{3.1}$$

converges uniformly on compact subsets of $[0,\infty)\backslash\{r_n\}$, where $r_n \to \infty$, that the sum function $\sum(\sin\gamma_j\lambda)/\gamma_j$ has a finite jump at each r_n, and that the series converges at each r_n.

$z(y)$ will denote a function for which $\lim_{y\to\infty} z(y)=0$.

The following is clear by Proposition 1.1 and (3.1); more is true from the discussion of capacity in Sect. 2.

Proposition 3.1. $\lim_{y\to\infty} z(y)\hat{f}_y = 0$, *a.e.*

Clearly,

$$\hat{f}_y(0) \leqq \|\hat{f}_y\|_\infty \leqq \|f_y\|_1 = \hat{f}_y(0). \tag{3.2}$$

When (3.2) is combined with the L^1-estimate in Sect. 1 one obtains –

Proposition 3.2. $y \ll \|(1/y)\hat{f}_y\|_\infty \ll y$.

It is also obvious that –

Proposition 3.3. $\hat{f}_y \in A(\mathbb{R}) \cap A'(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$, $\hat{f}_y \notin M(\mathbb{R})$, *and*

$$\lim_{y\to\infty} \|z(y)\hat{f}_y\|_2 = 0 = \lim_{y\to\infty} \|z(y)\hat{f}_y\|_{A'}.$$

Let E_n be a compact interval with center 0, let $V_n=[0,b_n]$, and define

$$\varphi_n=\frac{1}{|V_n|(|E_n|+|V_n|)}\chi_{V_n}*\chi_{E_n-V_n}, \tag{3.3}$$

where $|V|$ is the Lebesgue measure of V. $\varphi_n\geqq 0$ is a "trapezoid" centered at 0, $\int\varphi_n=1$ since

$$\int\varphi_n=\frac{1}{|V_n|(|E_n|+|V_n|)}\hat{\chi}_{V_n}(0)\hat{\chi}_{E_n-V_n}(0) \quad \text{and} \quad |E_n-V_n|=|E_n|+|V_n|,$$

φ_n vanishes off of $E_n+V_n-V_n$, and $\varphi_n=1/(|E_n|+|V_n|)$ on E_n. It is easy to compute

$$\|\varphi_n\|_A\leqq\frac{1}{(|V_n|(|E_n|+|V_n|))^{1/2}} \tag{3.4}$$

and

$$\|\varphi_n\|_2\leqq\frac{1}{(|E_n|+|V_n|)^{1/2}}. \tag{3.5}$$

For example, the latter inequality follows since

$$\int\left|\frac{1}{|V_n|}\chi_{V_n}*\chi_{E_n-V_n}\right|^2=|E_n|+2\int_0^{b_n}\left(\frac{1}{|V_n|^2}x^2-\frac{2}{|V_n|}x+1\right)dx$$
$$=|E_n|+\tfrac{2}{3}|V_n|.$$

Clearly, $\lim\varphi_n=\delta$ in the weak $*$ topology on $M(\mathbb{R})$ when $|E_n|+|V_n|\to 0$. Thus the following result shows that

$$\lim_{y\to\infty}\frac{1}{y}\hat{f}_y(y)=0 \quad \text{in the mean}. \tag{3.6}$$

Proposition 3.4. *Given $y_m\to\infty$. Then $\{E_n\}$ and $\{V_n\}$ can be chosen so that $|E_n|+|V_n|\to 0$ and*

$$\lim_{m\to\infty} z(y_m)\int\hat{f}_{y_m}(y)\varphi_m(y-y_m)dy=0.$$

Proof. From (3.5) and the L^2-bounds of Sect. 1, it follows that

$$|z(y_m)\int\hat{f}_{y_m}(y)\varphi_m(y-y_m)dy|\leqq\frac{Kz(y_m)}{(|E_m|+|V_m|)^{1/2}}.$$

The result is completed by choosing $|E_m|+|V_m|>z(y_m)$ for each given y_m. qed.

The following is relevant since it is not known if the Fourier series g_a converges everywhere.

Proposition 3.5. *Given $\lambda\geqq 1$ and $y_m\to\infty$. If $\{z(y_m)\hat{f}_{y_m}\}$ is equicontinuous at λ then* $\lim_{m\to\infty} z(y_m)\hat{f}_{y_m}(\lambda)=0.$

Proof. Let $a_{m,n}=z(y_m)\int \hat{f}_{y_m}(y)\varphi_n(y-\lambda)dy$ where $|E_n|+|V_n|\to 0$ (it is possible to use a simpler approximate identity for this proof).

Then

$$\lim_{n\to\infty} a_{m,n}=z(y_m)\hat{f}_{y_m}(\lambda) \tag{3.7}$$

since $z(y_m)\hat{f}_{y_m}$ is continuous and $\varphi_n\to\delta$ in the weak $*$ topology.

Also

$$\lim_{m\to\infty} a_{m,n}=0$$

since $|a_{m,n}|\leqq z(y_m)\|\hat{f}_{y_m}\|_2\|\varphi_n\|_2$ and $\{\|\hat{f}_{y_m}\|_2\}$ is bounded.

The convergence in (3.7) is uniform in n because of the equicontinuity.

Thus, by the Moore-Smith theorem, e.g., Benedetto [2, Appendix I], $\lim_m \lim_n a_{m,n}=\lim_n \lim_m a_{m,n}=0$, and the proof is complete. qed.

