Contemporary Mathematics

The construction of multiple dyadic minimally supported frequency wavelets on $\mathbb{R}^d$

John J. Benedetto and Manuel T. Leon

Abstract. Let $K^1,\ldots,K^L$ be Lebesgue measurable subsets of d-dimensional Euclidean space $\widehat{\mathbb{R}}^d$ with the property that the Lebesgue measure $|K^l|$ of each set K^l is finite. Set theoretic necessary and sufficient conditions are provided in order that the collection of inverse Fourier transforms of the characteristic functions of the K^l, $l=1,\ldots,L$, gives rise to a dyadic wavelet orthonormal basis for $L^2(\mathbb{R}^d)$. In addition, a general method to construct such families of sets is given, along with many specific examples, as well as the pseudo-code for the algorithm used to implement the construction. The results generalize the case $L=1$ treated in [**BL99**], and are a basic special case of the general theory for general expansive wavelets developed in [**Leo99**].

1. Introduction

Let $\mathbb{R}^d$ be d-dimensional Euclidean space, and let $L^2(\mathbb{R}^d)$ be the complex Hilbert space of square integrable functions f defined on $\mathbb{R}^d$, i.e.,

$$\|f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^2\ dx\right)^{\frac{1}{2}} < \infty.$$

The Fourier transform of f is designated by $\hat{f}$ and $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$, where the notation $\widehat{\mathbb{R}}^d$ is used to denote $\mathbb{R}^d$ considered as the domain of Fourier transforms. If $F \in L^2(\widehat{\mathbb{R}}^d)$ then $F^\vee$ denotes the inverse Fourier transform of F. Further, $\mathbf{1}_K$ denotes the characteristic function of the set K. We shall construct orthonormal bases (ONBs) $\{\psi^l_{m,n} : m \in \mathbb{Z}\ ,\ n \in \mathbb{Z}^d\ ,\ l = 1,\ldots,L\}$ for $L^2(\mathbb{R}^d)$, where

$$\psi^l_{m,n}(x) = 2^{\frac{md}{2}}\psi^l(2^m x - n) \tag{1.1}$$

and

$$\psi^l = \mathbf{1}^\vee_{K^l} \tag{1.2}$$

for some Lebesgue measurable subset $K^l \subseteq \widehat{\mathbb{R}}^d$. In particular, the Lebesgue measure $|K^l|$ of each K^l is finite. Our approach is general, depending only on an iteration of injective, measurable mappings; and our construction gives rise to effective computer implementation. An essential feature of our approach is that it is elementary.

This paper is a natural sequel to [**BL99**], where we made such constructions for the case $L=1$. In that case, the function $\psi = \psi^1$ is referred to as a *single dyadic orthonormal wavelet.* In the present case, the set $\{\psi^1,\ldots,\psi^L\}$ is referred to as a *dyadic wavelet collection.* In both cases, the ONBs $\{\psi_{m,n}\}$ and $\{\psi^l_{m,n}\}$ are called *dyadic wavelet orthonormal bases.* In these ONBs, the term "dyadic" is used because of the dyadic dilation " 2^m " in (1.1), and the term "wavelet" is used as the convention to describe systems determined by dilations and translations.

The wavelets constructed in both [**BL99**] and in this paper are characterized by (1.2), i.e., each wavelet is the inverse Fourier transform of a characteristic function of some set. In the case of a single dyadic orthonormal wavelet, such a set is called a *wavelet set*; and, in the case of a dyadic wavelet collection $\{\psi^1,\ldots,\psi^L\}$, the corresponding sequence $\{K^1,\ldots,K^L\}$ of sets is called a *wavelet collection of sets.* In light of the success of multiresolution analysis

1991 *Mathematics Subject Classification.* 42C15.

AFOSR Contract F49620-96-1-0193 first named author, Center for Auditory and Acoustic Research second named author.

(MRA) in wavelet theory, e.g., [**Mal89**], [**Dau92**], [**Mey92**], [**Mad92**], [**Str94**], [**HW96**], there was some doubt until recently about the existence of single dyadic orthonormal wavelets, and a fortiori about the added flexibility of those wavelet collections constructed here, see Section 4. That perception was changed with the groundbreaking work of Dai and Larson in the mid-90s, e.g., [**DL98**], and the results by Hernández, X. Wang, and Weiss on minimally supported frequency wavelets, e.g., [**HWW96**] and [**HWW97**].

Our introduction to the subject was the construction of specific single dyadic wavelets by Soardi and Weiland [**SW98**], and their construction was based on their characterization of wavelet sets, which itself was rooted in fundamental results due to Gripenberg [**Gri95**] and X. Wang [**Wan95**], as well as extensions to $\mathbb{R}^d$ by Frazier, Garrigós, K. Wang, and Weiss [**FGWW97**]. In 1998, after seeing our general construction and examples of wavelet sets, which is the content of [**BL99**], Bruno Torresani told us of Zakharov's unpublished preprint [**Zak96**]. In this work, Zakharov obtains specific constructions similar to those of Soardi and Weiland, and they are based on a characterization of wavelet sets rooted in the basic construction of the Shannon wavelet, e.g., pages 115-116 of [**Dau92**]. The work of Zakharov and Soardi-Weiland was done independently at about the same time.

In Section 3 of this paper we shall give two proofs characterizing wavelet collections of sets. One is based on Soardi and Weiland's approach and the other on Zakharov's approach. Equipped with our characterization, we proceed in Section 4 to give our basic construction of wavelet collections of sets, and, thus, to prove the existence of dyadic wavelet collections. To illustrate the versatility of our construction, Section 5 is devoted to a selection of examples of dyadic wavelet collections. Finally, in the Appendix, we present the pseudo-code for the algorithm derived from our construction.

For perspective, the next section (Section 2) provides some background from MRA wavelet theory as well as some perspective from the setting of expansive wavelet theory, that includes the dyadic case. In fact, there is a natural and broad sequel to this paper, viz., to extend the characterization of Section 3 and constructions and examples of Sections 4 and 5 to the general case dealing with expansive matrices. This is the content of Leon's paper [**Leo99**], cf., [**Cal99a**], [**Cal99b**].

Our construction and the analysis leading to it depend on the closely related ideas of tiling and of the relationship between Fourier analysis on groups and their quotients groups. We shall use the term "tiling" and "partition" interchangeably; and we shall deal exclusively with partitions of $\widehat{\mathbb{R}}^d$ by translation or dilation. For example, to say that $\{2^m K : m \in \mathbb{Z}\}$ is a partition of $\widehat{\mathbb{R}}^d$ means that $\bigcup 2^m K = \widehat{\mathbb{R}}^d$ and $(2^j K) \bigcap (2^m K)$ is a set of Lebesgue measure 0 when $j \neq m$.

We shall employ the usual notation in harmonic analysis and wavelet theory as found in [**Ben97**], [**Dau92**], [**Mey92**], and [**SW71**]. The Fourier transform of the function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is formally defined by

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \cdot \gamma} dx \ , \ \gamma \in \widehat{\mathbb{R}}^d,$$

where $\int$ denotes integration over $\mathbb{R}^d$; and the inverse Fourier transform $F^\vee$ of $F : \widehat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ is formally defined by

$$F^\vee(x) = \int F(\gamma) e^{2\pi i x \cdot \gamma} d\gamma \ , \ x \in \mathbb{R}^d.$$

Further, translation of a function f by x is designated by $\tau_x f$, i.e., $\tau_x f(y) = f(x - y)$.

2. Background

In 1986, Yves Meyer [**Mey86**] formulated the concept of a multiresolution analysis ("analyse graduée") of $L^2(\mathbb{R}^d)$ associated with (Γ, A), where $\Gamma \subseteq \widehat{\mathbb{R}}$ is a lattice and $A : \widehat{\mathbb{R}}^d \longrightarrow \widehat{\mathbb{R}}^d$ is a linear transformation with the properties that

$$A\Gamma \subseteq \Gamma \tag{2.1}$$

and

$$\forall \text{ eigenvalues } \lambda \text{ of } A, \ |\lambda| > 1. \tag{2.2}$$

Recall that a lattice $\Gamma \subseteq \mathbb{R}^d$ is the image of $\mathbb{Z}^d$ under some nonsingular linear transformation , i.e., Γ is a discrete subgroup of $\mathbb{R}^d$ consisting of integral linear combinations of the elements of a vector space basis of $\mathbb{R}^d$.

Definition 2.1. A *multiresolution analysis* (MRA) *of* $L^2(\mathbb{R}^d)$ *associated with* (Γ, A) is an increasing sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ and an element $\phi \in V_0$ with the following properties:

i. $\bigcap V_j = \{0\}$ and $\overline{\bigcup V_j} = L^2(\mathbb{R})$;
ii. $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1}$;
iii. $f \in V_0$ implies $\tau_\gamma f \in V_0$ for all $\gamma \in \Gamma$;
iv. $\{\tau_\gamma \phi\}$ is an ONB for V_0.

Meyer originally wrote condition (iv) in terms of bounded unconditional bases or exact frames, see [**BW94**].

Conditions (2.1) and (2.2) imply that $|det\ A|$ is an integer $q \geq 2$, and it is easy to see that $\Gamma/A\Gamma$ is a group of order $q = |det\ A|$. For example, if $\Gamma = \mathbb{Z}^d$ and $A = 2I$, where I is the $d \times d$ identity matrix, then $|det\ A| = 2^d$ and $\mathbb{Z}^d/2I\mathbb{Z}^d$ is a group of order 2^d . Based on Definition 2.1, Meyer developed a constructive algorithm leading to the following result.

THEOREM 2.2. *Given an MRA of $L^2(\mathbb{R}^d)$ associated with (Γ, A), and let $\phi \in V_0$ satisfy condition (iv) of Definition 2.1. If $q = |det\ A|$, then there exist $q-1$ functions $\psi^1, \ldots, \psi^{q-1}$ such that the set of elements of the form*

$$|det\ A|^{\frac{m}{2}} \psi^l(A^m x - \gamma)\ ,\ l = 1, \ldots, q-1\ ,\ m \in \mathbb{Z}\ ,\ \gamma \in \Gamma \tag{2.3}$$

is an ONB for $L^2(\mathbb{R}^d)$. In fact, each ψ^l is of the form

$$\psi^l(x) = \sum_{\gamma \in \Gamma} d_{l,\gamma} |det\ A|^{\frac{1}{2}} \phi(Ax - \gamma)$$

for some constructible sequence $\{d_{l,\gamma}\} \in l^2(\Gamma)$.

The set of elements defined by (2.3) is an MRA *wavelet orthonormal basis for $L^2(\mathbb{R}^d)$ associated with* (Γ, A). Expositions and valuable extensions of Meyer's MRA theory for (Γ, A) are due to Madych [**Mad92**], Gröchenig and Madych [**GM92**], A. Cohen and Daubechies [**CD93**], Lawton and Resnikoff [**LR91**], and Strichartz [**Str94**]. This approach is also important in multidimensional nonseparable filter design, and we refer to the work of Kovačević and Vetterli [**KV92**] in this regard.

EXAMPLE 2.3. If $\Gamma = \mathbb{Z}^d$ and $A = 2I$, and if $\{V_j\}$ and $\phi \in V_0$ give rise to an MRA associated with (Γ, A), then (2.3) in Theorem 2.2 defines a dyadic wavelet ONB whose corresponding dyadic wavelet collection $\{\psi^1, \ldots, \psi^L\}$ consists of $L = 2^d - 1$ elements. The fact that a dyadic wavelet ONB for $L^2(\mathbb{R}^d)$ requires $2^d - 1$ wavelets is also intuitively clear, e.g., [**Dau92**], [**Mey92**].

With the appearance of the Dai-Larson theory [**DL98**], there has been a flurry of further activity concerning wavelet ONBs whose elements are of the form

$$|det\ A|^{\frac{m}{2}} \psi^l(A^m x - \gamma), \tag{2.4}$$

where $l = 1, \ldots, L,\ m \in \mathbb{Z}, \gamma \in \Gamma$; only now an MRA structure is not necessarily associated with (Γ, A). In fact, Theorem 2.2 allows us to conclude that, in the case $L = 1$, the dyadic wavelet orthonormal bases

$$\{\psi_{m,n}\ :\ \psi = \mathbf{1}_K^\vee \in L^2(\mathbb{R}^d),\ d > 1\}$$

are *not* derived from MRAs. Of course, it is natural to investigate the existence of MRA dyadic wavelets of the form (1.1) and (1.2) in the case $L = 2^d - 1$.

Some of the recent work on non-dyadic wavelet ONBs whose elements are of the form (2.4) includes papers of Dai, Larson and Speegle, [**DLS97**],[**DLS98**], Baggett, Medina, and Merrill [**BMM00**], Calogero [**Cal99a**], [**Cal99b**], and [**Leo99**].

REMARK 2.4. In light of our dyadic results in this paper involving functions of the form $\mathbf{1}_K^\vee$, we point out that Gu and Han [**GH99**] proved that if $|det\ A| = 2$, then there is a measurable set $K \subseteq \widehat{\mathbb{R}}^d$ such that $\{2^{\frac{md}{2}} \mathbf{1}_K^\vee(A^m x - n)\ :\ m \in \mathbb{Z}\ ,\ n \in \mathbb{Z}^d\}$ is an ONB for $L^2(\mathbb{R}^d)$. This result can be viewed as a converse of the following theorem: if $\psi \in L^2(\mathbb{R}^d)$ is a single wavelet constructed from an MRA associated with $(\mathbb{Z}^d, A)$, then $|det\ A| = 2$, see [**BCMO95**] and [**Han97**] , cf., [**KV92**] and [**CD93**].

REMARK 2.5. a. The notion of congruence plays a role in several facets of wavelet theory including the results in this paper. Congruence criteria were used by Albert Cohen in 1990 to characterize the orthonormality of scaling functions defined by infinite products of dilations of a quadrature mirror filter, e.g., pages 182-186 of [**Dau92**]. The same notion of congruence also plays a fundamental role in recent work on self-similar tilings of $\mathbb{R}^d$ by Gröchenig, Haas, Lagarias, Yang Wang, et al., e.g., [**LW97**], [**LW00**].

b. The notion of congruence is intrinsically related to bijective restrictions of the canonical surjection $h : G \longrightarrow G/H$, where G is a locally compact group and H is a closed subgroup. An explanation of this relation is in Section 3 of [**Ben98**] in the context of Kluvánek's sampling theorem for locally compact abelian groups. Kluvánek's sampling formula for a signal f quantitatively relates the sampling rate with the measure of the subsets of a given bandwidth corresponding to the frequency content of f.

c. We have pointed out that our dyadic orthonormal wavelets of the form $\psi = \mathbf{1}_K^\vee \in L^2(\mathbb{R}^d)$ in [**BL99**] do not arise from an MRA, and there is never a question about the multiple wavelets of this paper arising from an MRA except when $L = 2^d - 1$, see Theorem 2.2 in the case of (1.1).

