A Wiener-Wintner theorem for 1/f power spectra

John J. Benedetto
Department of Mathematics
University of Maryland
College Park, MD 20742
jjb@math.umd.edu
(301) 405-5161(TEL) (301) 314-0827(FAX)

Sherry E. Scott
Department of Mathematics
Bowie State University
Bowie, MD 20715
sscott@bowiestate.edu
(301) 860-3356(TEL) (301) 860-3364(FAX)

Rodney Kerby
Department of Mathematics
Morgan State University
Baltimore, MD 21251
rkerby@morganstate.edu
(443) 885-4650(TEL) (410) 319-4323(FAX)

Abstract

Wiener's Generalized Harmonic Analysis (GHA) provides a theory of harmonic analysis for subspaces of tempered functions not accessible to the L^1, L^2, and Fourier series theories; and it does it in a way that is usually more quantitative than that provided by the theory of distributions. On the other hand, GHA does not yield an adequate spectral analysis of large classes of functions, including nonstationary processes and, in particular, $1/f$ noise. In this paper we adapt GHA to deal with $1/f$ noise by extending the Wiener-Wintner theorem to the case of $1/f$ power spectra.

Keywords and phrases: Wiener's Generalized Harmonic Analysis, $1/f$ noise, autocorrelation, power spectra, fractional Brownian motion.
2000 Mathematics Subject Classi£cation: 42A38, 42A99, 60H40.

1 Introduction

Wiener's Generalized Harmonic Analysis (GHA) originally addressed the problem of providing a quantitative Fourier analysis of locally integrable functions on $\mathbb{R}^d$ which were neither integrable nor square-integrable [27], [28]. Wiener's theory from about 1930 preceded Schwartz' more qualitative Fourier analysis of the space $\mathcal{S}^{'}(\mathbb{R}^d)$ of tempered distributions [23] by 20 years. The Wiener methods allowed for the Fourier analysis of functions in $L^\infty(\mathbb{R}^d)$, and included an analogue of the Plancherel theorem of the L^2 theory. This analogue, the so-called Wiener-Plancherel theorem, was originally proved on $\mathbb{R}$ [27], [28], but see [2] and [5] for the formulation in $\mathbb{R}^d$; and it is intimately connected with Wiener's fundamental work on Tauberian theorems, e.g., [12], [17], the spectral synthesis problem, e.g., [4], and power spectrum analysis, e.g., [22].

In this paper we shall focus on Wiener's approach to power spectrum analysis, and, in particular, on a joint result with Wintner, the Wiener-Wintner theorem [29]. This result, stated in Theorem 1.1, allows one to associate a power spectrum to a signal in $L^\infty(\mathbb{R}^d)$, which can be thought of as an "intelligent message" from $L^1(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$ combined with additive noise. Although Theorem 1.1 is deterministic in formulation, it is one of the ideas that led to Wiener's statistical approach to signal analysis, e.g., [30], [31], and to his formulation of Brownian motion which is one of the examples considered herein. In fact, Wiener's ideas in this area are duly noted by Shannon, who credits Wiener in [30] as giving "the £rst clear-cut formulation of communication theory as a statistical problem" [24, p. 53]. Theorem 1.1 itself has the celebrated and fundamental statistical formulation, the Wiener-Khintchine theorem, as well as various extensions, e.g., [22, Volume I, Section 4.8.2].

In order to state Theorem 1.1 we need the following formal background. For a given function $\phi \in L^2_{loc}(\mathbb{R})$, the *deterministic autocorrelation* P_ϕ is formally de£ned as

$$P_\phi(t) = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \phi(t+u)\overline{\phi(u)}du. \tag{1.1}$$

If the limit in (1.1) exists, P_ϕ is a positive-de£nite function; and so, by the Herglotz-Bochner theorem, there is a positive bounded Radon measure μ (see Section 2 for a de£nition) for which μ is the Fourier transform of P_ϕ. In this case, μ is the *power spectrum* of ϕ.

Typically, we can think of $\phi \in L^\infty(\mathbb{R})$ as noisy data containing an intelligent message, by which we mean a signal with relatively simple frequency components. Direct analysis of ϕ, e.g., a speech signal embedded in a noisy environment, may be impenetrable, whereas knowledge of μ may provide an accessible frequency structure of the intelligent message.

The Fourier transform of $f : \mathbb{R}^d \to \mathbb{C}$ is

$$\hat{f}(\gamma) = F(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \gamma \rangle} dx,$$

where $\gamma \in \widehat{\mathbb{R}}^d$. $\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as a spectral domain. The inverse Fourier transform of $F : \widehat{\mathbb{R}}^d \to \mathbb{C}$ is

$$F^{\vee}(x) = \int_{\widehat{\mathbb{R}}^d} F(\gamma) e^{2\pi i \langle x, \gamma \rangle} d\gamma.$$

Fourier transform theory is highly developed for the space of tempered distributions which includes $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, and $L^\infty(\mathbb{R}^d)$. On the other hand, there are many quantitative results for $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, including conditions for the equality $f = \check{F}$ in case $f \in L^1(\mathbb{R}^d)$ and the fact that $\widehat{L^2(\mathbb{R}^d)} = L^2(\widehat{\mathbb{R}}^d)$, as well as a methodology which is more calculable than what is generally available for distributions. GHA extends this methodology to $L^\infty(\mathbb{R}^d)$.

An important facet of that extension is the converse to the mapping of ϕ to P_ϕ to μ that we described above. This converse is the Wiener-Wintner theorem:

Theorem 1.1 (Wiener-Wintner theorem).
Let μ be a positive bounded Radon measure on $\widehat{\mathbb{R}}$. There is a constructible function $\phi \in L^\infty_{loc}(\mathbb{R})$ whose deterministic autocorrelation P_ϕ exists for each $t \in \mathbb{R}$ and which has the property that $\check{\mu} = P_\phi$ on $\mathbb{R}$.

Our main result is Theorem 3.1, which extends Theorem 1.1 to positive unbounded Radon measures $\mu \in L^1_{loc}(\widehat{\mathbb{R}})$ of the form

$$\mu_\alpha(\gamma) = \frac{1}{|\gamma|^\alpha}, \quad 0 < \alpha < 1. \tag{1.2}$$

In the case that the temporal data corresponding to (1.2) is considered as noise, then it would be $\frac{1}{f}$ - noise, where the frequency f is written as γ in this paper since we use f to designate functions.