Remark. In the same way, one sees that if $\{z(y_m)\hat{f}_{y_m}\}$ is a uniformly equicontinuous family on $\{y_m\}$ then

$$\lim z(y_m)\hat{f}_{y_m}(y_m)=0\,.$$

The following is Egorov's theorem, where (3.8) allows the case of an infinite measure space.

Proposition 3.6. *Given $y_m\to\infty$ and assume that*

$$\forall\varepsilon>0,\qquad \left|\bigcup_{n=1}^{\infty}\{\lambda:|z(y_n)\hat{f}_{y_n}(\lambda)|\geqq\varepsilon\}\right|<\infty\,. \tag{3.8}$$

Then for each $\delta>0$ there is a measurable set A, for which $|A|<\delta$, such that

$$\lim_m \sup\{|z(y_m)\hat{f}_{y_m}(\lambda)|:\lambda\notin A\}=0\,.$$

The following can be viewed as a means of quantifying the above equicontinuity criteria.

Proposition 3.7. *Given $y_m\to\infty$. If there is a positive sequence $\{\varepsilon_m\}$ tending to zero for which $\lim y_m\varepsilon_m^{1/2}=\infty$ and either*

$$\int_{|\lambda|\leqq\varepsilon_m} |\hat{f}_{y_m}(y_m)-\hat{f}_{y_m}(\lambda+y_m)|d\lambda\ll\varepsilon_m^{1/2} \tag{3.9}$$

or

$$\|\chi_m(\lambda)(\hat{f}_{y_m}(y_m)-\hat{f}_{y_m}(\lambda+y_m))\|_{A'}\ll\varepsilon_m^{1/2}\,, \tag{3.10}$$

where χ_m is the characteristic function of the interval $[-\varepsilon_m,\varepsilon_m]$, then

$$\lim_{m\to\infty}\frac{1}{y_m}\hat{f}_{y_m}(y_m)=0\,.$$

Proof. Set $\varepsilon_m = |E_m| + |V_m|$. The result follows from Proposition 3.4 and its proof, and by making the corresponding $L^1 - L^\infty$ and $A - A'$ estimates of

$$\frac{1}{y_m}\hat{f}_{y_m}(y_m) - \frac{1}{y_m}\int \hat{f}_{y_m}(\lambda)\varphi_m(\lambda - y_m)d\lambda$$

$$= \frac{1}{y_m}\int \varphi_m(\lambda)(\hat{f}_{y_m}(y_m) - \hat{f}_{y_m}(\lambda + y_m))d\lambda. \quad \text{qed.}$$

Remark. 1. (3.9) and (3.10) are optimal norm estimates. To compare them note that the integral in (3.9) without absolute values is

$$2\varepsilon_m \int f_{y_m}(x)\cos xy_m\left(1 - \frac{\sin \varepsilon_m x}{\varepsilon_m x}\right)dx$$

and this is the same as the Fourier transform of the function in (3.10) evaluated at the origin.

2. It is easy to see that $\hat{f}$, as an L^2-Fourier transform, is continuous a.e.

Example 3.1. The problem of estimating the error term of the prime number theorem given RH leads to the intermediate problem of finding functions $g(y) \ll \log y/\log\log y$ for which the asymptotic behavior of

$$\sum_{\gamma \leqq e^x} \frac{e^{ix\gamma}}{\gamma}$$

can be estimated sharply as $x \to \infty$, where the distribution function γ satisfies

$$\frac{\gamma(y)}{2\pi}\log\frac{\gamma(y)}{2\pi} - \frac{\gamma(y)}{2\pi} + g(y) = y, \qquad y \geqq 1.$$

For example, one takes the case $g(y)$ identically 0. Using Poisson summation one obtains

$$\sum_{\gamma \leqq e^x} \frac{e^{ix\gamma}}{\gamma} = \sum_{n=1}^{X} \frac{e^{ix\gamma(n)}}{\gamma(n)}$$

$$= \frac{1}{2}\left(\frac{e^{ix\gamma(1)}}{\gamma(1)} + \frac{e^{ix\gamma(X)}}{\gamma(X)}\right) + \sum_{k=-\infty}^{\infty} \int_1^X \frac{1}{\gamma(y)} \exp i(x\gamma(y) - 2\pi ky)dy,$$

where $X = \left[\frac{e^x}{2\pi}\log\frac{e^x}{2\pi} - \frac{e^x}{2\pi}\right]$ and $\gamma(X) \leqq e^x < \gamma(X+1)$. Clearly,

$$\gamma'(x) = \frac{2\pi}{\log(\gamma(x)/2\pi)} \quad \text{and} \quad \gamma''(x) = \frac{-4\pi}{\gamma(x)\log^3(\gamma(x)/2\pi)}.$$

Since $(\gamma(y)/2\pi)\log(\gamma(y)/2\pi) - (\gamma(y)/2\pi) = y$, one has

$$\int_1^X \frac{1}{\gamma(y)} \exp i(x\gamma(y) - 2\pi ky)dy$$

$$= \frac{1}{2\pi} \int_{\frac{\gamma(1)}{2\pi}}^{\frac{\gamma(X)}{2\pi}} \frac{\log y}{y} \exp 2\pi i\{yx - yk(\log y - 1)\}dy.$$