For perspective, Grochenig and Madych [**GM92**] have characterized MRAs of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$ in which the scaling function is of the form $\phi = \mathbf{1}_X$. The characterization is in terms of Cohen's criteria mentioned in part *a*, which in turn depends on a tiling $\{\tau_n \mathbf{1}_K : n \in \mathbb{Z}^d\}$ of $\widehat{\mathbb{R}}^d$ for some compact neighborhood K of the origin.

3. Characterization of a Wavelet Collection of Sets

As in (1.1), if $f \in L^2(\mathbb{R}^d)$, then $f_{m,n}$ denotes the function defined by

$$f_{m,n}(x) = 2^{\frac{md}{2}} f(2^m x - n),$$

for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^d$. In Theorem 3.6 we shall prove a set theoretic characterization of a wavelet collection of sets. The general construction of sets satisfying the conditions of this characterization is the content of Section 4, thereby giving rise to the dyadic wavelet orthonormal bases defined by (1.1) and (1.2). We shall give two distinct proofs of this characterization. The first proof is more difficult than the second, and can be considered analytical, whereas the second proof can be considered more geometrical. In particular, for the second proof it is convenient to have Definition 3.1 and Lemma 3.2. Further, the notion of τ -congruence in Definition 3.1 underlies the whole point of view of our approach.

Definition 3.1. Let A and B be subsets of $\widehat{\mathbb{R}}^d$. A and B are *τ-congruent*, denoted by $A \overset{\tau}{\simeq} B$, if there are countable partitions $\{A_p : p \in \mathbb{Z}\}$ and $\{B_p : p \in \mathbb{Z}\}$ of A and B, respectively, and a sequence $\{n_p : p \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that

$$\forall p \in \mathbb{Z}, \quad A_p = B_p + n_p.$$

The proof of Lemma 3.2 is elementary, see [**GM92**] as well as the proofs of Lemma 3.2 and part (a) of Theorem 3.1 in [**BL99**].

Lemma 3.2. *A set $K \subseteq \widehat{\mathbb{R}}^d$ is τ-congruent to the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ if and only if $\{K + k \ : \ k \in \mathbb{Z}^d\}$ is a partition of $\widehat{\mathbb{R}}^d$.*

We shall also require the following result, which is elementary to prove in the dyadic setting of this paper, requiring only a straightforward change of variables. The analogue of Lemma 3.3 in the case of matrices satisfying (2.1) and (2.2) is more difficult to prove, see Volume 1 of [**DFN92**] for relevant details. This analogue is required in the sequel [**Leo99**] by Leon.

Lemma 3.3.

a. If $f \in L^2(\mathbb{R}^d)$, then $f_{m,n} \in L^2(\mathbb{R}^d)$ for every $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^d$, and

$$\|f_{m,n}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)} .$$

b. If $f \in L^2(\mathbb{R}^d)$, then

$$\forall\ m \in \mathbb{Z} \text{ and } \forall\ n \in \mathbb{Z}^d,\ (f_{m,n})\hat{\ }(\gamma) = 2^{\frac{-dm}{2}} \hat{f}(2^{-m}\gamma)\ e^{-2\pi i n \cdot 2^{-m}\gamma} \text{ in } L^2(\widehat{\mathbb{R}}^d).$$

c. If $X \subseteq \widehat{\mathbb{R}}^d$ is Lebesgue measurable and $f \in L^1(2^d X)$, then $f(2\cdot) \in L^1(X)$ and

$$\int_{2^d X} f(x)\ dx = 2^d \int_X f(2y)\ dy.$$

The following theorem is difficult to prove, and the details are found in [**FGWW97**] for our setting on $\mathbb{R}^d$. Many of the essential ideas are the same as for the case $d = 1$, and the history and proof of the theorem on $\mathbb{R}$ is found in [**HW96**].

THEOREM 3.4. *A collection of functions $\psi^1, \dots, \psi^L \in L^2(\mathbb{R}^d)$ with $\|\psi^l\|_{L^2(\mathbb{R}^d)} = 1$ is a wavelet collection if and only if*

$$\sum_{l=1}^{L} \sum_{m\in\mathbb{Z}} |\widehat{\psi^l}(2^m\xi)|^2 = 1 \ a.e. \ on \ \widehat{\mathbb{R}}^d, \tag{3.1}$$

and

$$\forall n \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d \ , \ \sum_{l=1}^{L} \sum_{m=0}^{\infty} \widehat{\psi^l}(2^m\xi)\overline{\widehat{\psi^l}(2^m(\xi+n))} = 0 \ a.e. \ on \ \widehat{\mathbb{R}}^d. \tag{3.2}$$

Theorem 3.4 will be used in conjunction with with the following orthonormality criterion in the first proof of Theorem 3.6. The proof of the validity of this criterion is well-known and depends on a standard periodization argument and elementary application of Beppo Levi's Theorem, see Section 3.1 of [**HW96**].

THEOREM 3.5. *Let $\psi \in L^2(\mathbb{R}^d)$. The sequence $\{\psi_{m,n} : m \in \mathbb{Z} \ , \ n \in \mathbb{Z}^d\}$ is orthonormal if and only if*

$$\sum_{n\in\mathbb{Z}^d} |\widehat{\psi}(\xi+n)|^2 = 1 \ for \ a.e. \ \xi \in \widehat{\mathbb{R}}^d \tag{3.3}$$

and

$$\sum_{n\in\mathbb{Z}^d}^{\infty} \widehat{\psi}(2^m(\xi+n))\overline{\widehat{\psi}(\xi+n)} = 0 \ for \ a.e. \ \xi \in \widehat{\mathbb{R}}^d \ , \ m \geq 1. \tag{3.4}$$

In the second proof of Theorem 3.6 we shall use the notation $\delta(n)$, for $n \in \mathbb{Z}^d$, to denote the Kronecker δ:

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \in \mathbb{Z}^d \\ 0 & \text{if } n \in \mathbb{Z}^d \backslash \{0\} \ . \end{cases}$$

In the case $l \in \mathbb{Z}$, $\delta(l)$ denotes the usual Kronecker δ, equal to 1 if $l = 0$ and equal to 0 if $l \neq 0$.

Before giving the full characterization of wavelet collections of sets, let us point out that, for $K^1, \dots, K^L$ to be such collection, the functions $\psi^1, \dots, \psi^L$ must be orthonormal. Thus, in terms of the inner product in $L^2(\mathbb{R}^d)$,

$$\begin{aligned} 0 &= \langle \psi^h, \psi^l \rangle = \langle \hat{\psi}^h, \hat{\psi}^l \rangle = \langle \mathbf{1}_{K^h}, \mathbf{1}_{K^l} \rangle \\ &= \int_{\widehat{\mathbb{R}}^d} \mathbf{1}_{K^h}(y) \ \mathbf{1}_{K^l}(y) \ dy = \int_{K^h \bigcap K^l} dy = |K^h \bigcap K^l|. \end{aligned}$$

Therefore, the sets $K^1, \dots, K^L$ must be pairwise disjoint.

THEOREM 3.6. *Let $K^1, \dots, K^L$ be pairwise disjoint Lebesgue measurable subsets of $\widehat{\mathbb{R}}^d$. The family $\{K^l \ : \ l = 1, \dots, L\}$ is a wavelet collection of sets if and only if each $|K^l| = 1$ and the following conditions are satisfied:*

a. *For each fixed $l = 1, \dots, L$, $\{K^l + k \ : \ k \in \mathbb{Z}^d\}$ is a partition of $\widehat{\mathbb{R}}^d$;*

b. *If $K = \bigcup_{l=1}^{L} K^l$, then $\{2^j K : j \in \mathbb{Z}\}$ is a partition of $\widehat{\mathbb{R}}^d$.*

Proof 1: Assume that $\{K^1, \dots, K^L\}$ is a wavelet collection of sets. Then

$$\{\psi^l \ : \ l = 1, \dots, L \text{ and } (\psi^l)^\wedge = \mathbf{1}_{K^l}\}$$

is a dyadic wavelet collection, and, in particular, $\{\psi^l_{j,k} : j \in \mathbb{Z} \ , \ k \in \mathbb{Z}^d\}$ is orthonormal for each fixed $l = 1, \dots, L$. Thus, equation (3.3) holds for $\widehat{\psi^l}$ for each fixed $l = 1, \dots, L$, and this gives condition a.

Since $\{\psi^l\}$ is a dyadic wavelet collection, we can apply Theorem 3.4 and assert, by equation (3.1), that

$$1 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\widehat{\psi^l}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} |\widehat{\psi^l}(2^j\xi)|^2,$$

where the second equality holds since only finitely many terms are nonzero. This gives condition b because

$$\sum_{j\in\mathbb{Z}}\sum_{l=1}^{L}|\widehat{\psi}^l(2^j\xi)|^2=\sum_{j\in\mathbb{Z}}\sum_{l=1}^{L}\mathbf{1}_{K^l}(2^j\xi)=\sum_{j\in\mathbb{Z}}\mathbf{1}_K(2^j\xi).$$

Conversely, assume each $|K^l|=1$, and that conditions a and b are satisfied. Then, by condition b,

$$1=\sum_{j\in\mathbb{Z}}\mathbf{1}_K(2^j\xi)=\sum_{j\in\mathbb{Z}}\sum_{l=1}^{L}\mathbf{1}_{K^l}(2^j\xi)=\sum_{j\in\mathbb{Z}}\sum_{l=1}^{L}|\widehat{\psi}^l(2^j\xi)|^2=\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\widehat{\psi}^l(2^j\xi)|^2,$$

where last equality holds since only finitely many terms are nonzero. Thus, equation (3.1) holds. By condition a we have for each fixed $l=1,\dots,L$, and for all $j\geq 0$ and $k\in\mathbb{Z}^d\backslash\{0\}$, that

$$\widehat{\psi}^l(2^j\xi)\widehat{\psi}^l(2^j(\xi+k))=0 \quad \text{a.e. on} \quad \xi\in\widehat{\mathbb{R}}^d.$$

Thus, for all $j\in\mathbb{Z}$ and for all $k\in\mathbb{Z}^d\backslash 2\mathbb{Z}^d$,

$$\sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}^l(2^j\xi)\overline{\widehat{\psi}^l(2^j(\xi+k))}=0 \quad \text{a.e. on} \quad \widehat{\mathbb{R}}^d.$$

Hence, equation (3.2) holds. Since (3.1) and (3.2) are true we can apply Theorem 3.4 to complete the proof.

Proof 2: First, assume each $|K^l|=1$ and that conditions a and b are satisfied. Defining $\psi^l=\mathbf{1}_{K^l}^{\vee}$, we shall first prove that $\{\psi^l_{m,n}: l=1,\dots,L, m\in\mathbb{Z}\ ,\ n\in\mathbb{Z}^d\}$ is orthonormal, and then prove that

$$\forall f\in L^2(\mathbb{R}^d),\ \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{j=1}^{L}|\langle f,\psi^j_{m,n}\rangle|^2=\|f\|^2_{L^2(\mathbb{R}^d)}\,. \tag{3.5}$$

In light of the orthonormality, (3.5) allows us to conclude that $\{\psi^l_{m,n}\}$ is an ONB for $L^2(\mathbb{R}^d)$, e.g., pages 27-28 of [**GG81**].

To prove the orthonormality, we begin with the Parseval formula and Lemma 3.3 to compute

$$\begin{aligned}\langle\psi^l_{m,n},\psi^h_{j,k}\rangle &= \langle(\psi^l_{m,n})^\frown,(\psi^h_{j,k})^\frown\rangle \\ &= \int 2^{\frac{-dm}{2}}\widehat{\psi}^l(2^{-m}\lambda)\,e^{-2\pi i n\cdot 2^{-m}\lambda}\overline{2^{\frac{-dj}{2}}\widehat{\psi}^h(2^{-j}\lambda)\,e^{-2\pi i k\cdot 2^{-j}\lambda}}\,d\lambda \\ &= 2^{-\frac{d(m+j)}{2}}\int_{2^mK^l\bigcap 2^jK^h}e^{-2\pi i n\cdot 2^{-m}\lambda}\,e^{2\pi i k\cdot 2^{-j}\lambda}\,d\lambda,\end{aligned} \tag{3.6}$$

where the last equality is a consequence of the definition of ψ^l and ψ^h.

By the definition of K as $\bigcup_{l=1}^{L}K^l$ we see that

$$2^mK^l\bigcap 2^jK^h\subseteq 2^mK\bigcap 2^jK.$$

Thus, by condition b, either

$$|2^mK^l\bigcap 2^jK^h|=0 \tag{3.7}$$

or $m=j$; and even in the case $m=j$, we obtain

$$|2^mK^l\bigcap 2^mK^h|=0 \tag{3.8}$$

if and only if $l\neq h$ because the sets K^l are disjoint. Hence, in each of the cases (3.7) and (3.8), the right side of equation (3.6) vanishes. In the remaining case of $m=j$ and $l=h$, the right side of (3.6) is

$$2^{-dm}\int_{2^mK^l}e^{-2\pi i(n-k)\cdot 2^{-m}\lambda}\,d\lambda=\int_{K^l}e^{-2\pi i(n-k)\cdot\xi}\,d\xi=\int_{[-\frac{1}{2},\frac{1}{2}]^d}e^{-2\pi i(n-k)\cdot\xi}\,d\xi=\delta(n-k),$$

where the first inequality is a consequence of Lemma 3.3c and the second inequality follows from condition a and Lemma 3.2. Combining these cases, we see that

$$\langle\psi^l_{m,n},\psi^h_{j,k}\rangle=\delta((m-j)(l-h)(n-k)),$$

and hence $\psi^l_{m,n}$ is an orthonormal system.