Section 2 provides perspective, a rationale for addressing this problem, and some mathematical background and notation. Section 3 states the main theorem and an outline of our approach; and Section 4 completes the remaining details of the proof.

2 Preliminaries

We shall use the standard notation from harmonic analysis, for example as found in [23] and [26]. $C_c^\infty(\mathbb{R}^d)$ designates the space of compactly supported in£nitely differentiable functions on $\mathbb{R}^d$. $C_b(\mathbb{R}^d)$, resp., $C_c(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$, are linear subspaces of continuous functions on $\mathbb{R}^d$ which are bounded, resp., compactly supported and vanishing at infinity. The support of f is designated $supp\, f$.

Definition 2.1 (Radon measures).

a. A linear functional $\mu : C_c(\mathbb{R}^d) \to \mathbb{C}$ is a *Radon measure* if $\lim_{n\to\infty} \mu(f_n) = 0$ for every sequence $\{f_n\} \subseteq C_c(\mathbb{R}^d)$ satisfying the properties:

   i. $\exists K \subseteq \mathbb{R}^d$, compact, such that $\forall n$, $supp\, f_n \subseteq K$;

   ii. $\lim_{n\to\infty} \|f_n\|_\infty = 0$.

   The space of Radon measures on $\mathbb{R}^d$ is designated $M(\mathbb{R}^d)$.

b. The space $M^+(\mathbb{R}^d)$ of *positive Radon measures* consists of those $\mu \in M(\mathbb{R}^d)$ for which $\mu(f) \geq 0$ for every non-negative $f \in C_c(\mathbb{R}^d)$.

c. The linear space $M_b(\mathbb{R}^d)$ of *bounded Radon measures* is the dual space of $C_c(\mathbb{R}^d)$ or $C_0(\mathbb{R}^d)$ taken with the sup norm $\|\ldots\|_\infty$ topology. $M_b(\mathbb{R}^d)$ is a Banach algebra under convolution and with total variation norm $\|\ldots\|_1$.

By definition, we set $M_b^+(\mathbb{R}^d) = M_b(\mathbb{R}^d) \cap M^+(\mathbb{R}^d)$.

Remark 2.2 (The structure of measures on $\mathbb{R}$).

a. Let $BV(\mathbb{R})$ be the space of functions having bounded variation on $\mathbb{R}$, and let $BV_{loc}(\mathbb{R})$ be the space of functions on $\mathbb{R}$ having bounded variation on all finite intervals. A distributional formulation of the Riesz representation theorem allows us to assert that

$$M(\mathbb{R}) \;=\; \{f^{'} : f \in BV_{loc}(\mathbb{R})\}$$

   and

$$M_b(\mathbb{R}) \;=\; \{f^{'} : f \in BV(\mathbb{R})\},$$

   where $f^{'}$ is the first distributional derivative of f, e.g., [23].

b. Let $\mu \in M_b(\mathbb{R})$ and let $f^{'} = \mu$ distributionally. Then $\mu = g + \mu_{sc} + \sum a_t\delta_t$, where $g \in L^1(\mathbb{R})$, μ_{sc} is the continuous singular part of μ, and $\sum a_t\delta_t$ is the discrete part of μ, where $\sum |a_t| < \infty$ and δ_t is the Dirac δ-measure supported by t, e.g., see [6].

c. Clearly, $L^1(\mathbb{R}) \subseteq M_b(\mathbb{R})$ and $L^1_{loc}(\mathbb{R}) \subseteq M(\mathbb{R})$.

We shall need the following form of Paul Lévy's theorem, e.g., [16] for proof cf., [19, p.172] for a probabilistic formulation and proof as well as [19, pp. 91-93].

Theorem 2.3 (Lévy theorem).
Let $\{\mu_n : n = 1, ...\} \subseteq M_b^+(\widehat{\mathbb{R}}^d)$ *and let* $\mu \in M_b^+(\widehat{\mathbb{R}}^d)$.

a. *If* $\{\mu_n\}$ *converges to* μ *in the vague topology* $\sigma\left(M_b(\widehat{\mathbb{R}}^d), C_c(\widehat{\mathbb{R}}^d)\right)$ *and* $\lim_{n\to\infty} \|\mu_n\|_1 = \|\mu\|_1$, *then* $\{\check{\mu}_n\}$ *converges to* $\check{\mu}$ *uniformly on compact sets of* $\mathbb{R}^d$.

b. *If* $\{\check{\mu}_n\}$ *converges pointwise to a function* ϕ *on* $\mathbb{R}^d$ *and* ϕ *is continuous at 0, then there is a* $\mu \in M_b^+(\widehat{\mathbb{R}}^d)$ *such that* $\{\mu_n\}$ *converges to* μ *in the vague topology* $\sigma\left(M_b(\widehat{\mathbb{R}}^d), C_c(\widehat{\mathbb{R}}^d)\right)$ *and* $\lim_{n\to\infty} \|\mu_n\|_1 = \|\mu\|_1$.

We distinguish between the vague topology and the weak* topology $\sigma(M_b(\widehat{\mathbb{R}}^d), C_0(\widehat{\mathbb{R}}^d))$, e.g., [9]. Also, the term *narrow* topology is often used when combining a weak convergence with convergence of norms as in Theorem 2.3, see [4] for applications.

Remark 2.4 (The proof and extension to $\mathbb{R}^d$ of Wiener-Wintner).

a. In Wiener and Wintner's proof of Theorem 1.1 [29], a sequence $\{\mu_n\}$ of measures is constructed which converges in the vague topology $\sigma(M_b(\widehat{\mathbb{R}}), C_c(\widehat{\mathbb{R}}))$ to the given measure $\mu \in M_b^+(\widehat{\mathbb{R}})$ and which has certain desirable technical properties. These properties allow for the construction of a sequence $\{\phi_n\} \subseteq L^\infty(\mathbb{R})$ for which the following uniform convergence estimate is valid:

$$\forall n \geq 1, \quad \exists A_n \geq A_{n-1} \text{ such that } \forall t \in \mathbb{R} \text{ and } \forall R \geq A_n,$$
$$\left| \frac{1}{2R} \int_{-R}^{R} \phi_n(t+x)\overline{\phi_n(x)}dx - \check{\mu}_n(t) \right| < \frac{1}{2^{n+1}}.$$

With this estimate, one sets $R_n = (A_1 + 1)(A_2 + 2)...(A_n + n)$, and then de£nes ϕ on $\mathbb{R}$ by the rule that $\phi(t) = \phi_n(t)$ for $R_n < |t| < R_{n+1}$. It can then be checked that $\check{\mu} = P_\phi$, see also the detailed presentation by Bass [1] with a clari£cation by Kerby [16], cf., Bertrandias [8].

b. Theorem 1.1 was extended to $\mathbb{R}^d$ by Kerby [16] (1990) and Benedetto [5] (1991), the latter containing an application to spectral analysis. The proof of Wiener-Wintner in [5] requires a condition which Kerby showed can be removed. The proofs in [16] and [5] depend on essentially different ideas, the former inspired to some extent by [8] and the latter by [29]. Lévy's theorem, Theorem 2.3, plays a role in these approaches as well as in this paper.

