Wavelet Frames: Multiresolution Analysis and Extension Principles

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ABSTRACT After reviewing basic ideas of frame theory from a functional analysis point of view, we discuss two approaches for the construction of (affine) wavelet frames. The theory of Frame Multiresolution Analysis as introduced in [BeLi93] is presented in a streamlined form and the main result of the theory is completed. The interplay between redundancy and robustness in frame expansions is illustrated by a simple example. We then restate Ron and Shen's Unitary Extension Principle and give a simple direct proof different from the original derivation in [RoSh97a].

1 Introduction

Multiresolution analysis (MRA) and the theory of frames are highly developed subjects with origins in speech and image processing and the theory of non-harmonic Fourier series, respectively. The work of Mallat [Mal89] and Meyer [Mey90] is representative of the premier results on MRAs; and, after Duffin and Schaeffer's paper [DuSc52], the work of Daubechies, Meyer, and Grossmann [DGM86] and Daubechies [Dau92] contains the fundamental results about frames.

The frame concept for Hilbert spaces generalizes the notion of an orthonormal basis (ONB) in the sense that a frame $X = \{x_i : i \in I\}$ provides a stable representation for signals f by means of an expansion $f = \sum_i c_i(f)\, x_i$, but it is not necessarily an orthonormal or independent sequence.

Frames not only generalize ONBs, but also Riesz bases, which are defined as images of ONBs under bounded, invertible operators, and, hence, include all ONBs. Further, and important for their applicability, the coefficients $c_i(f)$ in a frame expansion of a signal f are computable and depend continuously on f, and redundancies are allowed in the frame itself. This latter property can lead to more freedom when constructing atoms, i.e., frame elements, $x_i \in X$ for specific types of expansions. It can also give additional "robustness" in the reconstruction of a signal from corrupted coefficients with respect to a frame expansion.

It is natural to combine the theories of frames and MRAs with the goal of attacking problems in signal processing. This has led to the notion of frame multiresolution analysis (FMRA), [BeLi93], [BeLi98]. There has also been a general development on the construction of so-called tight wavelet frames, spearheaded by the work of Ron and Shen, e.g., [RoSh95], [RoSh97a], [RoSh97b], [RoSh97c]. This paper is devoted to expositing and comparing these topics, and formulating some of the basic results in a new way.

After establishing notation in Section 2, we shall give an overview of the basic ideas of frame theory in Section 3. While there are no new results in Section 3, the presentation is completely in terms of functional analysis, which itself streamlines the theory and lays the groundwork for a possibly different point of view.

Section 4 is devoted to FMRAs. As mentioned, the theory appears in [BeLi98]. However, our functional analysis presentation gives another perspective on the theory, and proves a new result (Theorem 4.11) which adds insight into the constructive proofs of [BeLi98], as well as establishes a characterization and criterion for the existence of FMRA wavelet frames. The theory of FMRAs is essentially a local theory, beginning with criteria for constructing frames of integer translates, and then using the FMRA structure to build frames for a large space such as $L^2(\mathbb{R})$ (defined in Section 2). A feature of this approach is its compatibility with the theory of multirate systems in signal processing, and in particular with the construction of special types of filter banks. This relationship is the subject of Sections 5 and 6. In Section 5 a specific FMRA is analyzed, and quantitatively compared with orthogonal decompositions; and, in Section 6, we show that FMRAs can lead to perfect reconstruction multirate systems.

Section 7 is devoted to the construction of tight wavelet frames using the Unitary Extension Principle theorem originally proved by Ron and Shen. The contribution herein is twofold. First, we make an elementary refinement of the original Unitary Extension Principle theorem, which allows us to relate it to the aforementioned criteria for constructing frames of translates in the theory of FMRAs. More important, we give a straightforward, relatively short proof of the Unitary Extension Principle theorem using standard Fourier analysis. For comparison with FMRAs, we note that Ron and Shen's point of view is global in the sense that a wavelet frame is a global property involving all dilations and translations. In particular, a wavelet frame for $L^2(\mathbb{R})$ is not necessarily a wavelet frame for some dilation level subspace of $L^2(\mathbb{R})$.

2 Notation

H will denote a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|x\| = \langle x, x\rangle^{1/2}$ for each $x \in H$.

$L^2(\mathbb{R})$ is the Hilbert space of square-integrable functions on the real line $\mathbb{R}$, i.e., $L^2(\mathbb{R})$ is the set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ with norm

$$\|f\|_{L^2(\mathbb{R})} = \left(\int |f|^2\right)^{1/2} < \infty,$$

where integration "$\int_{\mathbb{R}}$" over $\mathbb{R}$ is designated "$\int$". The inner product $\langle\cdot,\cdot\rangle$ for $L^2(\mathbb{R})$ is defined as

$$\forall\, f, g \in L^2(\mathbb{R}), \quad \langle f, g\rangle = \int f\overline{g}.$$

$L^1(\mathbb{R})$ is the set of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^1(\mathbb{R})} = \int |f| < \infty.$$

We define the *Fourier transform* $\widehat{f}$ of $f \in L^1(\mathbb{R})$ by

$$\forall\, \gamma \in \widehat{\mathbb{R}}, \quad \widehat{f}(\gamma) = \int f(t)\, e^{-2\pi i t\gamma}\, dt, \tag{1.1}$$

where $\widehat{\mathbb{R}}$ is $\mathbb{R}$ considered as the domain of the function $\widehat{f}$. The Riemann-Lebesgue Lemma asserts that $\widehat{f}$ is a continuous function vanishing at $\pm\infty$ if $f \in L^1(\mathbb{R})$. The Parseval-Plancherel Theorem for $\mathbb{R}$ states that the Fourier transform extends from $L^1 \cap L^2(\mathbb{R})$ to a unitary operator on $L^2(\mathbb{R})$. This extension is designated the Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{R})$, and we use the notation of (1.1) for the $L^2(\mathbb{R})$-setting as well as for $L^1(\mathbb{R})$.