In this direction, it remains to prove (3.5), and the formal details for this proof are contained in the following calculation:

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}|\langle f,\psi^l_{m,n}\rangle|^2=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}|\langle \hat{f},\hat{\psi}^l_{m,n}\rangle|^2=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}|\int \hat{f}(\lambda)\overline{\hat{\psi}^l_{m,n}(\lambda)}\,d\lambda|^2 \tag{3.9}$$

$$=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}|\int \hat{f}(\lambda)\overline{2^{\frac{-dm}{2}}\hat{\psi}^l(2^{-m}\lambda)\ e^{-2\pi i n\cdot 2^{-m}\lambda}}d\lambda|^2$$

$$=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}2^{-dm}|\int \hat{f}(\lambda)\overline{\hat{\psi}^l(2^{-m}\lambda)}\ e^{2\pi i n\cdot 2^{-m}\lambda}d\lambda|^2$$

$$=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}2^{-dm}|\int_{2^mK^l}\hat{f}(\lambda)\ e^{2\pi i n\cdot 2^{-m}\lambda}\ d\lambda|^2=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}2^{-dm}|\int_{K^l}\hat{f}(2^m\eta)2^{dm}e^{2\pi i n\cdot\eta}\ d\eta|^2$$

$$=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L}2^{dm}|\int_{K^l}\hat{f}(2^m\eta)\ e^{2\pi i n\cdot\eta}\ d\eta|^2$$

$$=\sum_{m\in\mathbb{Z}}\sum_{l=1}^{L}2^{dm}\int_{K^l}|\hat{f}(2^m\eta)|^2d\eta=\sum_{m\in\mathbb{Z}}2^{dm}\int_{K=K^1\bigcup\cdots\bigcup K^L}|\hat{f}(2^m\eta)|^2d\eta$$

$$=\sum_{m\in\mathbb{Z}}2^{dm}\int_{2^mK}|\hat{f}(\xi)|^2\ 2^{-dm}\ d\xi=\|f\|^2_{L^2(\mathbb{R}^d)}\,.$$

The Parseval formula,

$$\int f(x)\overline{g(x)}\ dx=\int \hat{f}(\gamma)\overline{\hat{g}(\gamma)}\ d\gamma,$$

and Lemma 3.3b account for the first four equalities, and we use the definition of ψ^l for the fifth equality. Lemma 3.3c accounts for the next two equalities, the right side of second being the last term with three summation signs.

The next step, where the left side of (3.9) has three summation signs and the right side has two, is a critical one. It is at this step that we use condition a and Lemma 3.2 in order to invoke Parseval's Theorem,

$$(\int_{[-\frac{1}{2},\frac{1}{2}]^d}|F(\gamma)|^2)^{\frac{1}{2}}=(\sum_{n\in\mathbb{Z}^d}|F^\vee[n]|^2)^{\frac{1}{2}},$$

for Fourier series, where $\{F^{\vee}[n]\}$ is the sequence of Fourier coefficients of F. In fact, condition a and Lemma 3.2 allow us to assert that each $K^l \overset{\mathcal{T}}{\simeq} [-\frac{1}{2}, \frac{1}{2}]^d$. Consequently,

$$\text{(3.10)} \quad \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} \sum_{l=1}^{L} 2^{dm} \Big| \int_{K^l} \hat{f}(2^m\eta) e^{2\pi i n\cdot\eta} \, d\eta \Big|^2$$

$$= \sum_{m\in\mathbb{Z}} \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}^d} 2^{dm} \Big| \int_{[-\frac{1}{2},\frac{1}{2}]^d} F_m(\gamma) e^{2\pi i n\cdot\gamma} \, d\gamma \Big|^2$$

$$= \sum_{m\in\mathbb{Z}} \sum_{l=1}^{L} 2^{dm} \int_{[-\frac{1}{2},\frac{1}{2}]^d} |F_m(\gamma)|^2 \, d\gamma = \sum_{m\in\mathbb{Z}} \sum_{l=1}^{L} 2^{dm} \int_{K^l} |\hat{f}(2^m\lambda)|^2 \, d\lambda,$$

where F_m is a sum over $p \in \mathbb{Z}$ of products, where one factor has the form $\hat{f}(2^m(\gamma - p))$ and the other is an appropriate characteristic function, cf., [**BL99**] and [**Leo99**] for explicit calculations.

The final four equations of (3.9) follow from the definition of K, Lemma 3.3c, condition b, and the Plancherel Theorem, respectively.

This completes the prooof of the necesssary conditions for a wavelet collection of sets.

To prove the converse, assume that

$$\{\psi^l_{m,n} : l = 1, \ldots, L, m \in \mathbb{Z} \ , \ n \in \mathbb{Z}^d\}$$

is an ONB for $L^2(\mathbb{R}^d)$, where each ψ^l is of the form $\phi = \mathbf{1}_K^{\vee}$. Since $\psi^l = \psi^l_{0,0}$, and $\{\psi^l_{m,n}\}$ is orthonormal, the sets K^l must be disjoint because

$$\delta(l-h) = \langle \psi^l_{0,0}, \psi^h_{0,0} \rangle = \langle (\psi^l_{0,0})\widehat{\ }, (\psi^h_{0,0})\widehat{\ } \rangle \langle \mathbf{1}_{K^l}, \mathbf{1}_{K^h} \rangle$$

$$= \int \mathbf{1}_{K^l}(y) \mathbf{1}_{K^h}(y) \, dy = \int_{K^l \bigcap K^h} dy.$$

For any $n \in \mathbb{Z}^d$ and any $l = 1, \ldots, L$,

$$\delta(n) = \langle \psi^l_{0,0}, \psi^l_{0,n} \rangle = \langle \hat{\psi}^l_{0,0}, \hat{\psi}^l_{0,n} \rangle$$

$$= \int 2^{\frac{-d0}{2}} \hat{\psi}^l(2^{-0}\lambda) \, e^{-2\pi i 0\cdot 2^{-0}\lambda} \, \overline{2^{\frac{-d0}{2}} \hat{\psi}^l(2^{-0}\lambda) \, e^{-2\pi i n\cdot 2^{-0}\lambda}} \, d\lambda$$

$$= \int \hat{\psi}^l(\lambda) \, e^{-2\pi i 0\cdot\lambda} \overline{\hat{\psi}^l(\lambda) \, e^{-2\pi i n\cdot\lambda}} \, d\lambda = \int_{K^l} e^{2\pi i n\cdot\lambda} \, d\lambda.$$

In particular, taking $n = 0 \in \mathbb{Z}^d$, we see that each $|K^l| = 1$. By considering all values of $n \in \mathbb{Z}^d$, we can conclude that each $K^l \overset{\mathcal{T}}{\simeq} [-\frac{1}{2}, \frac{1}{2}]^d$, and, hence, condition a is a consequence of Lemma 3.2.

It remains to prove condition b. Let

$$H_0 = \widehat{\mathbb{R}}^d \setminus \bigcup_{m\in\mathbb{Z}} 2^m K,$$

where $K = \bigcup_{l=1}^{L} K^l$. We begin by computing

$$
\begin{aligned}
(3.11)\quad \|\mathbf{1}_{H_0}\|^2_{L^2(\widehat{\mathbb{R}^d})} &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L} |\langle \mathbf{1}_{H_0}, \hat{\psi}^l_{m,n}\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L} \Big| \int \mathbf{1}_{H_0}(\xi)\ \overline{2^{\frac{-dm}{2}}\hat{\psi}^l(2^{-m}\xi)\ e^{-2\pi i n\cdot 2^{-m}\xi}}\ d\xi\ \Big|^2 \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\sum_{l=1}^{L} \Big| \int_{2^m K^l} \mathbf{1}_{H_0}(\xi)\ \overline{2^{\frac{-dm}{2}} e^{-2\pi i n\cdot 2^{-m}\xi}}\ d\xi\ \Big|^2 = 0,
\end{aligned}
$$

where the first equality follows from our hypothesis of a wavelet collection of sets, where the next two equalities follow by Lemma 3.3b and the definition of $(\psi^l)\hat{}$, and where the last equality follows from the definition of H_0. Because of (3.11) we see that $|H_0| = 0$.

Thus, to complete the validity of condition b, it remains to show that

$$
(3.12)\qquad |2^j K \bigcap 2^m K| = 0
$$

for $j < m$. letting $H = 2^j K \bigcap 2^m K$, we compute

$$
\begin{aligned}
\|\mathbf{1}_{H}\|^2_{L^2(\widehat{\mathbb{R}^d})} &= \langle \mathbf{1}_H, \mathbf{1}_H\rangle = \int_{\widehat{\mathbb{R}}^d} \mathbf{1}_H(\xi)\ d\xi = \int_{2^j K \bigcap 2^m K} d\xi \\
&= \sum_{h=1}^{L}\sum_{l=1}^{L} \int_{2^j K^h \bigcap 2^m K^l} d\xi = \sum_{h=1}^{L}\sum_{l=1}^{L} \int_{\widehat{\mathbb{R}}^d} \hat{\psi}^h(2^{-j}\xi)\ \hat{\psi}^l(2^{-m}\xi)\ d\xi \\
&= \sum_{h=1}^{L}\sum_{l=1}^{L} 2^{\frac{d(j+m)}{2}} \int_{\widehat{\mathbb{R}}^d} 2^{\frac{-dj}{2}}\hat{\psi}^h(2^{-j}\xi)\ e^{-2\pi i 0\cdot 2^{-j}\xi} \\
&\qquad\times \overline{2^{\frac{-dm}{2}}\hat{\psi}^l(2^{-m}\xi)\ e^{-2\pi i 0\cdot 2^{-m}\xi}}\ d\xi \\
&= \sum_{h=1}^{L}\sum_{l=1}^{L} 2^{\frac{d(j+m)}{2}} \langle \hat{\psi}^h_{j,0}, \hat{\psi}^l_{m,0}\rangle \\
&= 2^{\frac{d(j+m)}{2}} \sum_{h=1}^{L}\sum_{l=1}^{L} \delta((j-m)(h-l)) = 2^{\frac{d(j+m)}{2}}\ \delta(j-m),
\end{aligned}
$$

and hence, (3.12) is obtained for $j \neq m$. Therefore condition b is true, and the proof of the theorem is complete. □

4. Construction

In this section we shall give a general procedure for constructing dyadic wavelet collections $\{\psi^1, \dots, \psi^L\}$ by constructing corresponding wavelet collections of sets. In fact, we shall construct sequences $\{K^1, \dots, K^L\}$ of pairwise disjoint sets for which each $|K^l| = 1$ and conditions a and b of Theorem 3.6 are satisfied.

We begin by taking any sequence $\{K_0^1, \dots, K_0^L\} \subseteq \widehat{\mathbb{R}}^d$ of Lebesgue measurable sets with the properties that they are pairwise disjoint, each K_0^l is τ-congruent to $[-\frac{1}{2}, \frac{1}{2}]^d$ for $l = 1, \dots, L$, and

$$
K_0 = K_0^1 \bigcup \cdots \bigcup K_0^L
$$

is a neighborhood of the origin contained in $[-N, N]^d$ for some fixed N. Further, suppose

$$
(4.1)\qquad T : [-N, N]^d \longrightarrow \widehat{\mathbb{R}}^d \backslash [-N, N]^d
$$

is a *measurable* mapping with the properties that it is *injective* and that

$$
\forall \gamma \in [-N, N]^d,\ T(\gamma) = \gamma + k_\gamma \text{ for some } k_\gamma \in \mathbb{Z}^d.
$$

(The property that T is measurable means that $T^{-1}(U)$ is Lebesgue measurable for any Lebesgue measurable set U contained in the range of T.)

Starting with any such sequence $\{K_0^1, \dots, K_0^L\}$ and the mapping T, we shall provide a method for constructing a wavelet collection of sets by iteratively transforming the K_0^l, $l = 1, \dots, L$, by means of T in a well-defined geometrical procedure.

CONSTRUCTION 4.1. We now describe our construction of wavelet collections of sets. Let

$$A_0 = K_0 \bigcap \left(\bigcup_{j \geq 1} 2^{-j} K_0 \right) \quad \text{and} \quad K_1 = (K_0 \backslash A_0) \bigcup T A_0.$$

Then,

$$K_0 \backslash A_0 \subseteq [-N, N]^d \text{ and } T A_0 \subseteq \widehat{\mathbb{R}}^d \backslash [-N, N]^d.$$

Moreover, if $A_0^l = A_0 \bigcap K_0^l$ and $K_1^l = (K_0^l \backslash A_0^l) \bigcup T A_0^l$, then the sets A_0^l are pairwise disjoint, the sets $T A_0^l$ are pairwise disjoint, and the sets K_0^l are pairwise disjoint for $l = 1, \dots, L$; and we have the decompositions

$$A_0 = \bigcup_{l=1}^{L} A_0^l \ , \ T A_0 = \bigcup_{l=1}^{L} T A_0^l \ , \ \text{and} \quad K_1 = \bigcup_{l=1}^{L} K_1^l.$$

Next, let

$$A_1 = K_1 \bigcap \left(\bigcup_{j \geq 1} 2^{-j} K_1 \right),$$

noting that

$$A_1 = K_1 \bigcap \left(\bigcup_{j \geq 1} 2^{-j} T A_0 \right)$$

and let

$$K_2 = ((K_0 \backslash A_0) \backslash A_1) \bigcup T A_0 \bigcup T A_1.$$

Then

$$(K_0 \backslash A_0) \backslash A_1 \subseteq K_0 \subseteq [-N, N]^d$$

and

$$T A_0 \bigcup T A_1 \subseteq \widehat{\mathbb{R}}^d \backslash [-N, N]^d.$$

Again, if $A_1^l = A_1 \bigcap K_1^l$ and $K_2^l = (K_1^l \backslash A_1^l) \bigcup T A_1^l$, then the sets A_1^l are pairwise disjoint, the sets $T A_1^l$ are pairwise disjoint, and the sets K_1^l are pairwise disjoint for $l = 1, \dots, L$; and we have the decompositions

$$A_1 = \bigcup_{l=1}^{L} A_1^l \ , \ T A_1 = \bigcup_{l=1}^{L} T A_1^l \ , \ \text{and} \quad K_2 = \bigcup_{l=1}^{L} K_2^l.$$

Notationally, we set $((K_0 \backslash A_0) \backslash A_1) = K_0 \backslash A_0 \backslash A_1$. Generally, for a given A_{n-1} and K_n, let

$$A_n = K_n \bigcap \left(\bigcup_{j \geq 1} 2^{-j} K_n \right) = K_n \bigcap \left(\bigcup_{j \geq 1} 2^{-j} (T A_0 \bigcup T A_1 \cdots \bigcup T A_{n-1}) \right)$$

and set

$$K_{n+1} = (K_0 \backslash A_0 \backslash A_1 \dots \backslash A_n) \bigcup (T A_0 \bigcup T A_1 \cdots \bigcup T A_n).$$

Then

$$K_0 \backslash A_0 \backslash A_1 \dots \backslash A_n \subseteq [-N, N]^d,$$

and

$$TA_0 \bigcup TA_1 \cdots \bigcup TA_n \subseteq [-2N, 2N]^d \backslash [-N, N]^d.$$

As expected, if we define $A_n^l = A_n \bigcap K_n^l$ and $K_{n+1}^l = (K_n^l \backslash A_n^l) \bigcup TA_n^l$, then the sets A_n^l are pairwise disjoint, the sets TA_n^l are pairwise disjoint, and the sets K_{n+1}^l are pairwise disjoint for $l = 1, \ldots, L$; and we have the decompositions

$$A_n = \bigcup_{l=1}^{L} A_n^l \ , \ TA_n = \bigcup_{l=1}^{L} TA_n^l \ , \text{ and } \quad K_{n+1} = \bigcup_{l=1}^{L} K_{n+1}^l.$$

We define K^l as

$$K^l = (\bigcap_{n=0,1,\ldots} (K_0^l \backslash A_0^l \backslash A_1^l \ldots \backslash A_n^l)) \bigcup (\bigcup_{n=0,1,\ldots} TA_n^l).$$

Denoting

$$K_n^{l-} = K_0^l \backslash A_0^l \backslash A_1^l \ldots \backslash A_{n-1}^l$$

and

$$K_n^{l+} = TA_0^l \bigcup TA_1^l \cdots \bigcup TA_{n-1}^l,$$

one has

$$K_n^l = K_n^{l-} \bigcup K_n^{l+}$$

and

$$K^l = (\bigcap_{n=0}^{\infty} K_n^{l-}) \bigcup (\bigcup_{n=0}^{\infty} K_n^{l+})$$

for $l = 1, \ldots, L$. Thus each set K^l is obtained by removing from K_0^l all of the A_i^ls and sending these disjoint subsets into $\widehat{\mathbb{R}}^d \backslash [-N, N]^d$ by means of the mapping T. Also, by construction, the sets K^l are pairwise disjoint, and

$$K = \bigcup_{l=1}^{L} K^l,$$

where K is defined as

$$K = (\bigcap_{n=0}^{\infty} K_n^-) \bigcup (\bigcup_{n=0}^{\infty} K_n^+)$$

and K_n^- and K_n^+ are defined as

$$K_n^- = K_0 \backslash A_0 \backslash A_1 \ldots \backslash A_{n-1}$$

and

$$K_n^+ = TA_0 \bigcup TA_1 \cdots \bigcup TA_{n-1}.$$

Thus, we set

$$K_n = K_n^- \bigcup K_n^+$$

as approximants to K.