Example 2.5 (Fractal noise stochastic processes).

Our extension Theorem 3.1 of Theorem 1.1 is to the case of $1/f$ noise, i.e., to the case of spectral data "$1/f = 1/|\gamma|^{\alpha}$", and our proof is relatively constructive. However, it would be misleading to think of our approach as implementable. In fact, we view our theorem as a nexus between the classical origins of GHA and the more recent theoretical developments dealing with 1/f - noise, especially those where wavelet theory plays a role, e.g., [10], [11], [32], [33]. Of course, this connection is based on parts of classical harmonic analysis which hopefully have not outlived their usefulness, especially as regards unresolved issues in the spectral analysis of fractal noise.

a. The autocorrelation in (1.1) is the deterministic analogue of the probabilistic autocorrelation $R_X(t,s) = E\{X(t,\omega)\overline{X(s,\omega)}\}$ used in the Wiener-Khintchine theorem. Here X is a stochastic process with time variables t and s, probability space variable ω, and expectation E. The Wiener(1930)-Khintchine(1934) theorem appplies in the case of wide sense stationary stochastic processes X. (Of course, the more dif£cult direction of the Wiener-Khintchine theorem on $\mathbb{R}$ is an immediate consequence of Bochner's theorem(1933) on $\mathbb{R}$, which is an extension of Herglotz' theorem (1911) in the Fourier series case. Details for the easier direction, constructing the appropriate process X, are found in [22, pp. 221-222].)
On the other hand there are many important non-stationary processes where experimental evidence indicates a spectrum of the form $1/|\gamma|^{\alpha}$ or where theoretical modelling goes beyond standard methods, but where an effective power spectrum can be de£ned and analyzed. As a speci£c example, we mention the study of epileptic seizure prediction in terms of electrocorticogram time series analysis found in [3].
Generally, and keeping in mind the goal of formulating a realistic notion of power spectrum for various non-stationary processes, we make the following simplistic ill-posed but realistic de£nition, see part b. A 1/f *-process* X is a stochastic process for which a natural notion of power spectrum μ arises experimentally and satis£es the following inequalities: $A/|\gamma|^{\alpha} \le \mu(\gamma) \le B/|\gamma|^{\alpha}$. The inequalities are to hold over some frequency band of interest for a £xed so-called *fractal parameter* $\alpha \in [0,2]$. The values $\alpha = 0$ and $\alpha = 2$ correspond, in the £rst approximation, to white noise and Brownian motion processes, respectively, e.g., [11], [21].

b. By de£nition a *fractional Brownian motion* with *Hurst exponent* $H \in (0,1)$ is a zero-mean Gaussian stochastic process $X = B_H$ such that $B_H(0,\omega) =$

0 and for which there is $\sigma^2 > 0$ such that

$$\forall s \in \mathbb{R}\,, \quad E\{|B_H(t,\omega) - B_H(t-s,\omega)|^2\} = \sigma^2 |s|^{2H}, \qquad (2.1)$$

e.g., [13], [14], [15], [20], [32], [33]. Classical Brownian motion corresponds to the case $H = 1/2$.
It is not dif£cult to prove that B_H is non-stationary and that, for each $s \in \mathbb{R}$, the increment $B_H(t,\omega) - B_H(t-s,\omega)$ is stationary, e.g., [18, pp. 211 ff.]. Then, by direct calculation of the distributional Fourier transform F of the power weight $|x|^{2H}$, e.g., [33], cf., [25], the power spectrum $\mu_{H,s}$ of the increment $B_H(t,\omega) - B_H(t-s,\omega)$ is

$$\mu_{H,s}(\gamma) = \frac{\sigma_H^2}{|\gamma|^{2H+1}} \left|1 - e^{-2\pi i s\gamma}\right|^2, \qquad (2.2)$$

where $\sigma_H^2 = \sigma^2 \lambda_H$ and $F(\gamma) = -\lambda_{_H} |\gamma|^{-(2H+1)}$, $\lambda_H > 0$. Consequently, when one divides out in (2.2) by the increment factor $\left|1 - e^{-2\pi i s\gamma}\right|^2 = \left|\widehat{(\delta - \tau_s\delta)}(\gamma)\right|^2$, where δ is the Dirac measure at 0 and $\tau_s\delta$ is its translation by s, it is not unnatural to de£ne the $power\,spectrum$ of B_H as

$$\mu_{B_H}(\gamma) = \frac{\sigma_H^2}{|\gamma|^{2H+1}}. \qquad (2.3)$$

Obviously this line of reasoning can be extended to consider temporal increment factors more general than $\delta - \tau_s\delta$ as well as non-Gaussian processes with stationary increments. Further, taking $H = 1/2$ in (2.3) is in line with our assertions in part *a* and at the beginning of this subsection about Brownian motion.

3 Theorem and outline of proof

3.1 Statement of theorem and elementary approximations

We shall prove the following result.

Theorem 3.1 (Wiener-Wintner theorem for $\mu(\gamma) = 1/|\gamma|^{\alpha}$).
Let $\mu(\gamma) = \frac{1}{|\gamma|^{\alpha}}, \quad 0 < \alpha < 1$ £xed and $\gamma \neq 0$. There is a constructible function $\phi \in L^{\infty}_{loc}(\mathbb{R})$ whose deterministic autocorrelation P_ϕ exists for each $t \in \mathbb{R}$, $t \neq 0$, and which has the property that $\check{\mu} = P_\phi$ on $\mathbb{R}$.

In the classical Wiener-Wintner theorem and its extension to $\mathbb{R}^d$, the given measure is bounded and positive. In our case, $\mu(\gamma) = \frac{1}{|\gamma|^\alpha}$, with $0 < \alpha < 1$, is a positive but unbounded measure in $L^1_{loc}(\mathbb{R})$. Thus, instead of dealing with μ directly, we consider auxiliary functions μ_n de£ned in equation (3.1).