The fact that $\frac{d}{dy}\{yx-yk(\log y-1)\}=0$ when $y=e^{x/k}$ and $k\neq 0$ leads one to the transformations $\xi=e^{x/k}$ and $y=\xi(1+\tau)$, and so

$$\sum_{\gamma\leqq e^x}\frac{e^{ix\gamma}}{\gamma}=\frac{1}{2}\left(\frac{e^{ix\gamma(1)}}{\gamma(1)}+\frac{e^{ix\gamma(X)}}{\gamma(X)}\right)+\int_1^X\frac{1}{\gamma(y)}e^{ix\gamma(y)}dy$$

$$+\sum_{k\neq 0}\frac{1}{2\pi}\int_{\frac{\gamma(1)}{2\pi\xi}-1}^{\frac{\gamma(X)}{2\pi\xi}-1}\frac{\log\xi(1+\tau)}{1+\tau}\exp\{-2\pi i\xi k(1+\tau)(\log(1+\tau)-1)\}d\tau\,.$$

Set $p_k(\tau)=p(\tau)=-2\pi k(1+\tau)(\log(1+\tau)-1)$ and $q_k(\tau)=q(\tau)=\log\xi(1+\tau)/(1+\tau)$. Then $p'(\tau)=0$ at $\tau=0$ independently of $k\neq 0$. Also, since $\gamma(1)>0$ is large enough one observes that $p(\tau)$ has no stationary points for $k<0$ and so the above integrals are calculated explicitly in this case using integration by parts. There are technical problems for summing over negative k which have to be dealt with by other summability methods.

4. Explicit Formulas

$\zeta(s)$ is the Riemann zeta function and $\varrho=\sigma+i\gamma$ designates a zero of ζ. $N(T)$ is the number of zeros ϱ of ζ for which $0<\sigma<1$ and $0<\gamma\leqq T$. The Riemann hypothesis, RH, states that if $\zeta(\varrho)=0$ and $\sigma\geqq 1/2$ then $\sigma=1/2$.

The von Mangoldt function Λ is defined as

$$\Lambda(n)=\begin{cases}\log p\,, & \text{if}\quad n=p^m \quad\text{and}\quad p\in P\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

where P is the set of primes and n and m are positive integers; and the Tchebychev function ψ is defined as

$$\psi^*(x)=\sum_{n\leqq x}{}'\Lambda(n)\,,$$

where the dash denotes that if x is a positive integer then $\Lambda(x)$ is to be taken with a factor 1/2.

For the case of RH, Littlewood's explicit formula [13] asserts

$$L(x)-\sum_{|\gamma|<x^{1/2}}\frac{x^\varrho}{\varrho}\ll x^{1/2}\log x\,,\qquad x>4\,,$$

where

$$L(x)=x-\psi^*(x)-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)-\log 2\pi\,,$$

cf., Littlewood [15].

Remark 1. (3.1) is valid for $\zeta(s)$ and for this case r_n has the form $k\log p$; this fact is a simple consequence of the Riemann-von Mangoldt explicit formula, which is more elementary than Littlewood's version. Interesting properties of condition (3.1) have been given by Rademacher [19] and Rubel and Straus [20].

Remark 2. One notes that for the case of $\zeta(s)$ and the imaginary parts γ of its zeros, (1.1) and (1.2) are true. (1.2) is clear from (1.1). The second inequality of (1.1) follows from an argument in elementary function theory or as an easy consequence of the Riemann-von Mangoldt formula which asserts: there is A and T_0 such that

$$\forall T \geqq T_0, \qquad \left| N(T) - \frac{1}{2\pi} T \log T + \frac{1+2\pi}{2\pi} T \right| < A \log T, \tag{4.1}$$

e.g., Ingham [9, p. 71] or Titchmarsh [23, Theorem 9.2]. The first inequality of (1.1) is due to Titchmarsh [23, Theorem 9.14]. If h is given and *large* then this first inequality is an immediate consequence of (4.1) since

$$\forall T \geqq T_0, \qquad N(T+h) - N(T) + \frac{1+\log 2\pi}{2\pi} h > \left(\frac{h}{2\pi} - 2A \right) \log T$$

if $h/2\pi > 2A$.

Propositions 4.1 and 4.2 are usually proved by means of explicit formulas less complicated than Littlewood's; and we refer to the texts in our bibliography as well as the monographs by Prachar, Davenport, and Huxley for details.

Proposition 4.1. *Assume RH and let $C(y) \geqq y$ increase to infinity for all large y. Then*

$$\sum_{0 < \gamma < e^{y/2}} \frac{\sin \gamma y}{\gamma} \ll C(y) \tag{4.2}$$

if and only if

$$\psi^*(x) - x \ll x^{1/2} C(\log x). \tag{4.3}$$

Proof. i) Clearly, one has $-\frac{1}{2} \log\left(1 - \frac{1}{x^2}\right) - \log 2\pi \ll 1$.

ii) By Littlewood's explicit formula and the hypothesis that $C(\log x) \geqq \log x$ for large x, (4.3) is obtained if and only if

$$\sum_{|\gamma| < \sqrt{x}} \frac{x^\varrho}{\varrho} \ll x^{1/2} C(\log x). \tag{4.4}$$

iii) Because of (4.4) it remains to verify that (4.2) holds if and only if

$$\sum_{|\gamma| < \sqrt{x}} \exp\{i\gamma \log x\}/(\tfrac{1}{2} + i\gamma) \ll C(\log x). \tag{4.5}$$

The left hand side of (4.5) has the form

$$A_x - 2i \sum_{0 < \gamma < \sqrt{x}} \frac{\gamma \sin(\gamma \log x)}{\frac{1}{4} + \gamma^2}, \tag{4.6}$$

where $A_x \ll 1$ because of the Riemann-von Mangoldt formula.