In order to prove that the sequence $\{K^1, \ldots, K^L\}$ constructed in Construction 4.1 is a wavelet collection of sets, we shall need the following two lemmas whose elementary proofs are in [**BL99**].

Lemma 4.2. *Let* $H \subseteq \widehat{\mathbb{R}}^d$.

(a) Let $a > 0$. $a^j H \bigcap a^m H = \emptyset$ *for all* $j, m \in \mathbb{Z}$, $j \neq m$, *if and only if* $a^{-j} H \bigcap H = \emptyset$ *for all* $j \geq 1$.

(b) $(H + k) \bigcap (H + n) = (H \bigcap (H + n - k)) + k$.

Lemma 4.2b is used in the proof of Lemma 4.3. The set A in Lemma 4.3 can be thought of as one of our sets K^l.

LEMMA 4.3. *Let $A, B \subseteq \widehat{\mathbb{R}}^d$ and assume $A \overset{\mathcal{T}}{\simeq} B$. Then condition a of Theorem 3.6 holds for A if and only if it holds for B.*

Lemma 4.2a and Lemma 4.3 are used in the proof of Theorem 4.4.

THEOREM 4.4. *Let $\{K_0^1, \ldots, K_0^L\} \subseteq \widehat{\mathbb{R}}^d$ and T satisfy the hypotheses for Construction 4.1, and let $\{K^1, \ldots, K^L\}$ be the sequence of sets constructed in Construction 4.1. Then $\{K^1, \ldots, K^L\}$ is a wavelet collection of sets, i.e.,*

$$\{\psi^l = \mathbf{1}_{K^l}^{\vee} : l = 1, \ldots, L\}$$

is a dyadic wavelet collection.

PROOF. The proof follows that of the single wavelet case presented in [**BL99**], and there are no new ideas.

By the construction, each $K^l \overset{\mathcal{T}}{\simeq} K_0^l$, and, by hypothesis, each $K_0^l \overset{\mathcal{T}}{\simeq} [-\frac{1}{2}, \frac{1}{2}]^d$. Thus, each $K^l \overset{\mathcal{T}}{\simeq} [-\frac{1}{2}, \frac{1}{2}]^d$, and, in particular, each $|K^l| = 1$. Clearly, $[-\frac{1}{2}, \frac{1}{2}]^d$ satisfies condition a of Theorem 3.6, and hence each K^l also satisfies condition a by Lemma 4.3.

The proof that $K = \bigcup_{l=1}^{L} K^l$ satisfies condition b of Theorem 3.6 is a bit more technical, but, on the other hand, is identical to the corresponding step in [**BL99**]. Lemma 4.2a is used in this step.

□

5. Examples

This section is devoted to examples based on the construction of Section 4. We shall deal with seven mappings T in the setting

$$T : [-N, N]^d \longrightarrow [-2N, 2N]^d \backslash [-N, N]^d \subseteq \widehat{\mathbb{R}}^d.$$

These are denoted by $T_1, \ldots, T_7$ for a given N. The mappings $T_1, \ldots, T_6$ are illustrated in Figure 1 and T_7 is illustrated in Figure 2. Figures 1 and 2 will be explained shortly. We shall also consider four different collections of sets $K_0^1, \ldots, K_0^L$ satisfying the initial conditions of the construction. They are shown in different gray scales in Figure 3. For example, the collection in the top left subfigure of Figure 3 has four elements, i.e. $L = 4$; and $K_0 = K_0^1 \bigcup \cdots \bigcup K_0^4 = [-1, 1]^2$ since each $K^l \overset{\mathcal{T}}{\simeq} [-\frac{1}{2}, \frac{1}{2}]^2$. The origin is the center of each of the subfigures of Figure 3, and thus we can and do take $N = 1.5$. Thus, in the bottom left subfigure of Figure 3, we have $L = 5$, and the length of the intersection of $K_0 = K_0^1 \bigcup \cdots \bigcup K_0^5$ with either axis is 3, whence $N = 1.5$ is suitable since $K_0 \subseteq [-N, N]^2$.

In order to explain the mappings T_j in Figures 1 and 2, let us look at T_1 in Figure 1. The central square, which is subdivided into four congruent subsquares, is $[-N, N]^2$. Suppose a set A is contained in $[-N, N]^2$ and TA is to be computed. If A_{+-} denotes that subset of A contained in the subsquare labelled $(2N, -2N)$, then TA_{+-} is the translation of A_{+-} by $(2N, -2N)$, and is therefore contained in the bottom right black square. The subsets A_{--}, A_{++}, A_{-+} of A are similarly defined, and TA_{--}, TA_{++}, TA_{-+} are the translations of A_{--}, A_{++}, A_{-+} by $(-2N, -2N)$, $(2N, 2N)$, $(-2N, 2N)$, respectively. In particular, TA_{--}, TA_{++}, TA_{-+} are subsets of the three other black squares. The other mappings are defined in an analogous way.

Figure 4 is the approximation K_5 of K corresponding to T_1 and to each of the four collections of sets in Figure 3. (The sets K_n and K were defined in Section 4.) The top left subfigure of Figure 4 corresponds to the collection depicted in the top left subfigure of Figure 3, etc. Of course, each K_n and K are decomposed in terms of the K_n^l and K^l, respectively; and this decomposition is reflected by the gray scale patterns representing K_5 in Figure 4.

The subfigures of Figure 5 illustrate the unions

$$K_5 \bigcup 2^{-1} K_5 \bigcup 2^{-2} K_5$$

(each one on different gray scale) corresponding to T_1 and to each of the four collections of sets in Figure 3. It illustrates condition b of Theorem 3.6.

The next six pairs of figures deal with the mappings $T_2, \ldots, T_7$ just as Figures 4 and 5 deal with T_1.

These figures, and the theory on which they are based, lead to geometrical questions concerning the complexity of K and its decompositions, and related issues about self-similarity or non-fractal behavior. Analytically, there are analogous questions about the slow decay of the dyadic wavelet collections we have constructed, as well as smoothing procedures for these wavelets. Notwithstanding a natural caution which such questions dictate when one thinks of applicability, we reiterate the elementary nature of our approach. In fact, our algorithm is general, its implementation is simple, and the computation of the approximants K_n^l is fast.

Appendix A. Pseudo-Code

Let $K_0^l \subseteq [-N_1, N_1]^d$ for $l = 1, \ldots, L$, and let $T : [-N_1, N_1]^d \longrightarrow [-N_2, N_2]^d \backslash [-N_1, N_1]^d$ be defined as required in Section 4. Define $f_n^l(\gamma) = \mathbf{1}_{K_n^l}(\gamma)$. The recursive functions f_n^l, for $l = 1, \ldots, L$, and $n = 1, 2, \ldots$ are given by:

```
f_n^l(γ)
{
    if(n == 0) return(1_{K_0^l}(γ));
    if(γ ∉ [−N_1, N_1]^d)
        {
        if(γ ∉ range(T)) return(0);
        if(f_{n−1}^l(γ)) return(1);
        γ = T^{−1}(γ);
        if(f_{n−1}^l(γ))
            while((λ = 2γ) & (λ ∈ [−N_2, N_2]^d))
                if(f_{n−1}^l(λ)) return(1);
            return(0);
        }
    if(tmp=f_{n−1}^l(γ))
        while((λ = 2γ) & (λ ∈ [−N_2, N_2]^d))
            if(f_{n−1}^l(λ)) return(0);
        return(tmp);
}
```

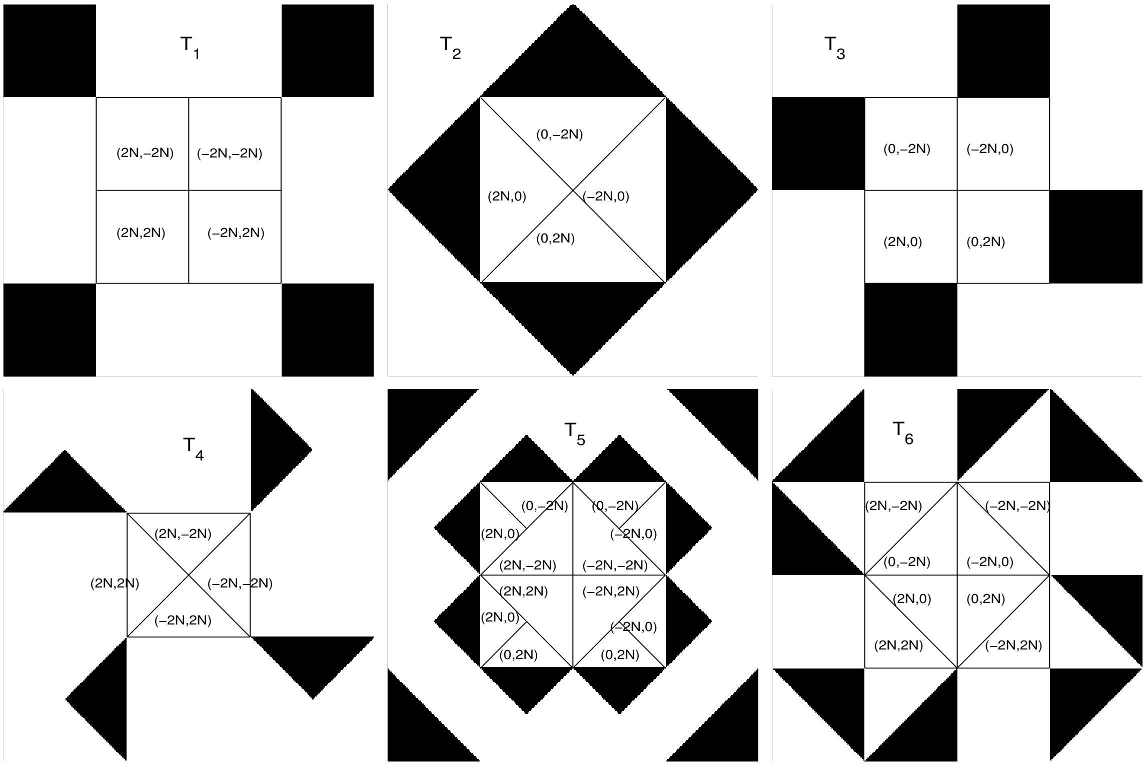

Figure 1. Examples of mappings $T_1, \ldots, T_6$

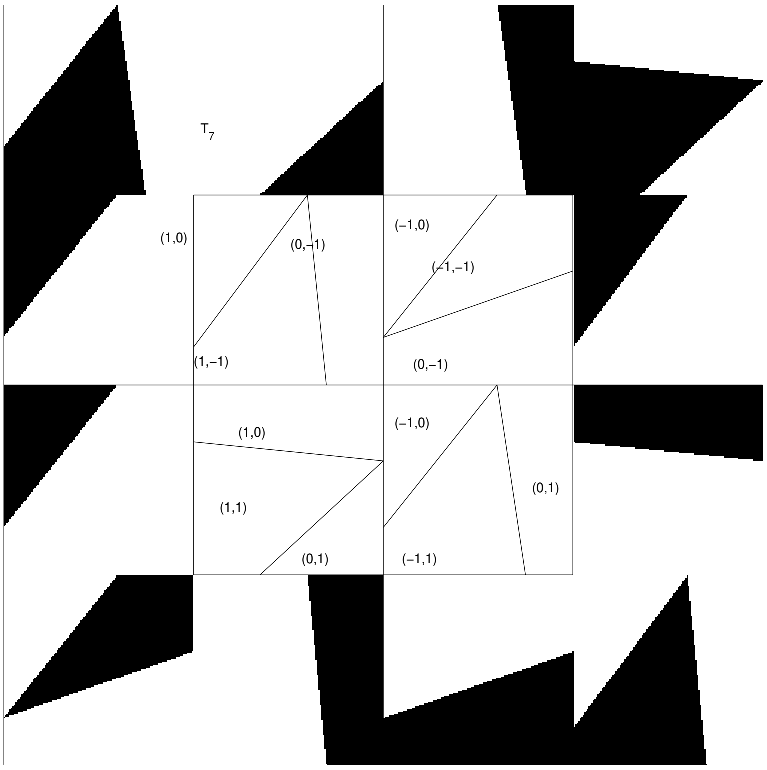

Figure 2. T_7

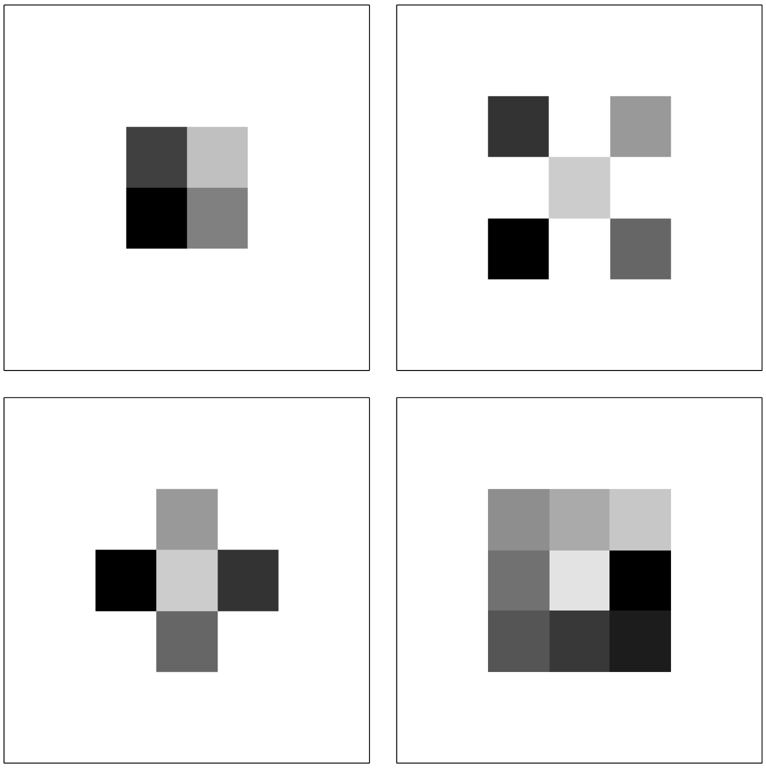

FIGURE 3. Four collections $K^1, \ldots, K^L$

References

[BCMO95] L Baggett, A. Carey, W. Moran, and P. Ohring. General existence theorems for orthonormal wavelets, an abstract approach. *Publ. RIMS, Kyoto Univ.*, 31:95–111, 1995.