We shall also deal with the Hilbert spaces $l^2(\mathbb{Z})$ and $L^2(\mathbb{T})$, where $\mathbb{Z}$ is the set of integers and $\mathbb{T}$ is the torus defined as $\widehat{\mathbb{R}}/\mathbb{Z}$. $l^2(\mathbb{Z})$ is the space of square-integrable sequences $c = \{c_k : k \in \mathbb{Z}\} \subseteq \mathbb{C}$ with norm

$$\|c\|_{l^2(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |c_k|^2\right)^{1/2},$$

and inner product

$$\forall\, c, d \in l^2(\mathbb{Z}), \quad \langle c, d\rangle = \sum_{k\in\mathbb{Z}} c_k \overline{d}_k.$$

$L^2(\mathbb{T})$ is the space of 1-periodic complex-valued Lebesgue measurable functions F on $\widehat{\mathbb{R}}$ with norm

$$\|F\|_{L^2(\mathbb{T})} = \left(\int_{\mathbb{T}} |F|^2\right)^{1/2} = \left(\int_{-1/2}^{1/2} |F|^2\right)^{1/2} < \infty,$$

and inner product

$$\forall\, F, G \in L^2(\mathbb{T}), \quad \langle F, G\rangle = \int_{\mathbb{T}} F\overline{G}.$$

Recall that F is 1-periodic on $\widehat{\mathbb{R}}$ if $F(\gamma) = F(\gamma+1)$ for all $\gamma \in \widehat{\mathbb{R}}$. This does not necessarily mean that 1 is the largest period of F.

If $c \in l^2(\mathbb{Z})$, $\widehat{c} \in L^2(\mathbb{T})$ denotes its 1-periodic Fourier transform

$$\widehat{c}(\gamma) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\gamma},$$

i.e., $\widehat{c}$ is a Fourier series with Fourier coefficients $c = \{c_k : k \in \mathbb{Z}\}$.

In the case of Hilbert spaces, convergence will be in H unless otherwise stated. In particular, if $H = L^2(\mathbb{R})$ or if we are dealing with Fourier series in $L^2(\mathbb{T})$, then convergence is in L^2-norm.

Translation by $y \in \mathbb{R}$ is denoted by τ_y, i.e., if $f : \mathbb{R} \to \mathbb{C}$ is a function, then $\tau_y f : \mathbb{R} \to \mathbb{C}$ is the function defined by $(\tau_y f)(t) = f(t-y)$, and, for $E \subseteq \mathbb{R}$, $\tau_y E = \{t + y : t \in E\}$.

The function $e_\gamma : \mathbb{R} \to \mathbb{C}$ is defined as

$$\forall\, t \in \mathbb{R}, \quad e_\gamma(t) = e^{-2\pi i t\gamma}.$$

Further, the *unitary dyadic dilation operator* D is defined on $L^2(\mathbb{R})$ as

$$\forall\, f \in L^2(\mathbb{R}), \qquad (Df)(t) = \sqrt{2} f(2t),$$

and, hence, $(D^j f)(t) = 2^{j/2} f(2^j t)$ for all $j \in \mathbb{Z}$. With this notation we have the following formulas:

$$\begin{aligned} \widehat{f}(\gamma) &= \int f e_\gamma, \\ (\tau_y f)^\wedge &= e_y \widehat{f}, \end{aligned}$$

and, as a manifestation of the uncertainty principle,

$$(Df)^\wedge = D^{-1}\widehat{f}.$$

We define the *periodization operator* P as follows: for a function f on $\mathbb{R}$, let

$$P(f) = \sum_{k\in\mathbb{Z}} \tau_k f.$$

Notice that $P(f)$ is an 1-periodic function; and, if g is 1-periodic and $gf \in L^1(\mathbb{R})$, then

$$\int gf = \int_{\mathbb{T}} gP(f).$$

Brackets are used to denote pullbacks, e.g.,

$$[f = 0] = \{t \in \operatorname{domain}(f) : f(t) = 0\}.$$

The Lebesgue measure of a set $E \subseteq \mathbb{R}$ is denoted by $|E|$. The characteristic or indicator function of a set E is $\mathbf{1}_E$.

If K_1 and K_2 are linear spaces and $L : K_1 \to K_2$ is a linear operator, then $\mathcal{R}(L) \subseteq K_2$ is the range of L and $\mathcal{N}(L) \subseteq K_1$ is its kernel, i.e.,

$$\mathcal{N}(L) = \{x \in K_1 : L(x) = 0 \in K_2\}.$$

$\mathbf{Id} : K_1 \to K_1$ denotes the identity map, and $\mathbf{Id}$ is also used to denote the identity matrix in Section 7.

In the case of a separable Hilbert space H and a subset $X \subseteq H$, $\operatorname{span} X \subseteq H$ is the set of finite linear combinations of elements from X; and $\overline{\operatorname{span}}\, X$ is the closure of $\operatorname{span} X$ in H. The notation $\operatorname{clos} X$ is also used for the closure of X. If $X \subseteq H$, then $X^\perp = \{y \in H : \langle x, y\rangle = 0 \text{ for all } x \in X\}$. A linear operator $L : H_1 \to H_2$, mapping the Hilbert space H_1 into the Hilbert space H_2, is a topological isomorphism if L is bijective and continuous. In this case, the open mapping theorem ensures the continuity of L^{-1}.

It will sometimes be convenient to use "$\Rightarrow$" to denote "implies" and "$\Leftrightarrow$" to denote "if and only if".