Now

$$\sum_{0<\gamma<\sqrt{x}} \frac{\gamma \sin(\gamma \log x)}{\frac{1}{4}+\gamma^2}$$

$$= \sum_{0<\gamma<\sqrt{x}} \frac{\sin(\gamma \log x)}{\gamma} + \sum_{0<\gamma<\sqrt{x}} \gamma \sin(\gamma \log x)\left(\frac{1}{\frac{1}{4}+\gamma^2} - \frac{1}{\gamma^2}\right)$$

$$= \sum_{0<\gamma<\sqrt{x}} \frac{\sin(\gamma \log x)}{\gamma} + B_x, \tag{4.7}$$

where $B_x \ll 1$.

The result follows by combining (4.5)–(4.7) and setting $y = \log x$. qed.

Proposition 4.2 (von Koch, 1901). *If RH is valid then*

$$\psi^*(x) - x \ll x^{1/2} \log^2 x. \tag{4.8}$$

Proof. Proposition 1.3a states

$$\sum_{0<\gamma<e^{y/2}} \frac{\sin \gamma y}{\gamma} \ll y^2 \tag{4.9}$$

and so the result follows from Proposition 4.1. qed.

Von Koch's theorem is proved by means of an explicit formula and routine estimates, e.g., Ingham [9]. Since Littlewood's explicit formula is strong the accompanying estimate, (4.9), in the above proof is particularly simple. Also, because of the rough estimate used to prove (4.9), it is to be expected that the right hand side of (4.9) can be lowered. If it were possible to prove

$$\sum_{0<\gamma<e^{y/2}} \frac{\sin \gamma y}{\gamma} \ll C(y)$$

for some $C(y) \geqq y$ for which $\lim_{y\to\infty} C(y)/y^2 = 0$, then Proposition 4.1 would yield

$$\psi^*(x) - x = o(x^{1/2} \log^2 x), \qquad x \to \infty,$$

which is an improvement over von Koch's theorem, cf., Proposition 5.2.

$\mathscr{S}(\mathbb{R})$ is Schwartz's space of infinitely differentiable rapidly decreasing functions and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions. $C_c^\infty(\mathbb{R})$ is the space of infinitely differentiable functions with compact support and $D(\mathbb{R})$ is the space of distributions. x_a^+ denotes the continuous function on $\mathbb{R}$ which is 0 on $(-\infty, a]$ and which is defined as $x_a^+(x) = x - a$ if $x > a$.

The Mellin transform of $F \in \mathscr{S}(\mathbb{R})$ is defined as

$$\Phi(s) = \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x} dx, \qquad s = \sigma + i\gamma,$$

and the functional W is defined as

$$W(F) = \sum \Phi(\varrho),$$

where the sum is taken over the zeros of the Riemann ζ function in the critical strip $0 \leqq \operatorname{Re} \varrho \leqq 1$. W was also defined in the introduction. If $S \in D(\mathbb{R})$ then the n-th primitive of S is denoted by $S_{(n)}$.

Proposition 4.3. *Assume RH. Then there is a second primitive $W_{(2)}$ of W for which $W_{(2)} \in L^\infty(\mathbb{R})$, and, in particular, $W \in \mathcal{S}'(\mathbb{R})$. In fact (computing directly), $W_{(2)}(x) = -\sum (1/(\operatorname{Im} \varrho)^2) e^{i(\operatorname{Im} \varrho)x}$.*

It can be shown that *RH is valid if, and only if, $W \in \mathcal{S}'(\mathbb{R})$*; and as such the condition in Theorem 6.1b is necessary and sufficient for the validity of RH. To prove that RH implies that $W \in \mathcal{S}'(\mathbb{R})$ one needn't use the Riemann-von Mangoldt formula; the result also follows from properties of *positive definite distributions.*

The following is Weil's explicit formula [24].

$W = W_0 + W_1 + (W_2 + W_3)$, where for $F \in C_c^\infty(\mathbb{R})$, $W_0(F) = -(\log \pi) F(0)$, $W_1(F) = \left\langle \operatorname{Re} \Gamma'/\Gamma\left(\frac{1}{4} + \frac{i}{2}\gamma\right), \hat{F}(\gamma)\right\rangle$,

$$W_2(F) = \int_{-\infty}^{\infty} F(x)(e^{x/2} + e^{-x/2})dx,$$

and

$$W_3(F) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}}(F(\log n) + F(-\log n)).$$

Technically speaking, Littlewood's explicit formula is a special case of Weil's, but, for most applications to $\zeta(s)$, Weil's formula does not seem more effective.

5. $\zeta(s)$ and the Series $\sum (d(m)/m) \sin mx$

Littlewood [14] has proved that if RH is valid then

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O\left(\frac{\log T}{\log\log T}\right), \qquad T \to \infty, \tag{5.1}$$

and so

$$\lim_{m \to \infty} \frac{d(m)}{(1/2\pi)\log m} = 1, \tag{5.2}$$

cf., Landau [6] and Titchmarsh [23, Sect. 14.13]. The proof of the Riemann-von Mangoldt yields

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \tfrac{7}{8} + S(T) + \varepsilon(T), \tag{5.3}$$

where $\varepsilon(T) \ll 1/T$ and $S(T) = (1/\pi) \arg \zeta(\frac{1}{2} + iT)$, and so the basic bound in (5.1) is the estimate (given RH)

$$S(T) \ll \log T / \log\log T, \tag{5.4}$$

cf., Selberg [21, Theorem 1].