[Ben97] J. J. Benedetto. *Harmonic Analysis and Applications.* CRC Press, Inc., Boca Raton, FL, 1997.

[Ben98] J. J. Benedetto. Frames, sampling, and seizure prediction. In K.-S. Lau, editor, *Advances in Wavelets.* Springer-Verlag, New York, 1998.

[BL99] J. J. Benedetto and M. Leon. The construction of single wavelets in d-dimensions. *The Journal of Geometric Analysis*, 1999. To appear.

[BMM00] L Baggett, H. Medina, and K. Merrill. Generalized multiresolution analysis, and a construction procedure for all wavelet sets in $\mathbb{R}^n$. *J. Fourier Analysis and Applications*, 2000. To appear.

[BW94] J. J. Benedetto and D. F. Walnut. Gabor frames for L^2 and related spaces. In J. J. Benedetto and M. W. Frazier, editors, *Wavelets: Mathematics and Applications*, pages 97–162. CRC Press, Inc., Boca Raton, FL, 1994.

[Cal99a] A. Calogero. A characterization of wavelets on general lattices. *The Journal of Geometric Analysis*, 1999. To appear.

[Cal99b] A. Calogero. Wavelets on general lattices, associated with general expanding maps of $\mathbb{R}^n$. *AMS Research Announcements*, 1999.

[CD93] A. Cohen and I. Daubechies. Non-separable bidimensional wavelet bases. *Revista Matematica Ibero-americana*, 9:51–137, 1993.

[Dau92] I. Daubechies. *Ten Lectures in Wavelets.* CBMS-NSF Series in Applied Math. 61. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[DFN92] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov. *Modern Geometry. Methods and Applications.* Springer-Verlag, New York, 1992.

[DL98] X. Dai and D. R. Larson. Wandering vectors for unitary systems and orthogonal wavelets. *Memoirs Amer. Math. Soc.*, 1998. Providence, RI.

[DLS97] X. Dai, D. R. Larson, and D. Speegle. Wavelet sets in $\mathbb{R}^n$. *J. Fourier Analysis and Applications*, 3:451–456, 1997.

[DLS98] X. Dai, D. R. Larson, and D. Speegle. Wavelet sets in $\mathbb{R}^n$, II. *AMS Contemporary Math.*, 216:15–40, 1998.

[FGWW97] M. W. Frazier, G. Garrigós, K. Wang, and G. Weiss. A characterization of functions that generate wavelet and related expansion. *J. Fourier Analysis and Applications, Special Issue Dedicated to M. De Guzmán. El Escorial 1996*, 3:883–906, 1997.

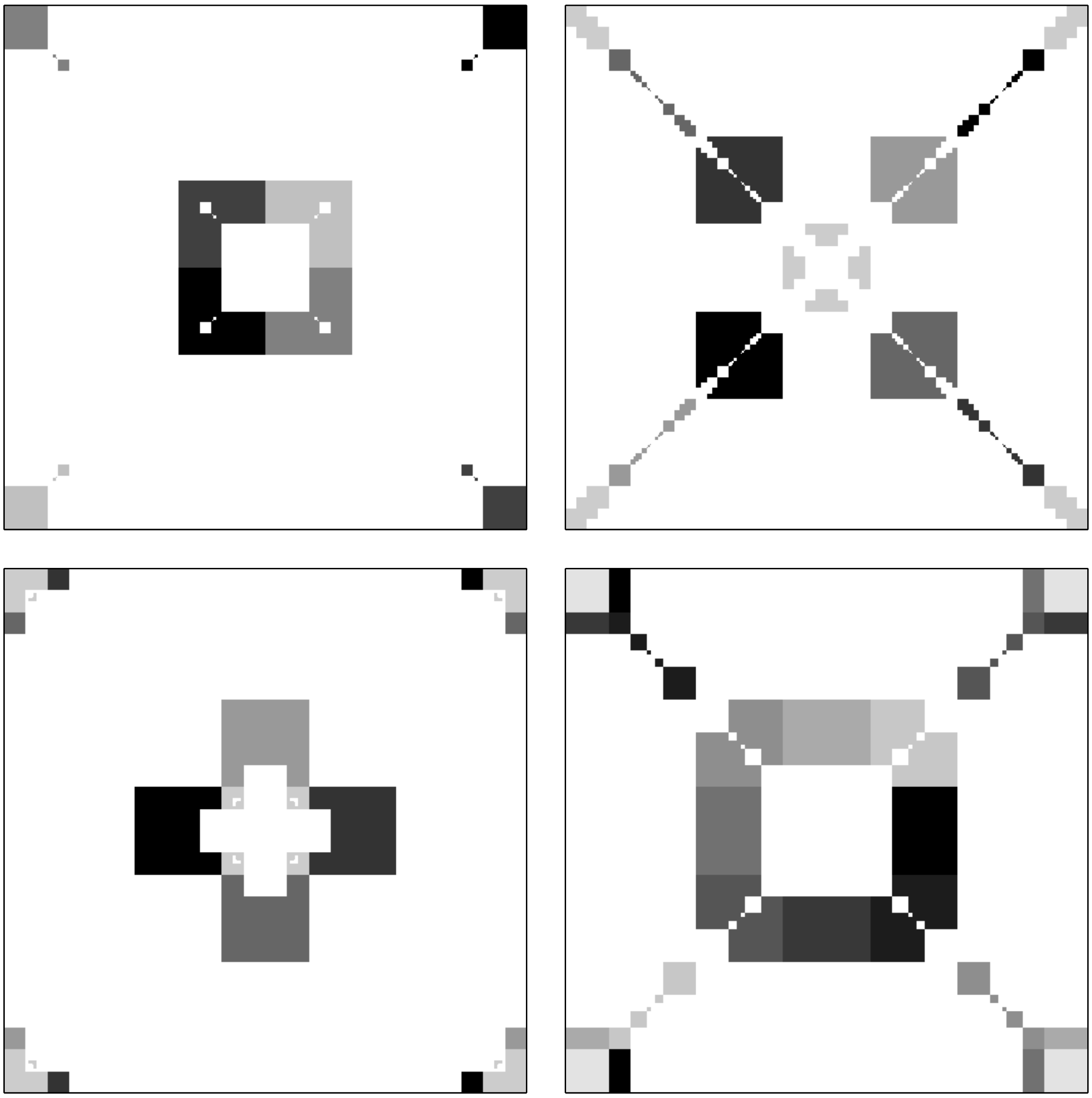

FIGURE 4. Examples of K_5 for T_1

[GG81] I. Gohberg and S. Goldberg. *Basic Operator Theory.* Birkhäuser, Boston, 1981.

[GH99] Q. Gu and D. Han. On multiresolution analysis (MRA) wavelets in $\mathbb{R}^n$. 1999. Preprint.

[GM92] K.-H. Gröchenig and W. R. Madych. Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbb{R}^d$. *IEEE Trans. Inf. Theory*, 38(2):556–568, 1992.

[Gri95] G. Gripenberg. A necessary and sufficient condition for the existence of a father wavelet. *Studia Math.*, 114:207–226, 1995.

[Han97] D. Han. *Unitary Systems, Wavelets, and Operator Algebras.* PhD thesis, Texas A & M University, 1997.

[HW96] E. Hernández and G. Weiss. *A First Course in Wavelets.* CRC Press, Inc., Boca Raton, FL, 1996.

[HWW96] E. Hernández, X. Wang, and G. Weiss. Smoothing minimally supported frequency wavelets. Part I. *J. Fourier Analysis and Applications*, 2:329–340, 1996.

[HWW97] E. Hernández, X. Wang, and G. Weiss. Smoothing minimally supported frequency (MSF) wavelets. Part II. *J. Fourier Analysis and Applications*, 3:23–41, 1997.

[KV92] J. Kovačević and M. Vetterli. Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$. *IEEE Trans. Inf. Theory*, 38(2):533–555, 1992.

[Leo99] M. T. Leon. On minimally supported frequency wavelets. 1999. Preprint. To be submitted.

[LR91] W.M. Lawton and H.L. Resnikoff. Multidimensional wavelet bases. Technical report, Aware, Inc., Bedford, MA, 1991.

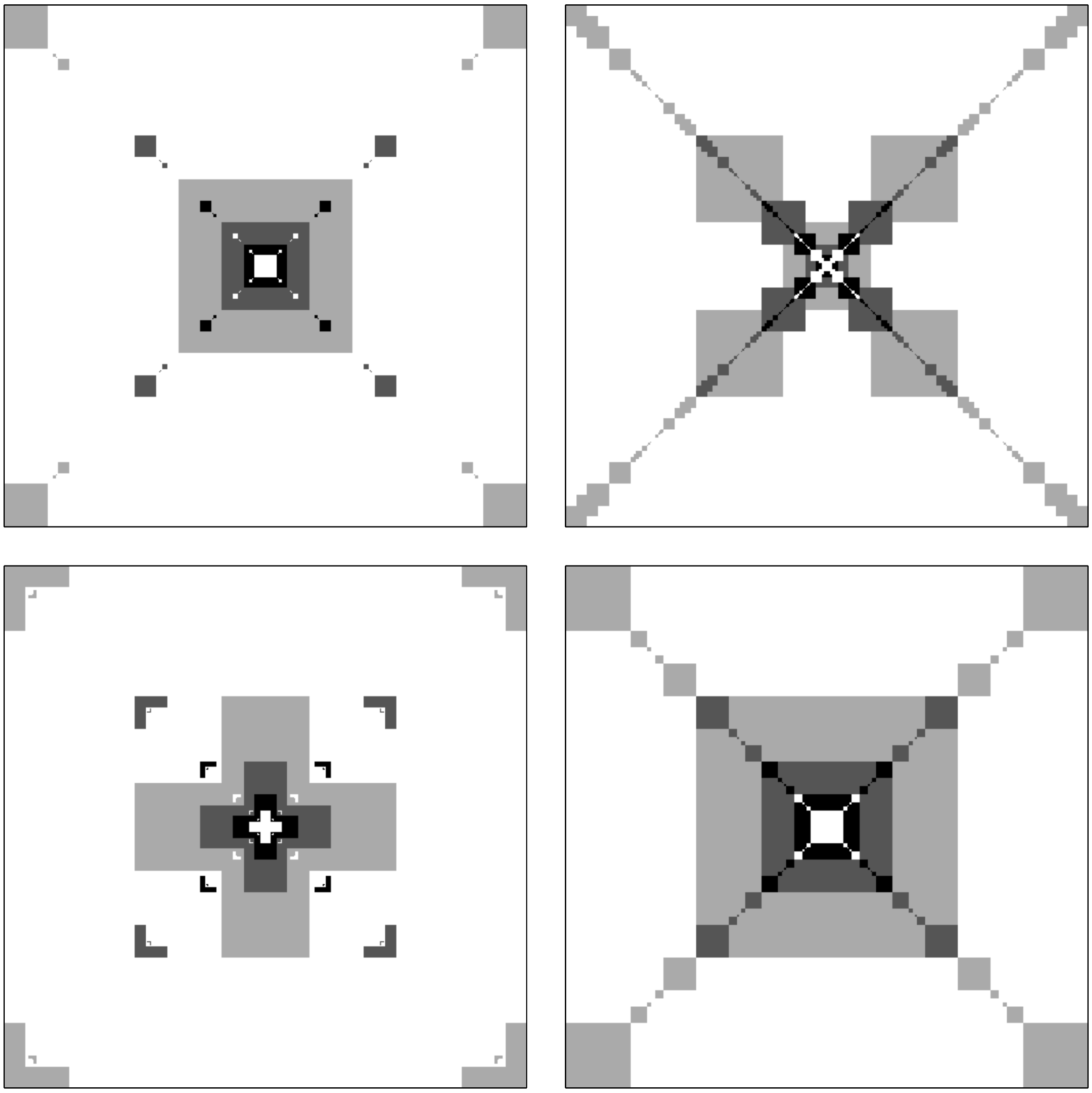

FIGURE 5. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_1

[LW97] J. C. Lagarias and Y. Wang. Integral self-affine tiles in $\mathbb{R}^n$, Part II: Lattice tilings. *J. Fourier Analysis and Applications*, 3:83–102, 1997.

[LW00] J. C. Lagarias and Y. Wang. Orthogonality criteria for compactly supported refinable functions and refinable function vectors. *J. Fourier Analysis and Applications*, 2000. To appear.

[Mad92] W. R. Madych. Some elmentary properties of multiresolution analysis of $L^2(\mathbb{R}^n)$. In C. K. Chui, editor, *Wavelets: A Tutorial in Theory and Applications*, pages 259–294. Academic Press, Inc, 1992.