3 Some Properties of Frames

A countable system $\{x_i : i \in I\}$ in a separable Hilbert space H is a *basis* if every element $x \in H$ has a unique representation $x = \sum_{i\in I} c_i x_i$, $c_i \in \mathbb{C}$. A basis is *unconditional* if the convergence of each representation is unconditional convergence in H. An unconditional basis $\{x_i : i \in I\}$ is a *bounded unconditional basis* if there are positive constants A and B such that $A \leq \|x_i\| \leq B$ for all $i \in I$. $\{x_i : i \in I\}$ is an *orthonormal basis* (ONB) if for all $i \in I$, $\|x_i\| = 1$, and for all $i \neq j$, $\langle x_i, x_j\rangle = 0$. A *Riesz basis* is the image of some orthonormal basis under a topological isomorphism. Clearly, ONBs are Riesz bases, and it is well-known that the concepts of Riesz bases and bounded unconditional bases are equivalent, e.g., [You80].

The concept of a frame, defined below, can be thought of as a generalization of a Riesz basis in the sense that it provides representations $x = \sum_{i\in I} c_i x_i$ where the coefficients can be computed. The idea itself was introduced by Duffin and Schaeffer [DuSc52] in the context of the Paley-Wiener theory, which culminated in the work of Beurling and Malliavin

[BeMa62], [BeMa67]. The idea to use frames in signal representation is due to Daubechies, Grossmann, and Meyer [DGM86]. Since then there has been a great deal of activity in the theory of frames, beginning with the fundamental exposition in [Dau92] and with many more specialized treatments, e.g., some chapters in the books [BeFr94] and [FeSt98], and many articles in the journals, Applied and Computational Harmonic Analysis and the Journal of Fourier Analysis and Applications.

The point of view of thinking of frames as a generalization of Riesz bases has resulted in interesting special decompositions. For example, Ron, Shen, and Gröchenig [RoSh97a], [RoSh97b], [GrRo98] have constructed frames of compactly supported smooth wavelets in $\mathbb{R}^d$. On the other hand, the non-uniqueness of signal expansions in frames which are not Riesz bases can lead to a type of robust representation, which we shall explain below.

Definition 3.1 Let $X = \{x_i : i \in I\}$ be a countable system in a separable Hilbert space H. If the map

$$\begin{array}{rcl} L : H & \rightarrow & l^2(I) \\ f & \mapsto & \{\langle f, x_i \rangle : i \in I\} \end{array}$$

is well-defined in the sense it takes values in $l^2(I)$, then L is the *Bessel map* associated with X. In the case the Bessel map L exists, then L is bounded by the uniform boundedness principle. X is a *frame* if the map L is a bounded invertible map of H onto a closed subspace of $l^2(I)$, i.e., if there exist constants A and B such that

$$\forall\, f \in H, \quad A\|f\|_H^2 \leq \|L(f)\|_{l^2(I)}^2 \leq B\|f\|_H^2.$$

The greatest possible such A is the *lower frame bound* and the least possible such B is the *upper frame bound.* If $A = B$, then the frame is called a *tight frame.*

Proposition 3.2 *Let $X = \{x_i : i \in I\} \subseteq H$ be a countable system with well-defined Bessel map $L : H \rightarrow l^2(I)$. The adjoint of L is given by*

$$\begin{array}{rrcl} L^* : & l^2(I) & \rightarrow & H \\ & c & \mapsto & \displaystyle\sum_{i \in I} c_i x_i. \end{array} \tag{1.2}$$

The proof of this proposition is a consequence of verifying (1.2) for finite sequences c, and then extending to all of $l^2(I)$ by the completeness of H.

Proposition 3.3 *Let $X = \{x_i : i \in I\} \subseteq H$ be a countable system with well-defined Bessel map $L : H \rightarrow l^2(I)$ and adjoint map $L^* : l^2(I) \rightarrow H$. Then*

$$(\mathcal{R}(L))^\perp = \mathcal{N}(L^*),$$

$$\overline{\mathcal{R}(L)} = (\mathcal{N}(L^*))^{\perp},$$

and

$$(\mathcal{R}(L^*))^{\perp} = \mathcal{N}(L).$$

This is a special case of a well-known result in Hilbert space theory, and it follows directly from the definitions of $\mathcal{N}, \mathcal{R}$, and $\perp$, e.g., Theorem II.11.4 in [GoGo81].

According to Proposition 3.3, we can visualize the relationship between L and L^* and between $\mathcal{R}(L)$ and $\mathcal{N}(L^*)$ as in Figure 1.

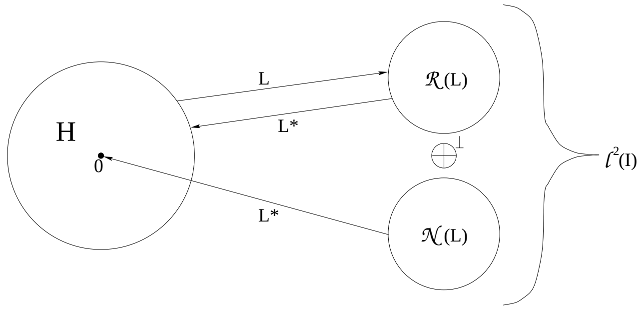

FIGURE 1. The Bessel map L and its adjoint L^*

We then have the following characterization of frames in terms of L^*, cf., Remark 3.9b in [BeWa94]:

Proposition 3.4 *Let $X = \{x_i : i \in I\} \subseteq H$ be a countable system with well-defined Bessel map $L : H \to l^2(I)$. Assume $\overline{\text{span}}\,\{x_i : i \in I\} = H$. Then*

$$\forall\, f \in H, \quad A\|f\|_H^2 \leq \|L(f)\|_{l^2(I)}^2 \leq B\|f\|_H^2 \tag{1.3}$$

if and only if

$$\forall\, c \in (\mathcal{N}(L^*))^{\perp}, \quad A\|c\|_{l^2(I)}^2 \leq \|L^*(c)\|_H^2 \leq B\|c\|_{l^2(I)}^2. \tag{1.4}$$

Proof: Let X be a frame, i.e., assume (1.3). Then $L : H \to l^2(I)$ is a bounded, invertible map onto $\mathcal{R}(L)$, and, in this case, $\mathcal{R}(L)$ is closed so that $\mathcal{R}(L) = (\mathcal{N}(L^*))^{\perp}$, where $L^* : l^2(I) \to H$. Thus, the adjoint map of $L : H \to \mathcal{R}(L)$ is $L^*_{|(\mathcal{N}(L^*))^{\perp}}$, and this latter map is invertible with bounds equal to those of L. (In fact, generally, if T is a bounded linear function between Hilbert spaces, then $\|T\| = \|T^*\|$; and if T is also invertible, then $(T^*)^{-1} = (T^{-1})^*$.) Hence, the assertion (1.4) is obtained. The converse is proven in an analogous way. $\square$

Definition 3.5 Let $X = \{x_i : i \in I\} \subseteq H$ be a frame with Bessel map L. The *frame operator* S is defined as

$$\begin{aligned} S : H &\to H \\ f &\mapsto \sum_{i \in I} \langle f, x_i \rangle x_i, \end{aligned}$$

i.e., S is the composition $S = L^* L$.

Since $\mathcal{R}(L) = (\mathcal{N}(L^*))^\perp$ in case X is a frame, the following result is obtained, cf., the proofs in [Dau92], [BeWa94].

Theorem 3.6 *The frame operator is a continuous, invertible operator on* H.

If $X = \{x_i : i \in I\}$ is a frame with frame bounds A and B, then we also have that

$$\langle S(f), f \rangle_H = \langle L(f), L(f) \rangle_{l^2(I)} = \|L(f)\|^2_{l^2(I)}.$$

Thus, S is a positive operator with $A \leq S \leq B$, and, in particular, S is Hermitian, i.e., $S = S^*$.

The frame operator S can be used to derive the following resolutions of identity from the system $X = \{x_i : i \in I\} \subseteq H$.

Theorem 3.7 *Let* $X = \{x_i : i \in I\} \subseteq H$ *be a frame with frame operator* S. *Then*

$$\forall\, f \in H, \quad f = \sum_{i \in I} \langle f, S^{-1}(x_i) \rangle x_i, \tag{1.5}$$

and

$$\forall\, f \in H, \quad f = \sum_{i \in I} \langle f, x_i \rangle S^{-1}(x_i). \tag{1.6}$$

In particular, if X *is a tight frame with frame bound* A, *then*

$$\forall\, f \in H, \quad f = A^{-1} \sum_{i \in I} \langle f, x_i \rangle x_i. \tag{1.7}$$

Proof: Because of Theorem 3.6 and the fact that $S = S^*$, the decompositions (1.5) and (1.6) are consequences of the calculations

$$f = S\left(S^{-1}(f)\right) = \sum_{i \in I} \langle S^{-1}(f), x_i \rangle x_i = \sum_{i \in I} \langle f, S^{-1}(x_i) \rangle x_i$$

and

$$f = S^{-1}\left(S(f)\right) = S^{-1}\left(\sum_{i \in I} \langle f, x_i \rangle x_i\right) = \sum_{i \in I} \langle f, x_i \rangle S^{-1}(x_i).$$

In the case X is a tight frame with frame bound A, $A^{-1/2}L$ is an isometry. Thus, $A^{-1}\,S = (A^{-1/2}L)^*(A^{-1/2}L)$ is the identity operator on H, whence (1.7) follows. $\square$
The right sides of (1.5) and (1.6) is a *frame decomposition* or *expansion* of the signal f.

Proposition 3.8 *Let $X = \{x_i : i \in I\} \subseteq H$ be a frame with Bessel map L and frame operator S. The maps*

$$\begin{aligned} LS^{-1} : H &\to \mathcal{R}(L) \\ f &\mapsto \{\langle f, S^{-1}(x_i)\rangle : i \in I\} \end{aligned}$$

and the restriction of L^* to $\mathcal{R}(L)$, viz.,

$$\begin{aligned} L^*_{|} : \mathcal{R}(L) &\to H \\ c &\mapsto \sum_{i\in I} c_i x_i, \end{aligned}$$

are continuous maps and are inverse to each other, i.e.,

$$(L^*_{|})^{-1} = LS^{-1}.$$

This follows from the previous discussion and the fact that

$$L^*_{|} LS^{-1} = SS^{-1} = \mathbf{Id}.$$

Definition 3.9 A frame is called *minimal* or *exact* if

$$\forall\, i \in I, \quad x_i \notin \overline{\mathrm{span}}\,\{x_j : j \in I \setminus \{i\}\}\,.$$

A frame is *redundant* or *non-exact* if it is not minimal.

Clearly, minimality of a frame is equivalent to $\mathcal{N}(L^*) = (\mathcal{R}(L))^\perp = \{0\}\,.$

Proposition 3.10 *Let $X = \{x_i : i \in I\} \subseteq H$ be a frame with Bessel map L. X is a Riesz basis if and only if it is minimal.*

Proof: Clearly, by the definition of Riesz basis and the Riesz-Fischer Theorem, Bessel maps L of Riesz bases are surjective, implying minimality.