Proposition 5.1. *Assume RH. Then*

$$\|g_{d,y}\|_{L^\infty(\mathbb{T})} \ll y^2/\log y. \tag{5.5}$$

Proof. $g_{d,y}(x)$ may be written as

$$\int_1^{M_y} \frac{\sin mx}{m} \int_m^{m+1} dN(t)$$

and so, because of (5.3) and the fact that

$$\int_m^{m+1} dN(t) = \int_m^{m+1} \left(\frac{\log t}{2\pi} - \frac{\log 2\pi}{2\pi}\right) dt + \int_m^{m+1} dS(t) + O(1/m), \qquad m\to\infty,$$

it follows that

$$g_{d,y}(x) = \frac{1}{2\pi} \sum_1^{M_y} \left(\int_m^{m+1} \log t\,dt\right) \frac{\sin mx}{m} + \sum_1^{M_y} \left(\int_m^{m+1} dS(t)\right) \frac{\sin mx}{m} + O(1), \qquad m\to\infty.$$

Using partial summation and the facts that the sequence $\left\{\int_m^{m+1} \log t\,dt\right\}$ increases and the sequence $\left\{\sum_1^{M_y} (\sin mx)/m\right\}$ is uniformly bounded, one computes

$$\sum_1^{M_y} \left(\int_m^{m+1} \log t\,dt\right) \frac{\sin mx}{m} \ll y + \sum_1^{M_y} \left|\int_{m+1}^{m+2} \log t\,dt - \int_m^{m+1} \log t\,dt\right| \ll y.$$

Thus, it is necessary to estimate

$$\sum_1^{M_y} \left(\int_m^{m+1} dS(t)\right) \frac{\sin mx}{m}. \tag{5.6}$$

By partial summation, (5.6) is bounded by

$$|S(e^{y/2})|/e^{y/2} + \left|\sum_1^{M_y} S(m) \left(\frac{\sin(m+1)x}{m+1} - \frac{\sin mx}{m}\right)\right|. \tag{5.7}$$

The first term in (5.7) is bounded independently of y and, because of (5.4), the sum in (5.7) is bounded by

$$\sum_1^{M_y} \frac{\log m}{m(m+1)\log\log m} \left| m\sin(m+1)x - m\sin mx - \sin mx\right| \ll \sum_1^{M_y} \frac{\log m}{m\log\log m}. \tag{5.8}$$

The right hand side of (5.8) is estimated by the integral

$$\int_A^{M_y} \frac{\log x\, dx}{x \log\log x} \ll \int_B^y \frac{u\, du}{\log u}. \tag{5.9}$$

It is well-known that

$$\int_2^y \frac{du}{\log u} = \frac{y}{\log y} + o\left(\frac{y}{\log y}\right), \qquad y \to \infty.$$

Thus the second integral in (5.9) is dominated by $y^2/\log y$ which gives the result. qed.

Remark. The bound $y^2/\log y$ on the integral in (5.9) can't be sharpened since

$$\int (u/\log u) du = u^2/\log u + \int (u/\log^2 u) du.$$

Remark. Selberg [21, 22] has given a strong Ω estimate (i.e., negation of little "o") for $S(T)$ so that, essentially, the term $\log T/\log\log T$ of (5.1) can't be replaced by anything smaller than $(\log T/\log\log T)^{1/2}$, cf., Titchmarsh [23, p. 189 and pp. 295–296] and Montgomery [18].

Propositions 1.1, 4.1, and 5.1 combine to yield –

Proposition 5.2. $\psi^*(x) = x + o(x^{1/2}\log^2 x)$, $x \to \infty$, *if and only if* $\lim_{y\to\infty} \frac{1}{y}\hat{f}_y(y) = 0$.

As far as the Fourier series g_a is concerned it remains to investigate whether or not it converges everywhere in the case of $\zeta(s)$ and RH.

Example 5.1. Because of Proposition 5.2 and conditions such as (3.8) it is relevant to estimate $\|\hat{f}_{y_m} - \hat{f}_{y_n}\|_\infty$. Note that if $\{\gamma_j\}$ is a linearly independent set [which is plausible for $\zeta(s)$] then this estimate is related, up to a factor, $1/\lambda$, to Kronecker's theorem.

Example 5.2. a) Clearly,

$$\frac{1}{y}\hat{f}_y(y) = \frac{1}{y}\int_0^1 \cos xy\left(\sum_{k=1}^{M_y} f(x+k)\cos ky\right)dx$$

$$- \frac{1}{y}\int_0^1 \sin xy\left(\sum_{k=1}^{M_y} f(x+k)\sin ky\right)dx.$$

b) Thus for the case of $\zeta(s)$, $\psi^*(x) = x + o(x^{1/2}\log^2 x)$, $x \to \infty$, is equivalent to each of the following conditions:

$$\frac{1}{y}\sum_{k=1}^{M_y} \frac{1}{k} \begin{matrix}\cos\\ \sin\end{matrix} ky \int_0^1 \begin{matrix}\cos\\ \sin\end{matrix} xy S(k+x)dx = o(1), \qquad y \to \infty,$$

$$\frac{1}{y}\sum_{k=1}^{M_y} \frac{1}{k} \begin{matrix}\cos\\ \sin\end{matrix} ky \int_0^1 \begin{matrix}\cos\\ \sin\end{matrix} xy \left(\int_{x+k}^{k+1} dN(t)\right)dx = o(1), \qquad y \to \infty. \tag{5.10}$$

It is to be expected that the total variation norm of the measure $dS(k+x)$ on $[0,1]$ is much larger than its pseudomeasure norm. In fact, it is easy to compute

$$\int_k^{k+1} |dS(t)| = \frac{1}{\pi}\log k + O\left(\frac{\log k}{\log\log k}\right), \qquad k \to \infty; \tag{5.11}$$

so that if $y_m = 2\pi m$ and the integral $\int_0^1$ in (5.10) is estimated in terms of (5.11) then all that can be said of (5.10) is that it is bounded.