[Mal89] S. Mallat. Multiresolution approximation and wavelet orthonormal bases of $L^2(\mathbb{R}^n)$. *Trans. Amer. Math. Soc.*, 315:69–88, 1989.

[Mey86] Y. Meyer. Ondelettes, fonctions splines, et analyses graduées. Technical report, University of Torino 1986 and Rapport CEREMADE 8703 (1987), 1986.

[Mey92] Y. Meyer. *Wavelets and Operators.* Cambridge University Press, 1992. Translated from the 1990 French Edition by D. H. Salinger.

[Str94] R. S. Strichartz. Construction of orthonormal wavelets. In J. J. Benedetto and M. W. Frazier, editors, *Wavelets: Mathematics and Applications*, pages 23–50. CRC Press, Inc., Boca Raton, 1994.

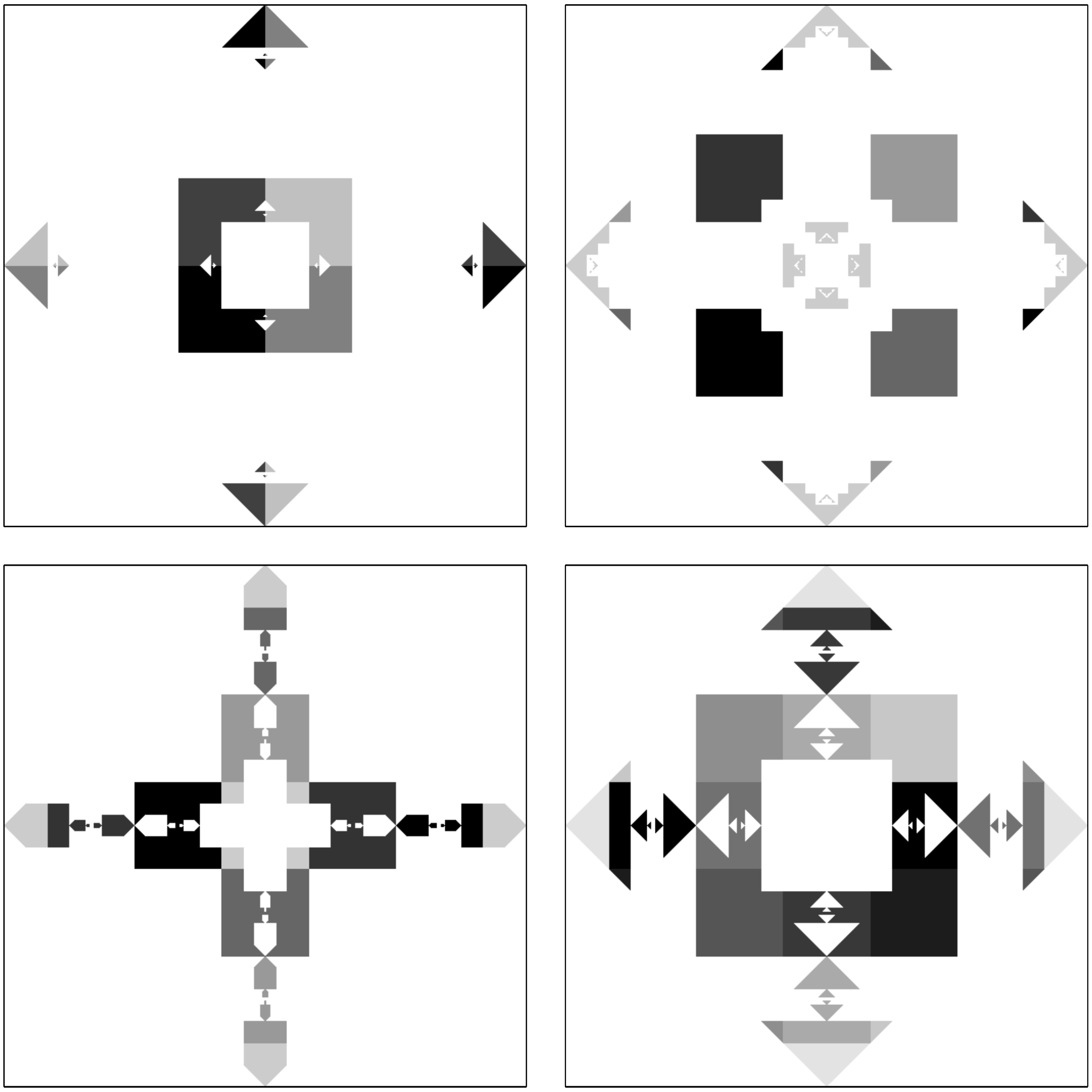

FIGURE 6. Examples of K_5 for T_2

[SW71] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton mathematical series, 32. Princeton University Press, Princeton, N.J., 1971.

[SW98] P. Soardi and D. Weiland. Single wavelets in n-dimensions. *J. Fourier Analysis and Applications*, 4:299–315, 1998.

[Wan95] X. Wang. *The study of wavelets from the properties of their Fourier Transform.* PhD thesis, Washington University in St. Louis, 1995.

[Zak96] V. Zakharov. Nonseparable multidimensional Littlewood-Paley like wavelet bases. 1996. Preprint.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742