The general set-up is as follows. Let $0 < \alpha < 1$ be £xed, so that

$$\mu(\gamma) = \frac{1}{|\gamma|^\alpha} \in L^1{}_{loc}(\mathbb{R}) \cap M_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$$

and let

$$\mu_n(\gamma) = nw_{\frac{2\pi}{n}}(\gamma)\mu(\gamma) \in L^1(\mathbb{R}) \cap M_b{}^+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R}), \tag{3.1}$$

where

$$nw_{\frac{2\pi}{n}}(\gamma) = \left(\frac{sin(\frac{\pi\gamma}{n})}{\frac{\pi\gamma}{n}}\right)^2.$$

Then

$$\check{\mu}(t) = C_\alpha |t|^{\alpha-1}, \tag{3.2}$$

where C_α is a constant that depends on α and where the 1 in the exponent denotes the dimension, see [25] for a proof of (3.2). Also,

$$\check{\mu}_n(t) = \Delta_n * \check{\mu}(t)$$

where

$$(nw_{\frac{2\pi}{n}})^\vee(t) = \Delta_n(t) = n\max(1 - n\,|t|\,, 0).$$

Since Δ_n is an approximate identity, we can prove the following result.

Proposition 3.2. *For* $t \neq 0$,

$$\lim_{n\to\infty} \check{\mu}_n(t) = \check{\mu}(t). \tag{3.3}$$

Proof. $\check{\mu}(x) = \frac{1}{|x|^{1-\alpha}}$ is continuous for each $x \neq 0$, and so, given $t \neq 0$ and $\epsilon > 0$, there exists η with $|t| > \eta > 0$ such that if $0 < |u| < \eta$ then $|\check{\mu}(t) - \check{\mu}(t-u)| < \epsilon$. Thus, if $\frac{1}{N} < \eta$, then, for all $n \geq N$,

$$|\check{\mu}(t) - \check{\mu}_n(t)| = |\check{\mu}(t) - \Delta_n * \check{\mu}(t)|$$

$$= \left|\int \Delta_n(u)\,(\check{\mu}(t) - \check{\mu}(t-u))\,du\right| \leq \int |\Delta_n(u)|\,|\check{\mu}(t) - \check{\mu}(t-u)|\,du$$

$$= \int_{|u|<\eta} |\Delta_n(u)|\, |\check{\mu}(t) - \check{\mu}(t-u)|\, du + \int_{|u|\geq\eta} |\Delta_n(u)|\, |\check{\mu}(t) - \check{\mu}(t-u)|\, du. \tag{3.4}$$

The £rst integral in (3.4) is less than ϵ by continuity, and the second integral in (3.4) is 0 since $\eta > \frac{1}{N} \geq \frac{1}{n}$ for all $n \geq N$, which in turn implies that the set $\{|u| \geq \eta\}$ is disjoint from the support of Δ_n. Therefore, for a given $t \neq 0$ and $\epsilon > 0$, there exists N such that for all $n \geq N$,

$$|\check{\mu}(t) - \check{\mu}_n(t)| < \epsilon.$$

□

3.2 The role of Lévy's theorem and the classical Wiener-Wintner theorem

Since μ_n is a bounded positive measure for each n, it is well known that there is a sequence $\{\mu_j^n\} \subset M_b^+(supp\, \mu_n)$ of positive £nitely supported discrete measures with corresponding signals ϕ_j^n such that the following properties hold, also see Remark 2.4.

- For each n, $\{\mu_j^n\}$ converges to μ_n in $\sigma\left(M_b(\hat{\mathbb{R}}), C_c(\hat{\mathbb{R}})\right)$ and $\lim_{j\to\infty} \left\|\mu_j^n\right\|_1 = \|\mu_n\|_1$. Thus, by Levy's theorem, $\lim_{j\to\infty} \check{\mu}_j^n = \check{\mu}_n$ uniformly on compact sets.
- For each n and each j, there exist A_j^n such that $\forall t$ and $\forall R \geq A_j^n$,

$$\left| \frac{1}{2R} \int_{-R}^{R} \phi_j^n(t+x)\overline{\phi_j^n(x)} dx - \check{\mu}_j^n(t) \right| < \frac{1}{2^{j+1}}\,.$$

Thus, if as in [16] we de£ne

$$\phi_n(t) = \begin{cases} \phi_j^n(t), & \text{when } R_j^n < |t| < R_{j+1}^n \\ 0, & \text{when } |t| < R_1^n, \end{cases} \tag{3.5}$$

where $R_1^n = (A_1^n + 1)$ and where $\{R_j^n : j = 1, ...\}$ is an increasing sequence for which each

$$R_j^n > A_j^n > j$$

and

$$\frac{(R_{j-1}^n + j)(M_{j+1}^n)^2}{R_j^n} < \frac{1}{2^j},$$

where $M_j^n = max_{1\leq k\leq j}\{\|\phi_k^n\|_\infty, j\}$, then, for each n and each $t \in \mathbb{R}$,

$$P_{\phi_n}(t) = \lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} \phi_n(t+x)\overline{\phi_n(x)}dx = \check{\mu}_n(t).$$

Now, with

$$\mu(\gamma) = \frac{1}{|\gamma|^\alpha} \in M_+(\mathbb{R})$$

and

$$\mu_n(\gamma) = nw_{\frac{2\pi}{n}}(\gamma)\mu(\gamma) \in {M_b}^+(\mathbb{R}),$$

we have the same type of set-up as in the classical Wiener-Wintner theorem and Benedetto and Kerby's extensions (see Remark 2.4). Namely, we have reduced our problem to constructing a signal ϕ from the ϕ_n which correspond to measures μ_n, where the μ_n are positive and bounded and thus satisfy the hypothesis of the classical Wiener-Wintner theorem (Theorem 1.1). The question then is how to construct the ϕ for the spectrum $\mu(\gamma) = \frac{1}{|\gamma|^\alpha}$? The natural direction is to construct the ϕ in the same manner as Wiener and Wintner, i.e., we try the following construction.