6. The Limitations of Explicit Formulas

Lemma 6.1.

a) $W_{0(2)}(x) = -(\log \pi) x_0^+(x)$;

b) $W_{1(2)}(x) = O(x), \quad x \to \infty$.

Proof. a) is clear.

b) i) Take $a, b > 0$ and write Γ'/Γ as

$$\Gamma'/\Gamma(a+ib\gamma) = -C - \frac{1}{a+ib\gamma} + \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{(m+a)+ib\gamma}\right),$$

where C is Euler's constant.

Consequently,

$$\begin{aligned}\operatorname{Re} \Gamma'/\Gamma(a+ib\gamma) &= -C - \frac{a}{b^2} \frac{C}{(a/b)^2+\gamma^2} + \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{a+m}{b^2} \frac{1}{((a+m)/b)^2+\gamma^2}\right) \\ &= -C - \frac{a}{b^2} \frac{1}{(a/b)^2+\gamma^2} + \sum_{m=1}^{\infty} \frac{a^2+am+\gamma^2 b^2}{b^2 m(((a+m)/b)^2+\gamma^2)}. \qquad (6.1)\end{aligned}$$

ii) The inverse Fourier transform G of the expression (6.1) is

$$G = -C\delta - \frac{a}{b^2} \cdot \frac{b}{a} e^{-|x|a/b} + \sum_{m=1}^{\infty} \frac{a^2+am}{b^2 m} \cdot \frac{b}{a+m} e^{-|x|(a+m)/b}$$
$$+ \delta'' * \sum_1^{\infty} \frac{1}{m} \cdot \frac{b}{a+m} e^{-|x|(a+m)/b}.$$

iii) Clearly, $G_{(2)}(x) = W_{1(2)}(x)$, $x \to \infty$.

The second primitives of the first and last terms of G are $-Cx_0^+(x)$ and $\sum [b/((m+a)m)] e^{-|x|(a+m)/b}$, respectively; and these are $O(x)$, $x \to \infty$.

iv) To estimate the second primitives of the remaining terms of G one calculates $(e^{-|x|r})_{(2)}$ as follows.

$$\begin{aligned}&\left\langle \left(\frac{1}{r^2} e^{-|x|r} + \frac{2}{r} x_0^+(x)\right), F(x) \right\rangle \\ &= \frac{1}{r^2} \int_{-\infty}^{\infty} F^{(2)}(x) e^{-|x|r} dx + \frac{2}{r} \int_0^{\infty} x F^{(2)}(x) dx \\ &= \frac{1}{r^2} \int_{-\infty}^{0} F^{(2)}(x) e^{xr} dx + \frac{1}{r^2} \int_0^{\infty} F^{(2)}(x) e^{-xr} dx - \frac{2}{r} \int_0^{\infty} F^{(1)}(x) dx \\ &= -\frac{1}{r} \int_{-\infty}^{0} F^{(1)}(x) e^{xr} dx + \frac{1}{r} \int_0^{\infty} F^{(1)}(x) e^{-xr} dx + \frac{2}{r} F(0) \\ &= \int_{-\infty}^{\infty} e^{-|x|r} F(x) dx.\end{aligned}$$

Thus, distributionally,

$$\left(\frac{1}{r^2}e^{-|x|r}+\frac{2}{r}x_0^+(x)\right)^{(2)}=e^{-|x|r}.$$

v) Because of part iv) the sum of the second primitives of the second and third terms of G [in part ii)] has the form

$$-\frac{1}{b}\left(\frac{b^2}{a^2}e^{-xa/b}+\frac{2b}{a}x\right)+\sum_{m=1}^{\infty}\frac{a}{bm}\left(\frac{b^2}{(a+m)^2}e^{-x(a+m)/b}+\frac{2b}{(a+m)}x\right),\qquad x>0.$$

Clearly, this expression is $O(x)$, $x\to\infty$, and so the proof is complete. qed.

If $S\in D(\mathbb{R})$ then $\check{S}$ will denote the operation $\langle\check{S}_x, F(x)\rangle=\langle S_x, F(-x)\rangle$. Define

$$V(F)=\sum_{n=1}^{\infty}\int_{n-1}^{n}\left(\frac{F(\log x)}{x^{1/2}}-\frac{\Lambda(n)}{n^{1/2}}F(\log n)\right)dx. \tag{6.2}$$

Lemma 6.2. a) $W_2+W_3=V+\check{V}$.

b) $V=T+U$, *where*

$$T(F)=\sum_{n=1}^{\infty}\int_{n-1}^{n}\left(\frac{F(\log n)}{x^{1/2}}-\frac{\Lambda(n)F(\log n)}{n^{1/2}}\right)dx,$$

$$U(F)=\sum_{n=1}^{\infty}\int_{n-1}^{n}\frac{F(\log x)-F(\log n)}{x^{1/2}}dx.$$

c) *There is a second primitive* $U_{(2)}$ *of* U *for which* $U_{(2)}\in L^\infty(\mathbb{R})$, *and, in particular,* $U\in\mathcal{S}'(\mathbb{R})$.