E-mail address: jjb@math.umd.edu, mleon@math.umd.edu

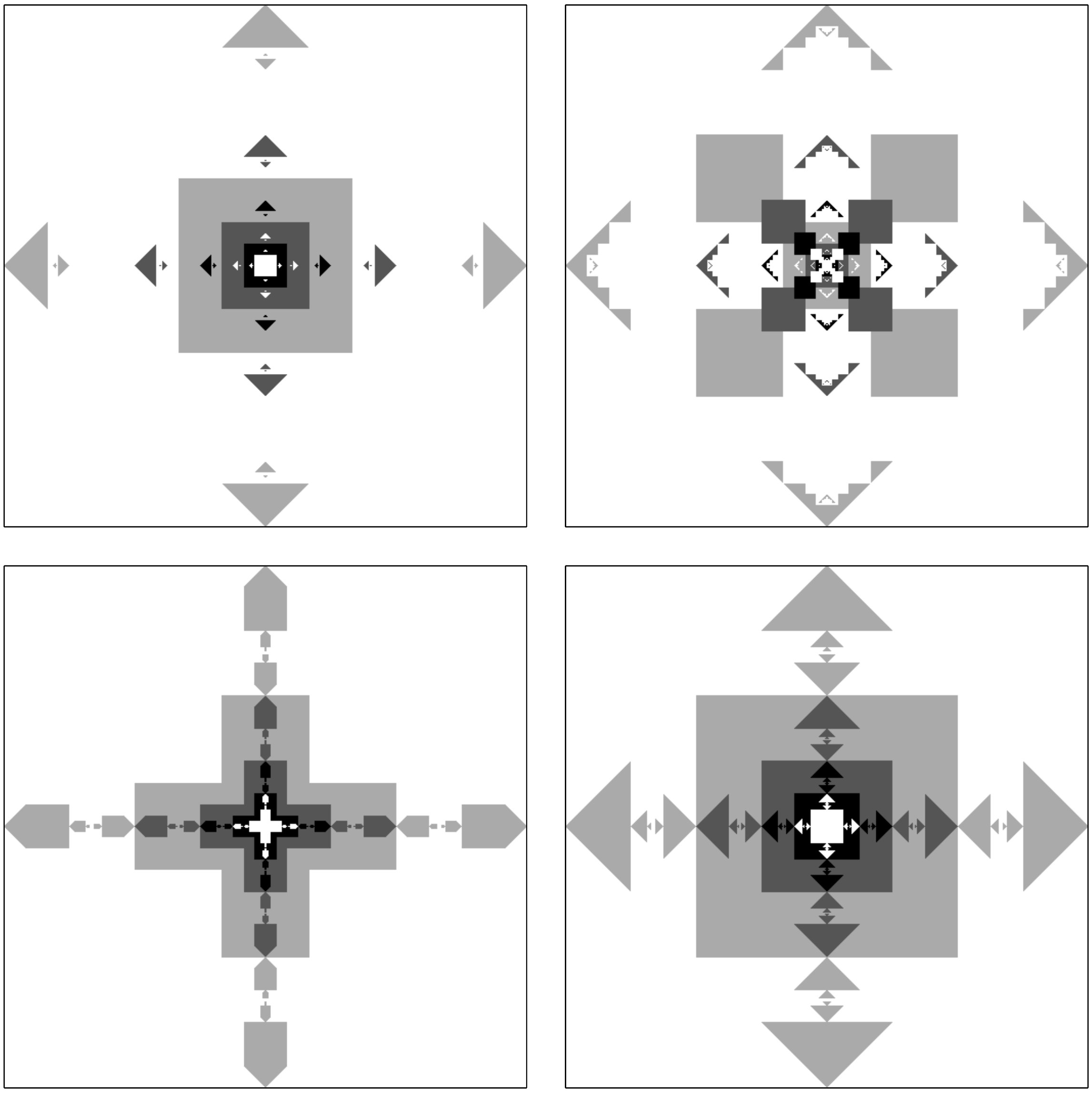

FIGURE 7. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_2

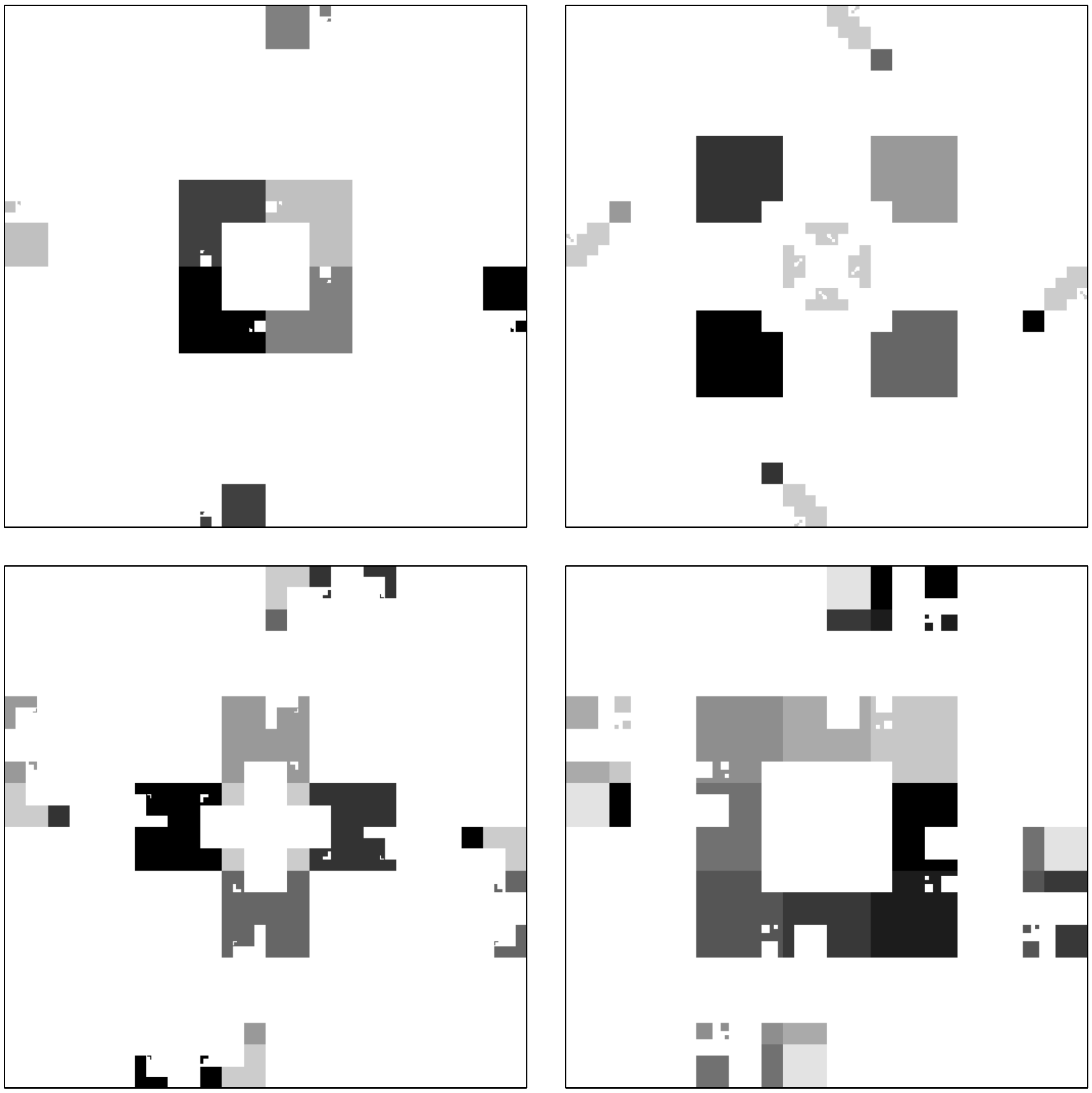

FIGURE 8. Examples of K_5 for T_3

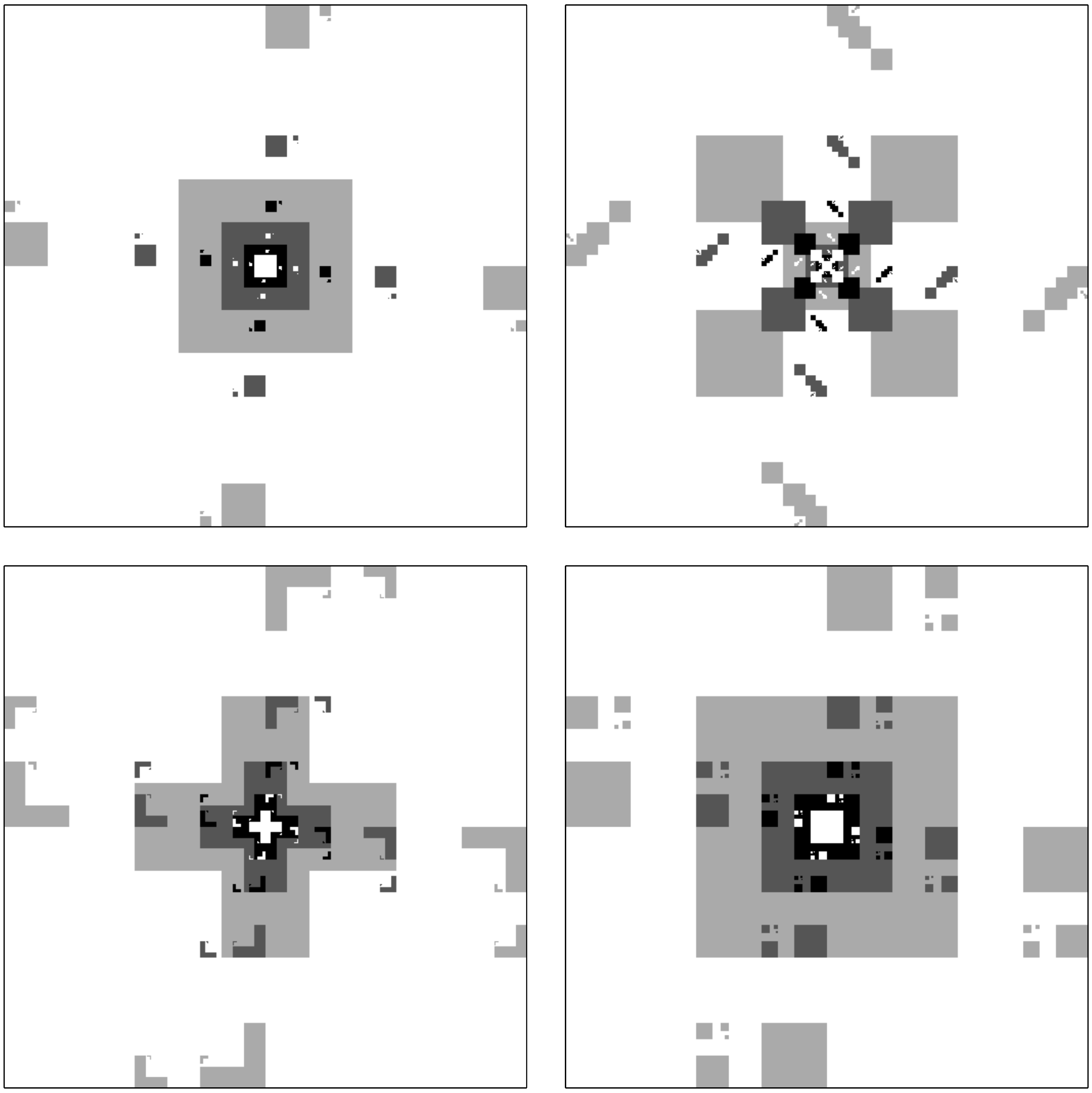

FIGURE 9. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_3

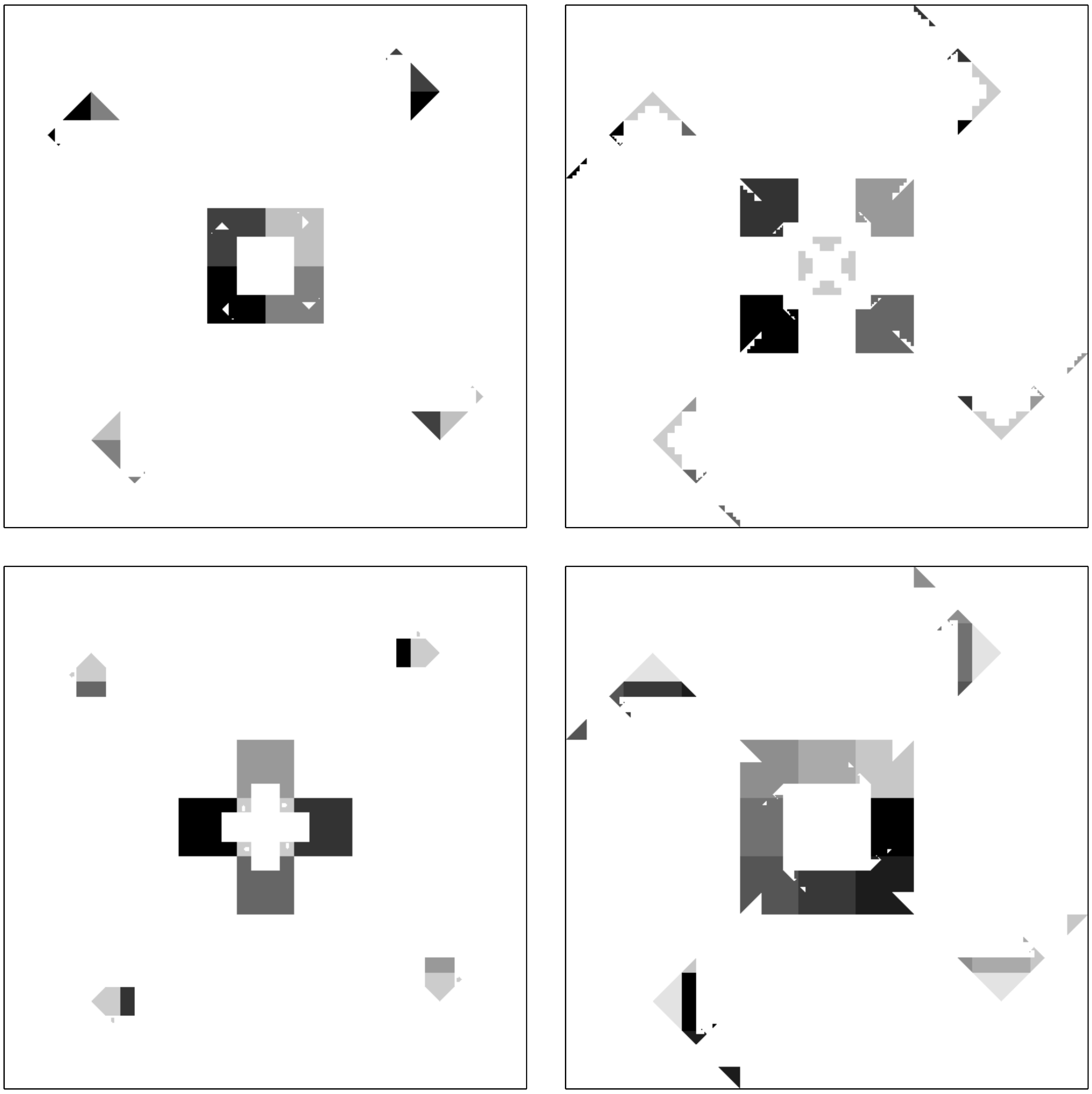

FIGURE 10. Examples of K_5 for T_4

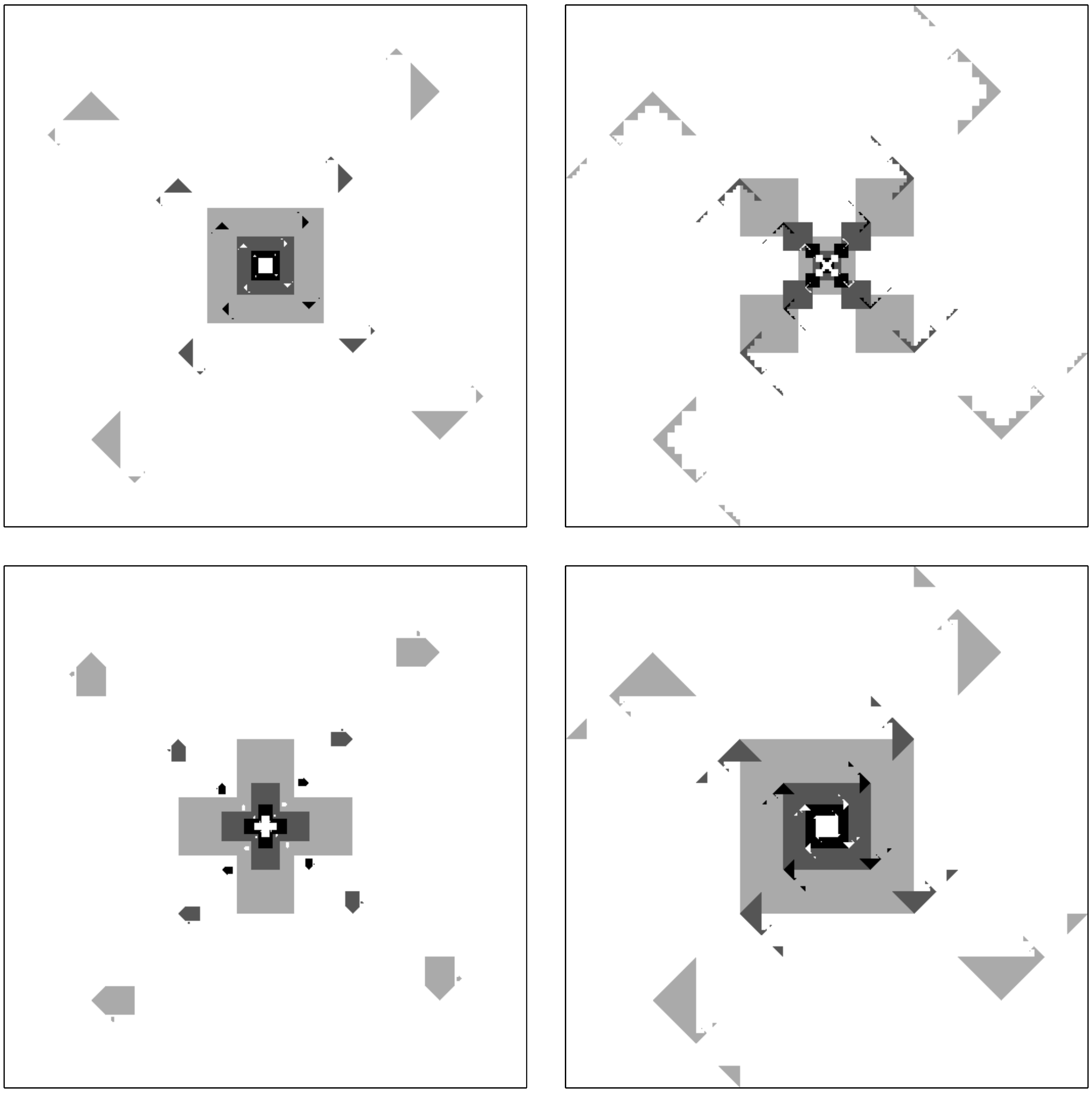