Recall that since μ_n is a bounded positive measure for each n, then the classical Wiener-Wintner theorem implies that there exists $\phi_n \in L^\infty_{loc}(\mathbb{R})$ such that

$$P_{\phi_n}(t) = \lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} \phi_n(t+x)\overline{\phi_n(x)}dx = \check{\mu}_n(t);$$

and, in fact, this convergence is uniform on compact sets. If we then choose appropriate S_n such that

$$\forall t \in [-S_{n-1}, S_{n-1}] \text{ and } \forall R \geq S_n,$$

$$\left|\frac{1}{2R}\int_{-R}^{R} \phi_n(t+x)\overline{\phi_n(x)}dx - \check{\mu}_n(t)\right| < \frac{1}{2^n}$$

we can de£ne

$$\phi(t) = \begin{cases} \phi_n(t), & \text{when } S_n < |t| < S_{n+1} \\ 0, & \text{when } |t| < S_1. \end{cases}$$

We can choose S_n so that $S_n - S_{n-1} > 2|t|$ for any given t. Hence, if $S_n - |t| < x < S_n$, then

$$S_{n-1} < S_n - |t| - |t| < x - |t| < |t+x| < |t| + |x| < |t| + S_n < S_{n+1}.$$

However, eventually $t + x$ will be such that $S_n < |t+x| < S_{n+1}$ with $S_{n-1} < x < S_n$ in which case, we have the following mixed term

$$\int \phi_n(t+x)\overline{\phi_{n-1}(x)}dx,$$

where the integral is taken over the set $\{x \in S_n \setminus S_n - |t| : t + x \in S_{n+1} \setminus S_n\}$. The problem arises as we try to make this integral small, since our ϕ_n are only in L^∞_{loc} not in L^∞. In order to resolve this problem, we de£ne ϕ by piecing together the L^∞ functions ϕ^n_j instead of the functions ϕ_n; thus, our ϕ is in L^∞_{loc}. Recall the ϕ^n_j are used in the de£nition of ϕ_n which is given in equation (3.5).

3.3 De£nition of $1/f$ noise ϕ

3.3.1 A goal

Fix $t \neq 0$. For each n, we shall choose S_n and p_n such that

$$\left| \frac{1}{2S_n} \int_{-S_n}^{S_n} \phi^n_{p_n}(t+x) \overline{\phi^n_{p_n}(x)} dx - \check{\mu}_n(t) \right|$$

is small.

3.3.2 Choice of p_n

Since for each n, $\lim_{j\to\infty} \check{\mu}^n_j = \check{\mu}_n$ uniformly on compact sets, we can choose an increasing sequence $\{p_n\}$ where for each n, $p_n > n$ and

$$\forall t \in [-R^1_n, R^1_n], \quad \left| \check{\mu}^n_{p_n}(t) - \check{\mu}_n(t) \right| < \frac{1}{2^{n+1}}.$$

Note:

1. $\{R^1_n\}$ is also an increasing sequence determined in the proof of the classical Wiener-Wintner theorem with the bounded Radon measure $\mu_1(\gamma) = w_{2\pi}(\gamma) \frac{1}{|\gamma|^\alpha}$. See Remark 2.4 a.

2. The p_n depend on n and on the compact set $[-R^1_n, R^1_n]$. Also the p_n are such that
$$\forall t \in [-R^1_n, R^1_n], \quad \left| \check{\mu}^n_{p_n}(t) - \check{\mu}_n(t) \right| < \frac{1}{2^{n+1}}. \tag{3.6}$$

3.3.3 Choice of radii S_n

By Wiener and Wintner's proof of Theorem 1.1 using the bounded Radon measure μ_n, see Remark 2.4 a, we have, for each n and p_n, that there exists $R^n_{p_n} > A^n_{p_n}$ such that

$$\forall R > R^n_{p_n} \text{ and } \forall t, \left| \frac{1}{2R} \int_{-R}^{R} \phi^n_{p_n}(t+x)\overline{\phi^n_{p_n}(x)}dx - \check{\mu}^n_{p_n}(t) \right| < \frac{1}{2^{p_n+1}}. \quad (3.7)$$

Thus, by (3.6), (3.7), and the triangle inequality,

$$\forall t \in [-R^1_n, R^1_n] \text{ and } \forall R > R^n_{p_n}, \left| \frac{1}{2R} \int_{-R}^{R} \phi^n_{p_n}(t+x)\overline{\phi^n_{p_n}(x)}dx - \check{\mu_n}(t) \right| < \frac{1}{2^n}, \quad (3.8)$$

since

$$\begin{aligned} &\left| \frac{1}{2R} \int_{-R}^{R} \phi^n_{p_n}(t+x)\overline{\phi^n_{p_n}(x)}dx - \check{\mu_n}(t) \right| \\ &= \left| \frac{1}{2R} \int_{-R}^{R} \phi^n_{p_n}(t+x)\overline{\phi^n_{p_n}(x)}dx - \check{\mu}^n_{p_n}(t) + \check{\mu}^n_{p_n}(t) - \check{\mu_n}(t) \right| \\ &< \frac{1}{2^{p_n+1}} + \frac{1}{2^{n+1}} < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}, \end{aligned}$$

where the last inequality follows since $p_n > n$.

Let $S_1 = R^1_{p_1} + 1$ and let

$$\begin{aligned} M_3 &= \max_{1 \le j \le 3} \{ \left\| \phi^j_{p_j} \right\|_\infty, 3 \} \\ &= \max\{ \left\| \phi^1_{p_1} \right\|_\infty, \left\| \phi^2_{p_2} \right\|_\infty, \left\| \phi^3_{p_3} \right\|_\infty, 3 \}. \end{aligned}$$

Because the p_n are already chosen based on the covering of the real line by the compact sets $[-R^1_n, R^1_n]$, we can choose S_n independently of p_n. We proceed as follows. We choose S_2 independently of p_2 such that $S_2 > R^2_{p_2}$ and $2^2(S_1 + 2)M_3^2 < S_2$. In general, we do the following. Let

$$M_{n+1} = \max_{1 \le j \le n+1} \{ \left\| \phi^j_{p_j} \right\|_\infty, n+1 \},$$

and choose S_n such that

(C1) $S_n > R^n_{p_n}$
and

(C2) $2^n(S_{n-1} + n)M^2_{n+1} < S_n$.

We can then de£ne ϕ as

$$\phi(t) = \begin{cases} \phi_{p_n}^n(t), & \text{when } S_n < |t| < S_{n+1} \\ 0, & \text{when } |t| < S_1. \end{cases}$$

Now that we have our ϕ it remains to prove that

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \phi(t+x)\overline{\phi(x)}dx = \check{\mu}(t), \tag{3.9}$$

where $\mu(\gamma) = 1/|\gamma|^\alpha$ and $0 < \alpha < 1$.