Proof. a) The change of variable $u=-x$ yields

$$\int_{-\infty}^{\infty}F(x)(e^{x/2}+e^{-x/2})dx=\int_{-\infty}^{\infty}e^{x/2}(F(x)+F(-x))dx; \tag{6.3}$$

and the change of variable $u=e^x$ in (6.3) yields

$$W_2(F)=\int_0^{\infty}\frac{1}{x^{1/2}}(F(\log x)+F(-\log x))dx. \tag{6.4}$$

Combining (6.4) with the definition of W_3 one obtains

$$\begin{aligned}(W_2+W_3)(F)&=\left(\int_0^{\infty}\frac{1}{x^{1/2}}F(\log x)dx-\sum_1^{\infty}\frac{\Lambda(n)}{n^{1/2}}F(\log n)\right)\\&\quad+\left(\int_0^{\infty}\frac{1}{x^{1/2}}F(-\log x)dx-\sum_1^{\infty}\frac{\Lambda(n)}{n^{1/2}}F(-\log n)\right)\\&=(V+\check{V})(F).\end{aligned}$$

b) This follows by adding and subtracting $F(\log n)/x^{1/2}$.

c) i) Define $U_n(F)=\int\limits_{n-1}^{n}\frac{1}{x^{1/2}}(F(\log x)-F(\log n))dx$. Thus, by the change of variables $y=\log x$,

$$U_n(F)=\int\limits_{\log(n-1)}^{\log n} F(y)e^{y/2}dy-2F(\log n)(n^{1/2}-(n-1)^{1/2})$$
$$=\langle\chi_{\log n}(x)\,e^{x/2},\,F(x)\rangle-\langle 2(n^{1/2}-(n-1)^{1/2})\delta_{\log n},F\rangle,$$

where $\chi_{\log n}$ is the characteristic function of the interval $(\log(n-1), \log n]$.

Therefore,

$$U_n=e^{x/2}\chi_{\log n}(x)-2(n^{1/2}-(n-1)^{1/2})\delta_{\log n}$$

and

$$U=\sum_1^\infty U_n=e^{x/2}-2\sum_1^\infty(n^{1/2}-(n-1)^{1/2})\delta_{\log n}.$$

Consequently,

$$U_{(2)}(x)=4e^{x/2}-2\sum_{n=1}^{\infty}(n^{1/2}-(n-1)^{1/2})x^{+}_{\log n}(x). \tag{6.5}$$

ii) On $(-\infty,0]$, one has $|U_{(2)}(x)|\leqq 4$.

To estimate $U_{(2)}(x)$ on $(0,\infty)$, let $x=\log m$, $m>1$.

Then, by the definition of x^+ and some routine calculations we obtain the estimate,

$$4(N+1)^{1/2}-2(N+1)^{1/2}\log(N+1)+\sum_2^N\frac{\log n}{n^{1/2}}$$
$$\leqq U_{(2)}(\log m)\leqq\sum_2^N\frac{\log n}{n^{1/2}}+C+4N^{1/2}-2N^{1/2}\log N$$

for all large m.

iii) Now let $x\in(\log m, \log(m+1))$. Then

$$U_{(2)}(x)\leqq K+2\sum_{n=1}^{m}(n^{1/2}-(n-1)^{1/2})\left(\log\frac{m+1}{n}-\log\frac{m}{n}\right)$$
$$=K+2\left(\log\left(1+\frac{1}{m}\right)\right)m^{1/2}\leqq K+2/m^{1/2}.$$

Similarly, one obtains a lower bound for $U_{(2)}(x)$. qed.

Theorem 6.1. *Assume RH and set* $a_n=2(n^{1/2}-(n-1)^{1/2})-\Lambda(n)/n^{1/2}$. *Then*

a) $f(x)=T_{(2)}(x)=\sum_1^\infty a_n x^{+}_{\log n}(x)=O(x),\quad x\to\infty,$

b) $\sum_1^m a_n\log\frac{m}{n}=O(\log m),\quad m\to\infty.$

Proof. a) $W_{(2)}$ is an element of $L^\infty(\mathbb{R})$ because of Proposition 4.3 and RH. Thus, by Weil's explicit formula and Lemmas 6.1 and 6.2, the result will follow from

$$T_{(2)}(x) = \sum_1^\infty a_n x^+_{\log n}(x). \tag{6.6}$$

The computation,

$$\begin{aligned}\langle T, F\rangle &= \sum_{n=1}^\infty \int_{n-1}^n \left(\frac{F(\log n)}{x^{1/2}} - \frac{\Lambda(n)F(\log n)}{n^{1/2}}\right)dx \\ &= \sum_{n=1}^\infty [2(n^{1/2}-(n-1)^{1/2}) - \Lambda(n)/n^{1/2}]F(\log n) \\ &= \sum_{n=1}^\infty a_n \langle \delta_{\log n}, F\rangle,\end{aligned}$$

yields $T = \sum_1 a_n \delta_{\log n}$ and (6.6) follows.

b) Let $x = \log m$ so that if $\log n \leqq x$ then $x^+_{\log n}(x) = \log\frac{m}{n}$. qed.

Obviously, $f(x)/x \in L^\infty(\mathbb{R})$.

The verification of the next result follows by partial summation.