Conversely, assume X has a surjective Bessel map L. Let $E = \{e_i : i \in I\}$ be an orthonormal basis for H and let K be the Bessel map $H \to l^2(I)$ associated with E. Then L^*K is a bounded invertible operator of H onto itself and $L^*K(e_i) = x_i$, i.e., X is the image of an orthonormal basis under a topological isomorphism, and so X is a Riesz basis. $\square$

We also have the following well-known fact originally proved by Vitali in 1921, see Remark 3.3 of [BeWa94].

Proposition 3.11 *If $X = \{x_i : i \in I\} \subseteq H$ is a tight frame with frame bound 1 and $\|x_i\| = 1$ for all i, then X is an orthonormal basis.*

Proof: By the definition of frame bound, we have

$$\forall\, j \in I, \quad \sum_{i\in I} |\langle x_j, x_i\rangle|^2 = 1,$$

whence $\langle x_j, x_i\rangle = 0$ if $i \neq j$. $\square$

Remark 3.12
When X is not minimal, $\mathcal{N}(L^*)$ is a nonzero subspace of $l^2(I)$; so that given a signal $f \in H$, there are different coefficient sequences $c, d \in l^2(I)$ for which $L^*(c) = L^*(d) = f$. We then have $(c-d) \in \mathcal{N}(L^*)$. In this setting, we see that if $c = LS^{-1}(f)$, then c is the coefficient sequence with minimal $l^2(I)$-norm among all sequences d with frame expansion $L^*(d) = f$. This follows from the fact that $LS^{-1}(f) \in \mathcal{R}(L) = (\mathcal{N}(L^*))^{\perp}$, cf., Theorem 3.5 in [BeWa94].

Moreover, suppose such coefficients c for a signal f are corrupted with additive noise η. For example, this could occur at the coding level from the effects of quantization, in which case η is often modeled as white noise, e.g., [Vai93]. When we use L^* to attempt to recover the signal f from its corrupted coefficient sequence $c + \eta$, only the component of η in $\mathcal{R}(L) = (\mathcal{N}(L^*))^{\perp}$ has any effect on the reconstruction. In this sense, redundancy in a frame can lead to more robustness.

It should be mentioned, however, that if a frame is non-exact, then signals will be redundantly represented in their frame expansion coefficients. If the coefficient sequence of a given frame expansion of f is, for example, $LS^{-1}(f)$ (as in Theorem 3.7), then one can think of more of its terms being non-negligibly different from 0 than in a Riesz basis or even orthonormal basis decomposition. A more concrete illustration of this will be given in Section 5.

4 The Frame Multiresolution Analysis Approach

In this section we shall deal with the separable Hilbert space $H = L^2(\mathbb{R})$.

A standard way to construct a multiresolution analysis of $L^2(\mathbb{R})$ is to find a scaling function $\phi \in L^2(\mathbb{R})$ such that its integer translates $\{\tau_k\phi : k \in \mathbb{Z}\}$ give an orthonormal basis for their closed span V_0, and such that the dilations V_j of V_0, where $f \in V_j$ if and only if $D^{-j}f \in V_0$, $j \in \mathbb{Z}$, form a nested sequence of subspaces with dense union and trivial intersection in $L^2(\mathbb{R})$. To obtain a wavelet basis, one then decomposes $V_{j+1} = V_j \oplus^{\perp} W_j$ as an orthogonal direct sum, and defines a wavelet ψ whose integer shifts form an orthonormal basis for W_0. Then $L^2(\mathbb{R}) = \mathrm{clos}(\bigoplus_j^{\perp} W_j)$, and $\{D^j\tau_k\psi : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

In the above scheme, the hypothesis, that $\{\tau_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, can be weakened to assume that it is only a Riesz basis for

V_0. Moreover, rather than orthogonal direct sums $V_{j+1} = V_j \oplus^\perp W_j$, one can consider splittings into merely direct sums $V_{j+1} = V_j \oplus W_j$. In this latter case, we face additional global stability questions if $\{D^j \tau_k \psi : j, k \in \mathbb{Z}\}$ is to be a Riesz basis for $L^2(\mathbb{R})$, cf., [CoDa92].

In light of Propositions 3.3 and 3.10, the requirement that $\{\tau_k \phi : k \in \mathbb{Z}\}$ and $\{\tau_k \psi : k \in \mathbb{Z}\}$ should be Riesz bases for V_0 and W_0, respectively, entails that the set of coefficient sequences in signal decompositions in terms of these basis vectors exhausts all of the square integrable coefficient sequences, e.g., the Bessel map $f \mapsto \{\langle f, \tau_k \phi \rangle : k \in \mathbb{Z}\}$ is a surjection of V_0 onto $l^2(\mathbb{Z})$. As a consequence, the filters of a corresponding multirate system must cover the full band $\mathbb{T}$ to achieve perfect reconstruction, see [Dau92], [Mal89], [Mal98], [VeKo95], and [Vai93] for this type of reconstruction as well as the basic theory giving the relationship between wavelets and multirate systems.

In this section, we shall show that replacing Riesz bases by redundant frames in the above scheme, where we have used orthogonal direct sum decompositions $V_{j+1} = V_j \oplus^\perp W_j$, can give rise to stable decompositions of $L^2(\mathbb{R})$. Further, the corresponding multirate systems have narrow band analysis and synthesis filters, that provide perfect reconstruction of narrow band sequences, see Section 6 for this type of reconstruction and Section 5 where we study a simple generic example. The aforementioned development for redundant frames was first done in 1993 [BeLi93], [BeLi98], cf., [KiLi97] for an analogous analysis in terms of non-orthogonal direct sum splittings.