3.4 Outline of the proof of (3.9)

Step 1: Given $t \neq 0$, $\epsilon > 0$, $\{R_n^1\}$ as in Note 1 in Section 3.3.2, and $\{S_n\}$ as in Section 3.3.3, we assume that q satis£es the following:

(1) $|t| \leq q < R_q^1$,
(2) $\forall k \geq q,\ S_k - S_{k-1} > 2\,|t|$,
and
(3) $\forall k \geq q, |\check{\mu_k}(t) - \check{\mu}(t)| < \epsilon$.

Then we let $R > S_{q+1}$ and let $s = s(R)$ have the property that $S_s \leq R < S_{s+1}$. We write the mean on the left side of (3.9) as follows:

$$\frac{1}{2R}\int_{-R}^{R} \phi(t+x)\overline{\phi(x)}dx = \frac{1}{2R}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|} \phi(t+x)\overline{\phi(x)}dx$$

$$+\frac{1}{2R}\int_{S_s-|t|\setminus S_{s-1}+|t|} \phi(t+x)\overline{\phi(x)}dx + \frac{1}{2R}\int_{S_s\setminus S_s-|t|} \phi(t+x)\overline{\phi(x)}dx \tag{3.10}$$

$$+\frac{1}{2R}\int_{R\setminus S_s} \phi(t+x)\overline{\phi(x)}dx,$$

where $\int_{A\setminus B} = \int_{[-A,-B]\cup[B,A]}$. Note that assumption (2) at the beginning of this Step is used to split the integral, i.e., to order the S_n, and is used similarly throughout the proof.

Step 2: By choice of S_n, i.e., by the condition (C2) in Section 3.3.3, we can make the £rst and third integrals on the right side of the calculation in Step 1 small, in particular, less than $\frac{1}{2^s}$.

Step 3: For the second integral, we use condition (C1) in Section 3.3.3, equation (3.8), assumption (3) in Step 1, and the triangle inequality to obtain for R large enough, i.e., s large enough, that

$$\frac{1}{2R}\int_{S_s-|t|\setminus S_{s-1}+|t|}\phi(t+x)\overline{\phi(x)}dx = \frac{S_s}{R}\check{\mu}(t) + a(R,t),$$

where $|a(R,t)| < 6\epsilon$. Note that this step also uses assumption (1) of Step 1.

Step 4: We rewrite the remaining integral as follows:

$$\frac{1}{2R}\int_{R\setminus S_s}\phi(t+x)\overline{\phi(x)}dx$$

$$= \frac{1}{2R}\int_{R\setminus(R-|t|)}\phi(t+x)\overline{\phi(x)}dx + \frac{1}{2R}\int_{R-|t|\setminus(S_s+|t|)}\phi(t+x)\overline{\phi(x)}dx$$

$$+\frac{1}{2R}\int_{S_s+|t|\setminus S_s}\phi(t+x)\overline{\phi(x)}dx. \tag{3.11}$$

Again the £rst and last integrals are made small, i.e., less than $< \frac{1}{2^s}$, as in Step 2, and, then, as in Step 3, we obtain for R large enough, i.e., s large enough, that

$$\frac{1}{2R}\int_{R-|t|\setminus S_s+|t|}\phi^s_{p_s}(t+x)\overline{\phi^s_{p_s}(x)}dx = \left(1-\frac{S_s}{R}\right)\check{\mu}(t) + w(R,t)$$

where $|w(R,t)| < 6\epsilon$.

Step 5: By the previous steps, we have

$$\frac{1}{2R}\int_{-R}^{R}\phi(t+x)\overline{\phi(x)}dx = \check{\mu}(t) + z(R,t),$$

where $|z(R,t)| < 12\epsilon + \frac{4}{2^s}$.

Step 6: Finally, we use a lim sup argument to obtain

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\phi(t+x)\overline{\phi(x)}dx = \check{\mu}(t).$$

4 The proof

Proof. Throughout the proof we refer to the steps given in the previous section. We £rst make the assumptions stated in Step 1 of Subsection 3.4 and split up the integral as written there. We then estimate the resulting integrals as follows.

First and third integrals of (3.10) $\left(\frac{1}{2R}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|} \quad \text{and} \quad \frac{1}{2R}\int_{S_s\setminus S_s-|t|}\right)$

For the integral $\frac{1}{2R}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|}\phi(t+x)\overline{\phi(x)}dx$, assumption (2) in Step 1 and the fact that

$$|x| < S_{s-1} + |t|$$

implies

$$|t+x| \le |t| + |x| \le |t| + S_{s-1} + |t| < S_s.$$

Thus, as stated in Step 2, the £rst integral can be estimated as follows:

$$\left|\frac{1}{2R}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|}\phi(t+x)\overline{\phi(x)}dx\right| < \frac{(S_{s-1}+s)M_s^2}{S_s} < \frac{1}{2^s}.$$

Similarly, for the third integral, we have

$$S_s - |t| < |x| < S_s.$$

Therefore, using assumption (2) in Step 1,

$$S_{s-1} < S_s - |t| - |t| < |x| - |t| < |t+x| < |t| + |x| < |t| + S_s < S_{s+1},$$

and hence, the third integral can be estimated as

$$\left|\frac{1}{2R}\int_{S_s\setminus S_s-|t|}\phi(t+x)\overline{\phi(x)}dx\right| < \frac{sM_s^2}{S_s} < \frac{1}{2^s}.$$

The second integral of (3.10) $\left(\frac{1}{2R}\int_{S_s-|t|\setminus S_{s-1}+|t|}\right)$

In the second integral, we have

$$S_{s-1} + |t| < |x| < S_s - |t|\,,$$

which implies

$$S_{s-1} < S_{s-1} + |t| - |t| < |x| - |t| < |t+x| < |t| + |x| < |t| + S_s - |t| = S_s.$$

Hence, from the de£nition of ϕ, we have

$$\frac{1}{2R}\int_{S_s-|t|\setminus S_{s-1}+|t|}\phi(t+x)\overline{\phi(x)}dx$$

$$= \frac{1}{2R}\int_{S_s-|t|\setminus S_{s-1}+|t|}\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx,$$

which can be written as

$$\frac{2(S_s-|t|)}{2R}\left[\frac{1}{2(S_s-|t|)}\int_{-(S_s-|t|)}^{S_s-|t|}\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx\right]$$

$$-\frac{2(S_{s-1}+|t|)}{2R}\left[\frac{1}{2(S_{s-1}+|t|)}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|}\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx\right]. \quad (4.1)$$