Theorem 6.2. *The estimate,*

$$\sum_1^m a_n = O(\log m), \quad m \to \infty, \tag{6.7}$$

is equivalent to the estimate,

$$\psi(x) = x + O(x^{1/2}\log x), \quad x \to \infty. \tag{6.8}$$

Example 6.1. a) Assume RH. If it were possible to use the Weil explicit formula to verify (6.8), without having additional information about $\{a_n\}$, then it would be necessary to make estimates such as $W_{(1)}(x) = O(x)$, $x \to \infty$, cf., the proof of Theorem 6.1 as well as the statements of Propositions 4.1 and 4.2. A similar remark applies to the verification of (2) in the introduction, instead of (6.8).

b) There are certain harmonic analysis implications about the estimate, $W_{(1)}(x) = O(x)$, $x \to \infty$. These are discussed now and in part c).

$W_{(2)}$ is the Fourier transform of a discrete measure. $W_{(1)} \in \mathcal{S}'(\mathbb{R}) \setminus (L^\infty(\mathbb{R}) \cup L^2(\mathbb{R}))$ and it is a B^2-almost-periodic function. Clearly,

$$\forall F \in \mathcal{S}(\mathbb{R}), \quad \langle W_{(1)}, F\rangle = -i \sum_{-\infty}^\infty (1/\gamma_j)\hat{F}(\gamma_j).$$

Fix $\varepsilon > 0$ and choose $K > 0$ so that $\gamma_j < 2\pi K/\log K$ if $j < K$. Define

$$\Phi_N(x) = \chi_{[\varepsilon,\infty)}(x)\frac{1}{x}\sum_{n=K}^N (1/\gamma_n)e^{ix\gamma_n},$$

noting that, formally, $W_{(1)}(x)$ is given by $-i\sum(1/\gamma_j)e^{i\gamma_j x}$.

It is easy to see that $\{\Phi_N\} \subseteq L^\infty(\mathbb{R})$ *converges in the weak * topology if, and only if, for each* $g \in L^1(\mathbb{R})$, *for which* $g=0$ *on* $(-\infty, \varepsilon]$ *and* $xg(x) \in L^1(\mathbb{R})$,

$$\sum_K^\infty (1/\gamma_n)\hat{g}(\gamma_n) \tag{6.9}$$

converges.

Similarly, $\{\Phi_N\} \subseteq L^\infty(\mathbb{R})$ *converges in the weak * topology if, and only if, for each* $g \in L^1(\mathbb{R})$, *for which g is locally absolutely continuous on* $(-\infty, \varepsilon]$, $xg(x) \in L^1(\mathbb{R})$, *and* $g' \in L^1(-\infty, \varepsilon)$, one *can conclude that the series of* (6.9) *converges.*

Note that the series of (6.9) converges whenever $g \in C_c(\varepsilon, \infty)$.

c) There is a natural playoff between the various hypotheses on g in part b) and the divergence of (6.9). Taking the simplest model, $\gamma_n = n/\log n$, this playoff is illustrated in the following two examples.

i) Let $g = \sum_{n \geqq K} g_n$ where $\hat{g}_n = \varphi_n$ and

$$\varphi_n(\gamma) = \frac{1}{(\log n)^2} \exp\{-r(n)(\gamma - n/\log n)^2\}, \qquad r(n) > 0.$$

Thus $\hat{g} \geqq 0$ and

$$g_n(x) = \frac{e^{-ixn/\log n}}{2(\log n)^2 \sqrt{\pi r(n)}} \exp\{-x^2/(4r(n))\}.$$

Setting $f(x) = xg(x)$ one obtains

$$\int_{-\infty}^{\infty} |f(x)| dx \leqq 2\sqrt{\pi} \sum_{n=K}^{\infty} \frac{(r(n))^{1/2}}{(\log n)^2}$$

and hence $f \in L^1(\mathbb{R})$ if $r(n) \leqq 1/n^2$. On the other hand, $\|g_n\|_{L^1(\mathbb{R})} = 1/(\log n)^2$ and so there are difficulties in examining directly the integral, $\int \Phi_N f$. If $1/(\log n)^2$ is replaced by $1/(n(\log n)^2)$ in the definition of φ_n then $g, f \in L^1(\mathbb{R})$ and $\lim \int \Phi_N f$ exists.

ii) Let $g = \sum_{n \geqq K} g_n$ where $\hat{g}_n = \psi_n$ and $\psi_n(\gamma) = (1/\log n)^2 t_n * \varrho_n(\gamma)$, where t_n is a trapezoid of height 1 centered at $n/\log n$ and $\{\varrho_n\}$ is a C_c^∞-approximate identity. Then choose t_n and ϱ_n so that $\operatorname{supp}\psi_n \cap \operatorname{supp}\psi_m = \emptyset$ [as opposed to part c)i)]. Note that the distance between $n/\log n$ and $(n+1)/\log(n+1)$ is about $1/\log n$; and that

$$\|(1/\log n)^2 t_n * \varrho_n\|_{A(\mathbb{R})} \leqq \frac{1}{(\log n)^2} \|t_n\|_{L^2(\mathbb{R})} \|\varrho_n\|_{L^2(\mathbb{R})}, \tag{6.10}$$

by a simple application of Hölder's inequality and the Plancherel theorem. Because of (6.10) one can make a good estimate of $\sum \|\psi_n\|_{A(\mathbb{R})}$ and $\sum \|\psi_n'\|_{A(\mathbb{R})}$ in order to guarantee that g and f are integrable. The playoff mentioned at the beginning of part c) is manifested as follows. If a translate of $\operatorname{supp}\varrho_n$ is contained in $\operatorname{supp} t_n$ then the divergence of (6.9) is obtained; but in this case, $\sum \|\psi_n'\|_{A(\mathbb{R})}$ diverges.

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