4.1 Frames Generated by Integer Shifts of a Function

Let $\phi \in L^2(\mathbb{R})$ and define the function $\Phi : \widehat{\mathbb{R}} \to \mathbb{C}$ by

$$\Phi(\gamma) = \sum_{k \in \mathbb{Z}} |\widehat{\phi}(\gamma + k)|^2, \quad \gamma \in \widehat{\mathbb{R}}, \tag{1.8}$$

noting that Φ is a 1-periodic function on $\widehat{\mathbb{R}}$, and, in fact, $\Phi \in L^1(\mathbb{T})$ by the Parseval-Plancherel Theorem. The set $[\Phi > 0]$ is well-defined up to nullsets; and it is easy to see that $\widehat{\phi} = 0$ on $[\Phi = 0]$. Using our periodization notation, we can write

$$\Phi = P(|\widehat{\phi}|^2).$$

We shall now characterize when $\{\tau_k \phi : k \in \mathbb{Z}\}$ is a frame for its closed span V_0 in terms of a boundedness condition on Φ off of its zero set (Theorem 4.2). The result was proved in 1992 by Benedetto and Li [BeLi93] on $\mathbb{R}$ with their details and an elementary extension to $\mathbb{R}^d$ given in [BeWa94]. Theorem 4.2 was also proved by deBoor, DeVore, and Ron [BDR94a] for finitely generated shift invariant spaces in terms of their notion of quasi-stable bases. Ron and Shen [RoSh95] noted the formulation of frames in terms of quasi-stability, and, as such, also stated Theorem 4.2. More recently, the result has been proved by Di-Rong [DiR97], Kim and Lim [KiLi97], and Casazza, Christensen, and Kalton [CCK98]. There is no hypothesis for Theorem 4.2 other than $\phi \in L^2(\mathbb{R})$, whereas in the original proof the mild condition, $\Phi \in L^2(\mathbb{T})$, was required.

Proposition 4.1 *Let $X = \{\tau_k \phi : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})$ and define $\Phi = P(|\widehat{\phi}|^2)$. The Bessel map L associated with X has norm $\|L\| \leq B^{1/2}$ if and only if $\Phi \leq B$ a.e.*

Proof: First recall that in general $\|L\|=\|L^*\|$. Let c be a finitely supported sequence. Then

$$\|L^*(c)\|^2 = \|(L^*(c))^\wedge\|^2 = \int_{\mathbb{R}} |(\sum_{k\in\mathbb{Z}} c_k e_k)\widehat{\phi}|^2 = \int_{\mathbb{T}} |\sum_{k\in\mathbb{Z}} c_k e_k|^2 \Phi.$$

By the Parseval-Plancherel Theorem, $\|c\|^2_{l^2(\mathbb{Z})} = \int_{\mathbb{T}} |\sum_{k\in\mathbb{Z}} c_k e_k|^2$. Thus, if $\Phi \leq B$ a.e. on $\mathbb{T}$, then $\|L\| \leq B^{1/2}$. For the converse, consider for $\delta > 0$ the set $\Lambda = [\Phi \geq B + \delta]$. Now, for any measurable set $\Lambda \subseteq \mathbb{T}$, there exists a sequence $\{p_n\}$ of trigonometric polynomials with $\|p_n\|^2_{L^2(\mathbb{T})} \leq |\Lambda|$ such that $\{p_n\}$ converges uniformly to $\mathbf{1}_\Lambda$ except on a set of arbitrarily small measure. Thus, if the measure $|\Lambda|$ of Λ was positive, there would be a finitely supported sequence c with $\|c\|^2_{l^2(\mathbb{Z})} \leq |\Lambda|$ such that $\|L^*(c)\|^2 > |\Lambda|(B + \delta/2)$, and hence $\|L^*\| > B^{1/2}$. □

Theorem 4.2 *Let $X = \{\tau_k \phi : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})$ and assume that $\Phi = P(|\widehat{\phi}|^2) \in L^\infty(\mathbb{T})$. Then X has a well-defined Bessel map L (by Proposition 4.1), and*

$$\mathcal{N}(L^*) = \{c \in l^2(\mathbb{Z}) : \widehat{c} = 0 \text{ on } [\Phi > 0]\}. \tag{1.9}$$

Moreover, X is a frame for $V_0 = \overline{\text{span}}\,\{\tau_k \phi : k \in \mathbb{Z}\}$ with lower frame bound $A > 0$ and upper frame bound $B < \infty$ if and only if

$$A = \inf\{a : |[\Phi \leq a] \cap [\Phi > 0]| > 0\} \quad \text{and} \quad \text{ess sup}\, \Phi = B < \infty.$$

In particular, X is a frame for V_0 if and only if Φ is essentially bounded above by B and below by A off of its zero set $[\Phi = 0]$.

Proof: By definition of V_0, $\{\tau_k \phi : k \in \mathbb{Z}\}$ is complete in V_0. Pick $c \in l^2(\mathbb{Z})$ and compute

$$\begin{aligned} \|L^*(c)\|^2_{L^2(\mathbb{R})} &= \|\sum_{k\in\mathbb{Z}} c_k\, \tau_k \phi\|^2_{L^2(\mathbb{R})} = \|\sum_{k\in\mathbb{Z}} c_k\, e_k\, \widehat{\phi}\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |\sum_{k\in\mathbb{Z}} c_k\, e_k\, \widehat{\phi}|^2 = \int_{\mathbb{T}} |\sum_{k\in\mathbb{Z}} c_k\, e_k|^2 \Phi. \end{aligned}$$

Thus,

$$\mathcal{N}(L^*) = \left\{c \in l^2(\mathbb{Z}) : \widehat{c} = 0 \text{ on } [\Phi > 0]\right\}.$$

The theorem now follows from Proposition 3.4, since

$$\|c\|_{l^2(\mathbb{Z})} = \|\sum_{k\in\mathbb{Z}} c_k\, e_k\|_{L^2(\mathbb{T})}$$

by the Parseval-Plancherel Theorem. □

Proposition 4.1 and Theorem 4.2 carry over to the d-dimensional setting without any modifications. In light of Propositions 3.10 and 3.11, we also recover the familiar characterizations $0 < A \leq \Phi \leq B < \infty$ or $\Phi = 1$ when the integer translates of ϕ form a Riesz basis or orthonormal basis, respectively, for their closed span, e.g., [BeWa94].