Observe that $t \in [-R^1_{s-1}, R^1_{s-1}]$ and $S_s - |t| > S_{s-1} > R^{s-1}_{p_{s-1}}$ (see condition C1), and so the estimate in equation (3.8) holds. Thus, using the reasoning given in Step 3, we estimate the £rst summand in equation (4.1) using the following:

$$\left|\frac{1}{2(S_s-|t|)}\int_{-(S_s-|t|)}^{S_s-|t|}\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx - \check{\mu}(t)\right| < \frac{1}{2^{s-1}} + \epsilon,$$

which can be rewritten in the form,

$$\frac{1}{2(S_s-|t|)}\int_{-(S_s-|t|)}^{S_s-|t|}v\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx = \check{\mu}(t) + \alpha_{s-1}(t)\left(\frac{1}{2^{s-1}}+\epsilon\right),$$

where $|\alpha_{s-1}(t)| < 1$.

Similarly, since $t \in [-R^1_{s-1}, R^1_{s-1}]$, $S_{s-1}+|t| > S_{s-1} > R^{s-1}_{p_{s-1}}$ and $s-1 \geq q$, the reasoning in Step 3 applies again and we can estimate the second summand of (4.1) using the following:

$$\frac{1}{2(S_{s-1}+|t|)}\int_{-(S_{s-1}+|t|)}^{S_{s-1}+|t|}\phi^{s-1}_{p_{s-1}}(t+x)\overline{\phi^{s-1}_{p_{s-1}}(x)}dx = \check{\mu}(t)+\beta_{s-1}(t)\left(\frac{1}{2^{s-1}}+\epsilon\right),$$

where $|\beta_{s-1}(t)| < 1$.

Therefore, we can express our second integral of (3.10) as the difference

$$\frac{(S_s - |t|)}{R}\left(\breve{\mu}(t) + \alpha_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right)\right) - \frac{(S_{s-1} + |t|)}{R}\left(\breve{\mu}(t) + \beta_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right)\right)$$

$$\begin{aligned} &= \frac{S_s - |t|}{R}\breve{\mu}(t) + \frac{S_s - |t|}{R}\alpha_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right) - \frac{(S_{s-1} + |t|)}{R}\breve{\mu}(t) \\ &- \frac{(S_{s-1} + |t|)}{R}\beta_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right), \end{aligned} \tag{4.2}$$

where $|\alpha_{s-1}(t)|,\ |\beta_{s-1}(t)| < 1$. Hence, we can estimate the second integral of (3.10) by estimating each summand on the right side of equation (4.2).

By assumption 1 in Step 1 and the fact that $S_s < R$, the second and fourth terms on the right side of equation (4.2) are estimated as follows.

$$\left|\frac{S_s - |t|}{R}\alpha_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right)\right| + \left|\frac{(S_{s-1} + |t|)}{R}\beta_{s-1}(t)\left(\frac{1}{2^{s-1}} + \epsilon\right)\right|$$

$$< 2\,(\frac{1}{2^{s-1}} + \epsilon).$$

The third term on the right side of equation (4.2) satis£es

$$\left|\frac{(S_{s-1} + |t|)}{R}\breve{\mu}(t)\right| < \frac{S_{s-1} + s}{S_s}\breve{\mu}(t) < \frac{1}{2^s}\breve{\mu}(t)$$

by condition (C2). In estimating the £rst term on the right side of equation (4.2), we have

$$\frac{S_s}{R}\breve{\mu}(t) - \frac{|t|}{R}\breve{\mu}(t),$$

where

$$\frac{|t|}{R}\breve{\mu}(t) < \frac{s}{S_s}\breve{\mu}(t) < \frac{1}{2^s}\breve{\mu}(t).$$

Combining these estimates, we see that for R large enough, i.e., s large enough, the right side of equation (4.2) can be written as

$$\frac{S_s}{R}\breve{\mu}(t) + a(R, t),$$

where $|a(R, t)| < 6\epsilon$. Therefore, we have proved that for a given $\epsilon < 0$, we can choose R large enough, i.e., s large enough, such that

$$\frac{1}{2R}\int_{S_s-|t|\backslash S_{s-1}+|t|}\phi(t+x)\overline{\phi(x)}dx=\frac{S_s}{R}\check{\mu}(t)+a(R,t), \tag{4.3}$$

where $|a(R,t)|<6\epsilon$.

The last integral of 3.10 $\left(\frac{1}{2R}\int_{R\backslash S_s}\right)$

We rewrite the remaining integral as given in equation (3.11) in Step 4.

If, as in the first integral on the right side of (3.11),

$$R-|t|<|x|<R,$$

then, by assumption (2) in Step 1, we have

$$S_{s-1}<S_s-2\,|t|<R-|t|-|t|<|x|-|t|<|t+x|<R+|t|<S_{s+1}+|t|<S_{s+2},$$

or if, as in the last integral on the right side of (3.11), we take

$$S_s<|x|<S_s+|t|\,,$$

then we have

$$S_{s-1}<S_s-|t|<|x|-|t|<|t+x|<S_s+2\,|t|<S_{s+1}.$$

Hence, the first and last integrals on the right side of equation (3.11) are bounded by $\frac{sM_{s+1}^2}{S_s}$, which by condition (C2) satisfies

$$\frac{sM_{s+1}^2}{S_s}<\frac{1}{2^s}.$$

In the second integral on the right side of (3.11), we take

$$S_s+|t|<|x|<R-|t|\,;$$

and we have

$$S_s=S_s+|t|-|t|<|x|-|t|<|t+x|<R-|t|+|t|=R<S_{s+1}.$$

Thus, the second integral on the right side of equation (3.11) satisfies

$$\frac{1}{2R}\int_{R-|t|\backslash S_s+|t|}\phi(t+x)\overline{\phi(x)}dx$$

$$=\frac{1}{2R}\int_{R-|t|\backslash S_s+|t|}\phi^s_{p_s}(t+x)\overline{\phi^s_{p_s}(x)}dx$$

which, as before, by the reasoning given in Step 3, can be expressed as

$$\frac{R-|t|}{R}\left(\check{\mu}(t)+\gamma_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right)-\frac{S_s+|t|}{R}\left(\check{\mu}(t)+\tau_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right)$$