FIGURE 11. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_4

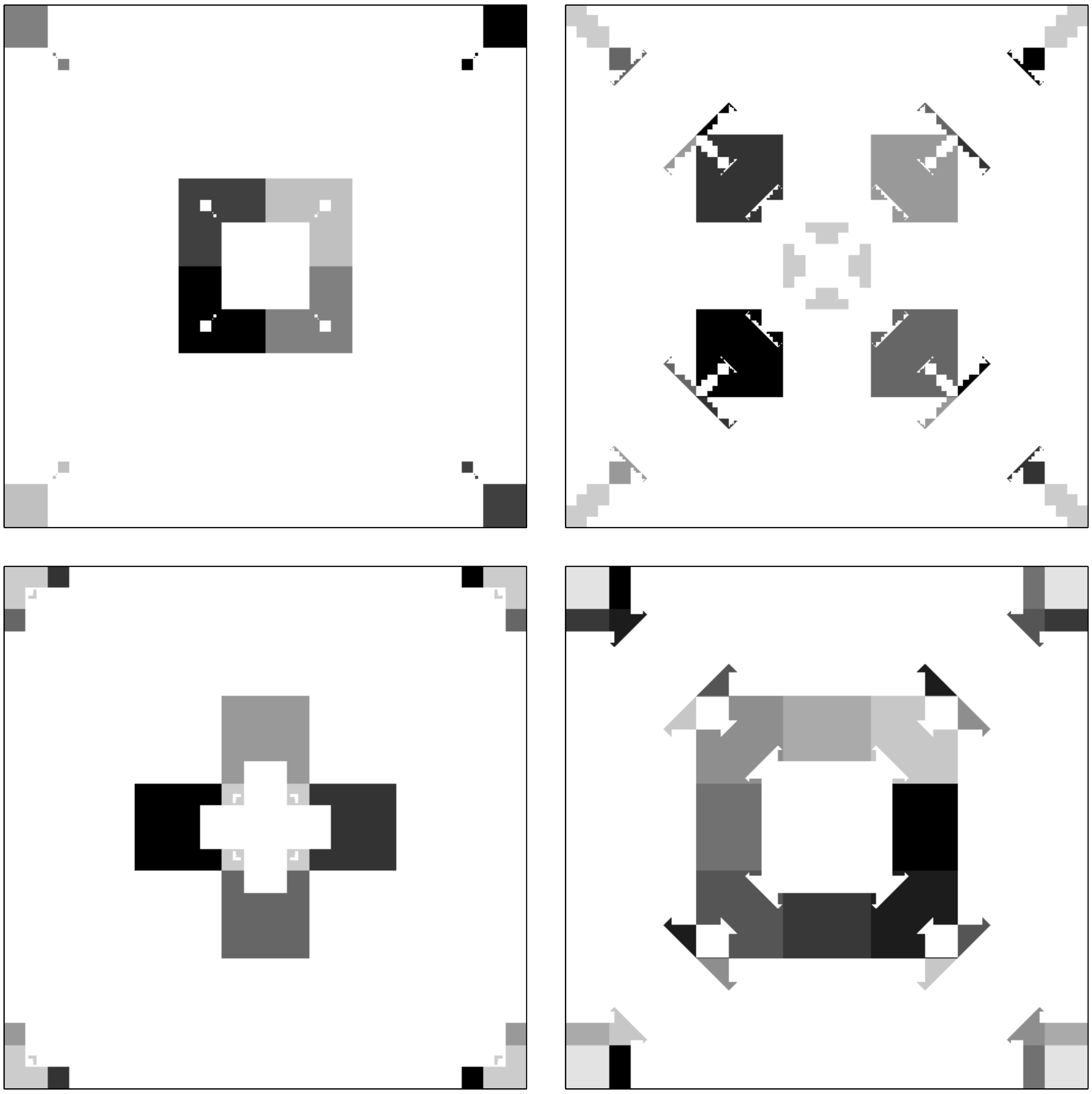

FIGURE 12. Examples of K_5 for T_5

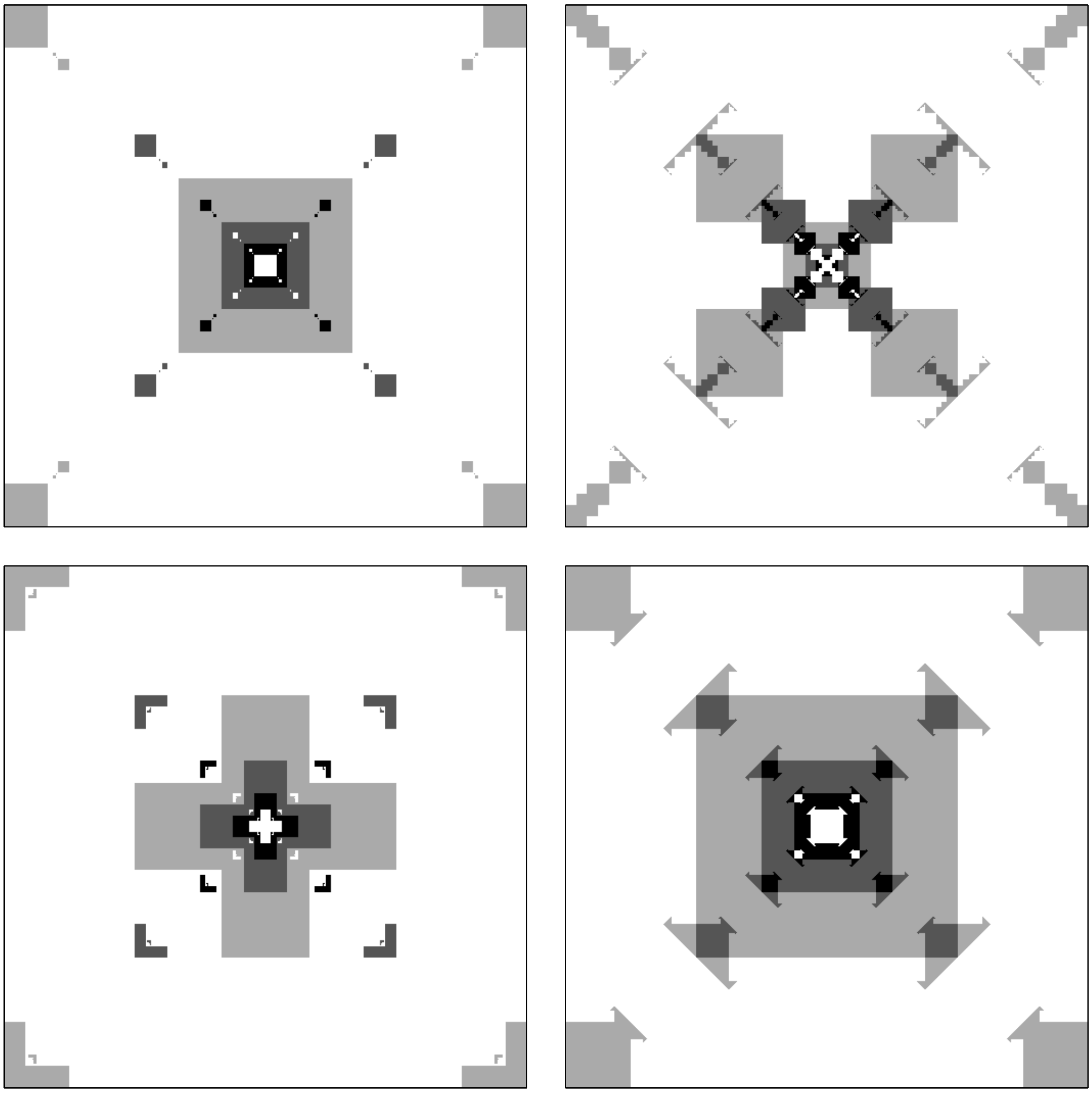

FIGURE 13. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_5

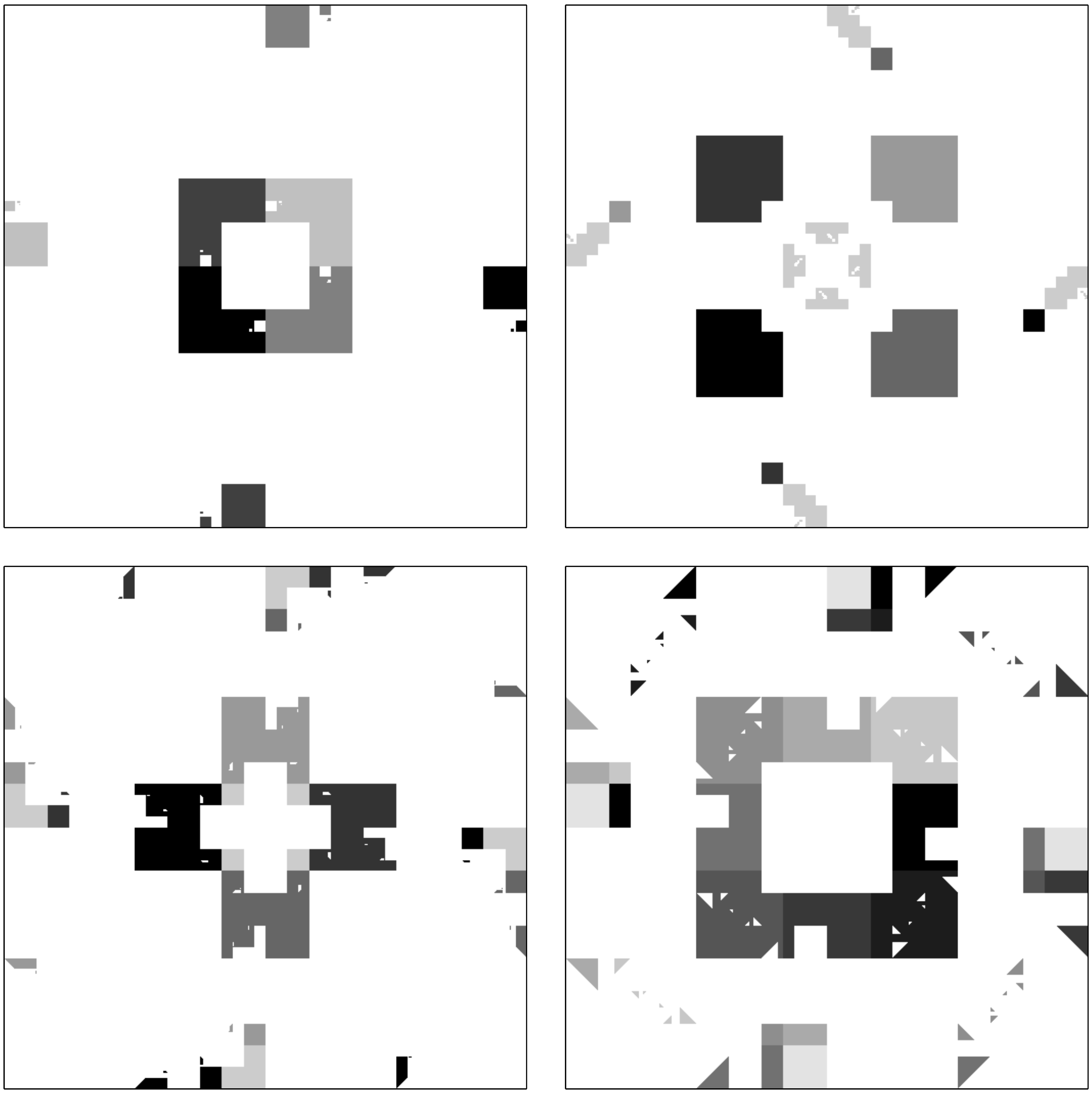

FIGURE 14. Examples of K_5 for T_6

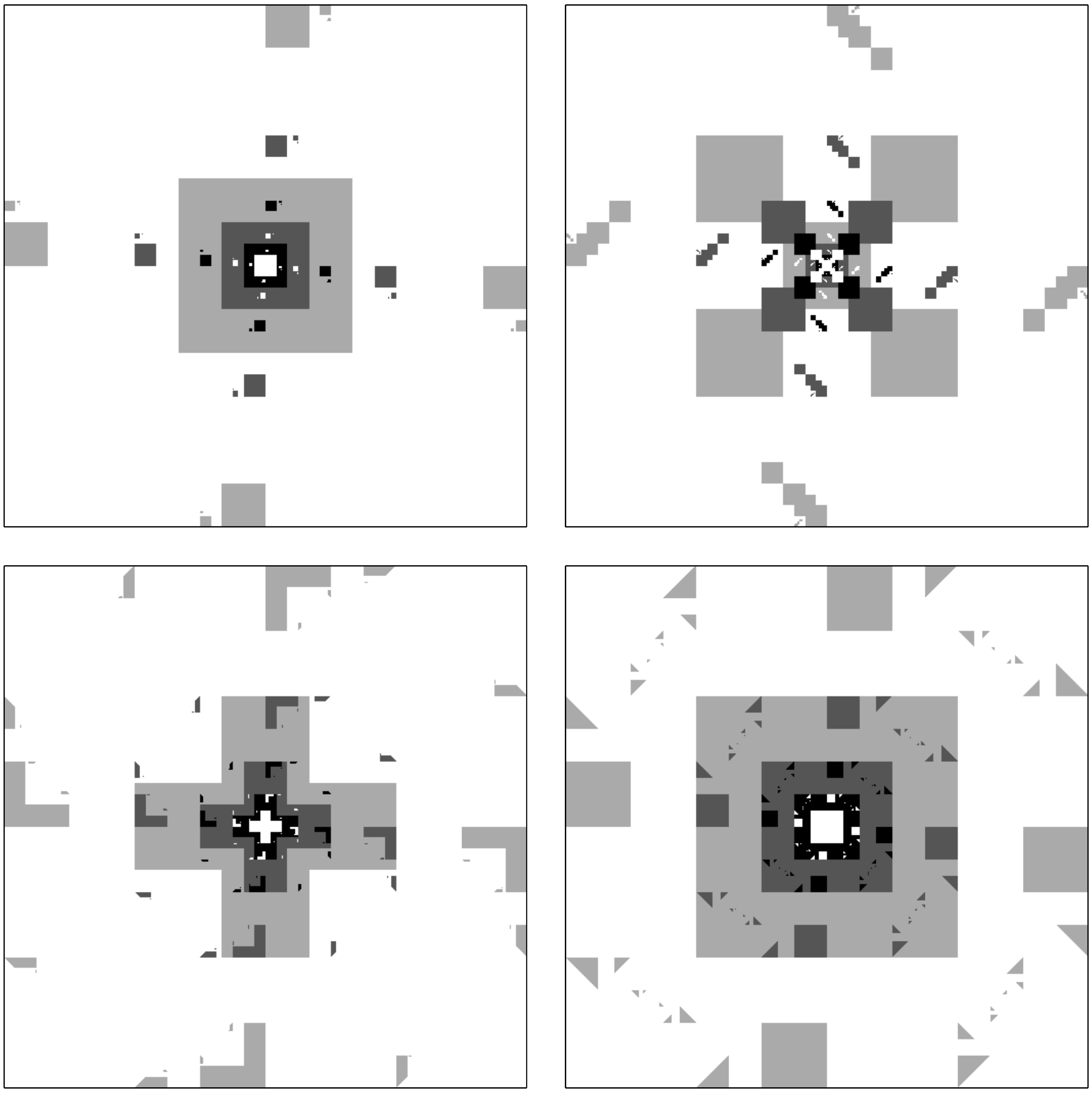

FIGURE 15. Concentric $K_5, 2^{-1}K_5$ and $2^{-2}K_5$ for T_6

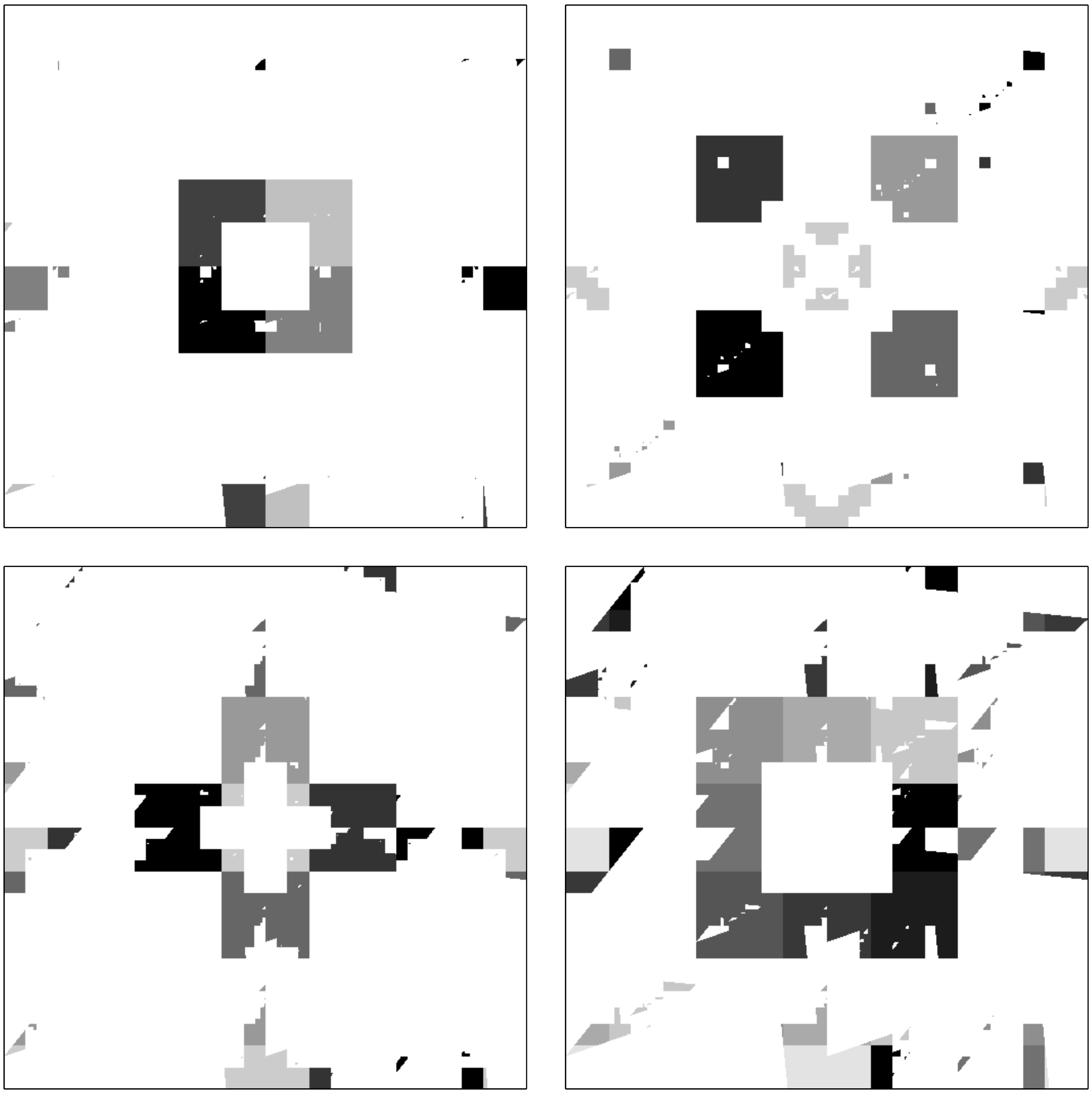

FIGURE 16. Examples of K_5 for T_7