We remark that if $\{\tau_k\phi : k \in \mathbb{Z}\}$ generates a non-exact frame for its span, i.e., if $[\Phi = 0]$ is non-trivial, then Φ must be discontinuous. In that case, we have the following two facts.

- ϕ cannot have compact support, since then $\widehat{\phi}$ would be entire and have only isolated zeros.
- If $\phi \in L^1(\mathbb{R})$, then $\widehat{\phi}$ is continuous, and so $\widehat{\phi}$ cannot be bounded by $\frac{C}{x^{(1/2)+\epsilon}}$ for any ϵ positive; in fact, if there was such a bound, then convergence in (1.8) would be absolute and hence Φ would be continuous. In this sense, a generator ϕ giving rise to non-exact frames must always be poorly-localized in time or frequency.

Analogous drawbacks hold in higher dimensions. To circumvent these issues, whether in one or more dimensions, Li has introduced the concept of affine pseudoframes, cf., [Li98].

4.2 Frame Decompositions from $\{\tau_k\phi : k \in \mathbb{Z}\}$

Proposition 4.3 *Suppose $\{\tau_k\phi : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})$ is a frame for its closed span V_0. Then*

$$f \in V_0 \Leftrightarrow \widehat{f} = F\,\widehat{\phi},$$

for some $F \in L^2(\mathbb{T})$ depending on $f \in L^2(\mathbb{R})$. In particular, for such an f, $\widehat{f} = 0$ almost everywhere on the set $[\widehat{\phi} = 0]$.

Proof: Since $\{\tau_k\phi : k \in \mathbb{Z}\}$ is a frame for V_0, $f = \sum_{k\in\mathbb{Z}} c_k\tau_k\phi$ for some sequence $c \in l^2(\mathbb{Z})$. Taking the Fourier transform of this equation gives

$$\widehat{f} = \sum_{k\in\mathbb{Z}} c_k e_k\,\widehat{\phi},$$

and so $F = \sum_{k\in\mathbb{Z}} c_k e_k$, i.e., F is the Fourier transform of a coefficient sequence c of f with respect to $\{\tau_k\phi : k \in \mathbb{Z}\}$. The fact, $F \in L^2(\mathbb{T})$, follows from the Parseval Theorem. $\square$

We remark that in view of (1.9), f determines F uniquely only on the set $[\Phi > 0]$.

Proposition 4.4 *Suppose $\{\tau_k\phi : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})$ is a frame for its closed span V_0. Define*

$$\widehat{\theta} = \begin{cases} \widehat{\phi}/\Phi & \text{on } [\Phi > 0] \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Then $\theta \in V_0$, and $\theta = S^{-1}(\phi)$.

Proof: We have that $\theta \in V_0$ from Proposition 4.3, since $0 < A \leq \Phi \leq B$ on $[\Phi > 0]$ and Φ is 1-periodic.

Because $\Phi \leq B$, and $\operatorname{supp}\widehat{\phi} \subseteq \operatorname{supp}\Phi$ considered as a subset of $\mathbb{R}$, the following formal computations can be made rigorous without difficulty.

$$\begin{aligned} (S(\theta))^{\wedge} &= \left(\sum_{k\in\mathbb{Z}} \langle \theta, \tau_k\phi\rangle \tau_k\phi\right)^{\wedge} = \left(\sum_{k\in\mathbb{Z}} \langle \widehat{\theta}, e_k\,\widehat{\phi}\rangle_{L^2(\mathbb{R})} e_k\right)\widehat{\phi} \\ &= \left(\sum_{k\in\mathbb{Z}} \langle \sum_{n\in\mathbb{Z}} \tau_n(\widehat{\theta}\overline{\widehat{\phi}}), e_k\rangle_{L^2(\mathbb{T})} e_k\right)\widehat{\phi} \\ &= \left(\sum_{k\in\mathbb{Z}} \langle \mathbf{1}_{[\Phi>0]}, e_k\rangle e_k\right)\widehat{\phi} = \mathbf{1}_{[\Phi>0]}\,\widehat{\phi} = \widehat{\phi}. \quad \square \end{aligned}$$

Clearly, we have that $\tau_k S = S\tau_k$ and $\tau_k S^{-1} = S^{-1}\tau_k$ for all $k \in \mathbb{Z}$. Thus, the map

$$f \mapsto \sum_{k\in\mathbb{Z}} \langle f, \tau_k\theta\rangle \tau_k\phi \tag{1.11}$$

is the orthogonal projection of $L^2(\mathbb{R})$ onto $V_0 = \overline{\text{span}}\,\{\tau_k\phi : k \in \mathbb{Z}\}$, since it is the identity on V_0, and since for all $k \in \mathbb{Z}$, $\langle g, \tau_k\theta\rangle = 0$ for every $g \in V_0^{\perp}$.

Remark 4.5

Let $\{x_i : i \in I\}$ be a frame for H, let $f \in H$, and set

$$c = \left\{\langle f, S^{-1}(x_i)\rangle : i \in I\right\}.$$

In the frame expansion $f = L^*(c)$, the component of c in $\mathcal{N}(L^*)$ is zero, i.e., $c \in (\mathcal{N}(L^*))^{\perp}$. This follows since $c = \left\{\langle f, S^{-1}(x_i)\rangle : i \in I\right\} = LS^{-1}(f)$ and $LS^{-1}(f) \in \mathcal{R}(L) = (\mathcal{N}(L^*))^{\perp}$, as we noted in Remark 3.12.