$$=\check{\mu}(t)+\frac{-|t|}{R}\check{\mu}(t)+\frac{R-|t|}{R}\left(\gamma_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right)$$
$$-\frac{S_s+|t|}{R}\check{\mu}(t)-\frac{S_s+|t|}{R}\left(\tau_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right),$$

which simpli£es as

$$\left(1-\frac{S_s}{R}\right)\check{\mu}(t)-\frac{2\,|t|}{R}\check{\mu}(t)+\frac{R-|t|}{R}\left(\gamma_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right)-\frac{S_s+|t|}{R}\left(\tau_s(t)\left(\frac{1}{2^s}+\epsilon\right)\right),$$

where $|\gamma_s(t)|\,,|\tau_s(t)|<1$. Thus, as stated in Step 4, for R large enough we have

$$\frac{1}{2R}\int_{R-|t|\setminus S_s+|t|}\phi^s_{p_s}(t+x)\overline{\phi^s_{p_s}(x)}dx=\left(1-\frac{S_s}{R}\right)\check{\mu}(t)+w(R,t),$$

where $|w(R,t)|<6\epsilon$.

The lim sup argument

From the previous proof, and as in Step 5, we have that for all ϵ there is an R large enough, i.e., s large enough, such that

$$\left|\frac{1}{2R}\int_{-R}^{R}\phi(t+x)\overline{\phi(x)}dx-\check{\mu}(t)\right|=|z(R,t)|<16\epsilon.$$

Hence, for all ϵ,

$$\limsup_{R\to\infty}\left|\frac{1}{2R}\int_{-R}^{R}\phi(t+x)\overline{\phi(x)}dx-\check{\mu}(t)\right|\le 16\epsilon,$$

and so

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\phi(t+x)\overline{\phi(x)}dx=\check{\mu}(t).$$

□

ACKNOWLEDGEMENT. The £rst named author gratefully acknowledges support from DARPA Grant MDA-972011003 and the General Research Board of the University of Maryland.

References

[1] J. Bass, Fonctions de Corrélation, Fonctions Pseudo-aléatoires, et Applications, Masson, Paris, 1984.

[2] J.J. Benedetto, G. Benke, and W.R. Evans, An n-dimensional Wiener-Plancherel formula, Adv. in Appl. Math., 10, 457-487, 1989.

[3] J. J. Benedetto and D. Colella, Wavelet analysis of spectrogram seizure chirps, Proc. SPIE, 2569, 512-521, 1995.

[4] J.J. Benedetto, Spectral Synthesis, Academic Press, New York, 1975.

[5] J. J. Benedetto, A multidimensional Wiener-Wintner theorem and spectrum estimation, Trans. Amer. Math. Soc., 327, 833-852, 1991.

[6] J.J. Benedetto, Harmonic Analysis and Applications, CRC Press, Boca Raton, FL, 1997.

[7] J. P. Bertrandias, Espaces de fonctions bornées et continues en moyenne asymptotique d'order p, Mém. Soc. Math., 5, 3-105, France, 1966.

[8] J. P. Bertrandias, Opérateurs subordinatifs sur des espaces de fonctions bornées en moyenne quadratique, J. Math. Pures et Appl., 52, 27-63, 1973.

[9] N. Bourbaki, Intégration, 2nd edition, Chapters 1 - 4, Hermann, Paris, 1965.

[10] S. Cambanis and C. Houdré, On the continuous wavelet transform of second-order random processes, IEEE Trans. Inf. Theory, 41, 628-642, 1995.

[11] D. Colella, Detection of signals in $\frac{1}{f}$ noise using wavelets, The First IEEE Regional Conference on Aerospace Control Systems, Proceedings, 151-155, 1993.

[12] H. G. Feichtinger and H.J. Schmeisser, Weighted versions of Beurling's Tauberian theorem, Math. Ann., 275, 353-363, 1986.

[13] P. Flandrin, On the spectrum of fractional brownian motion, IEEE Trans. Inf. Theory 35(1)(1989), 197-199.

[14] P. Flandrin, Wavelet analysis and synthesis of fractional Brownian motion, IEEE Trans. Inf. Theory IT-38(2)(1992), 910-917.

[15] S. Jaffard, Some open problems about multifractal functions, preprint.

[16] R. Kerby, The Correlation Function and the Wiener-Wintner Theorem in Higher Dimensions, PhD thesis, University of Maryland, College Park, 1990.

[17] K. -S. Lau, Extension of Wiener's Tauberian identity and multipliers on the Marcinkiewicz space, Trans. Amer. Math. Soc., 277(2), 489-506, June 1983.

[18] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press, San Diego, 1998.

[19] P. Malliavin, Intégration et Probabilités, Analyse de Fourier et Analyse Spectrale, Masson, Paris, 1982.

[20] Y. Meyer, F. Sellan, and M. S. Taqqu, Wavelets, generalized white noise and fractional integration: the Synthesis of fractional Brownian motion, J. Fourier Anal. Appl., 5(5), 465-494, 1999.

[21] A. Papoulis, Signal Analysis, McGraw-Hill Book Company, New York,1977.

[22] M. Priestley, Spectral Analysis and Time Series, Volumes 1 and 2, Academic Press, New York, 1981.

[23] L. Schwartz, Théorie des Distributions, Hermann, Paris, 1966 (1950, 1951).

[24] C.E. Shannon, The Mathematical Theory of Communication, The Univ. of Illinois Press, 1949.

[25] R. Strichartz, A Guide to Distribution Theory and Fourier Transforms, CRC Press, Boca Raton, 1994.

[26] E. M. Stein and G. Weiss, An Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton , NJ, 1971.

[27] N. Wiener, Generalized harmonic analysis, Acta. Math, 55, 1930

[28] N. Wiener, The Fourier Integral and Certain of Its Applications, Cambridge University Press, London, 1933.

[29] N. Wiener and A. Wintner, On singular distributions, J. Math. Phys., 17, 233-246, 1939 (Collected Works, Vol. II, P. Masani, Ed.).

[30] N. Wiener, Time Series, The MIT Press, Cambridge, MA, 1949.

[31] N. Wiener, Nonlinear Problems in Random Theory, Technology Press of MIT, Cambridge, 1958.

[32] G. Wornell, Estimation of fractal signals from noisy measurements using wavelets, IEEE Trans. on Signal Processing, 1992.

[33] G. Wornell, Wavelet-based representations for the $\frac{1}{f}$ family of fractal processes, IEEE Trans. Inf. Theory 81(10), 1428-1450, 1993.