Let $H = V_0 \subseteq L^2(\mathbb{R})$, where $V_0 = \overline{\text{span}}\,\{\tau_k\phi : k \in \mathbb{Z}\}$ for some $\phi \in L^2(\mathbb{R})$, and assume $\{\tau_k\phi : k \in \mathbb{Z}\}$ is a frame for V_0. Then the aforementioned assertion $c \in (\mathcal{N}(L^*))^\perp$ can also be formulated and proved in the following way. Let $f \in V_0 \setminus \{0\}$. Then $f = \sum_{k\in\mathbb{Z}} c_k \tau_k \phi$, where $c_k = \langle f, S^{-1}(x_k)\rangle = \langle f, \tau_k\theta\rangle$, see Proposition 4.4 for the definition of θ. Hence, $\widehat{f} = F\widehat{\phi}$, where $F = \sum_{k\in\mathbb{Z}} c_k e_k$; and we compute

$$\begin{aligned}
\widehat{f} &= \left(\sum_{k\in\mathbb{Z}} \langle f, \tau_k\theta\rangle_{L^2(\mathbb{R})} \tau_k\phi\right)^{\wedge} = \left(\sum_{k\in\mathbb{Z}} \langle F\widehat{\phi}, e_k\,\widehat{\theta}\rangle_{L^2(\widehat{\mathbb{R}})} e_k\right)\widehat{\phi} \\
&= \left(\sum_{k\in\mathbb{Z}} \langle F\,\mathbf{1}_{[\Phi>0]}, e_k\rangle_{L^2(\mathbb{T})} e_k\right)\widehat{\phi} = \left(F\,\mathbf{1}_{[\Phi>0]}\right)\widehat{\phi}.
\end{aligned}$$

Recalling from Theorem 4.2 that

$$\mathcal{N}(L^*) = \left\{d \in l^2(\mathbb{Z}) : \widehat{d} = 0 \text{ on } [\Phi > 0]\right\},$$

the fact $\widehat{f} = (F\mathbf{1}_{[\Phi>0]})\widehat{\phi}$, verified in (1.12), allows us to conclude that $c \in (\mathcal{N}(L^*))^\perp$. Thus, *the frame representation of f in terms of the coefficients $\{\langle f, \tau_k\theta\rangle : k \in \mathbb{Z}\}$ "squeezes" the support of the Fourier transform of this coefficient sequence into the set* $[\Phi > 0]$.

Because of calculation (1.12) and its interpretation in terms of "squeezing" the support of a Fourier series whose coefficients are those of a given representation on $\mathbb{R}$, the possibility arises of designing useful signal reconstruction multirate systems by means of narrowband filters, see Section 6.

4.3 Frame Multiresolution Analysis (FMRA)

The concept of multiresolution analysis is extended to include frames in the following way.

Definition 4.6 A *frame multiresolution analysis (FMRA)* $\{V_j, \phi\}$ of $L^2(\mathbb{R})$ is an increasing sequence of closed linear subspaces $V_j \subseteq L^2(\mathbb{R})$ and an element $\phi \in V_0$ for which the following hold:

$$\overline{\cup_j V_j} = L^2(\mathbb{R}) \quad \text{and} \quad \cap_j V_j = \{0\},$$

$$f \in V_j \Longleftrightarrow Df \in V_{j+1},$$

$$f \in V_0 \ \Rightarrow \tau_k f \in V_0 \quad \text{for all } k \in \mathbb{Z},$$

$$\{\tau_k\phi : k \in \mathbb{Z}\} \text{ is a frame for the subspace } V_0. \tag{1.12}$$

An FMRA is called *redundant* or *exact* according as to the nature of the frame in (1.12).

It is easy to verify that if $\{\tau_k\phi : k \in \mathbb{Z}\}$ is a frame for V_0, then the system $\left\{D^j\tau_k\phi : k \in \mathbb{Z}\right\}$ is a frame for V_j with the same frame bounds.

Given an FMRA for $L^2(\mathbb{R})$, define W_0 to be the orthogonal complement of V_0 in V_1, and set $W_j = \left\{f \in L^2(\mathbb{R}) : D^{-j}f \in W_0\right\}$.

Analogous to an MRA, there are two fundamental problems associated with FMRAs: first, finding FMRAs, and, second, given an FMRA, constructing a *wavelet* ψ such that $\{\tau_k\psi : k \in \mathbb{Z}\}$ is a frame for W_0, with some frame bounds A and B. Then, as in the standard MRA setting, $L^2(\mathbb{R}) = \mathrm{clos}(\bigoplus_j^{\perp} W_j)$ as an orthogonal direct sum, and $\left\{D^j\tau_k\psi : j,k \in \mathbb{Z}\right\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B.

In this context, let us first recall the following result.

Theorem 4.7 *Let $\phi \in L^2(\mathbb{R})$, and set $V_0 = \overline{\mathrm{span}}\,\{\tau_k\phi : k \in \mathbb{Z}\}$ and $V_j = \left\{D^jf : f \in V_0\right\}$ for each $j \in \mathbb{Z}$. Suppose $V_0 \subseteq V_1$. If $|\widehat{\phi}| > 0$ almost everywhere on some neighborhood of 0, then $\overline{\cup_j V_j} = L^2(\mathbb{R})$ and $\cap_j V_j = \{0\}$.*

An explicit proof of Theorem 4.7 is in [BDR93], and there are several ways of deriving it. For example, the density property can be proved using Wiener's result that the closed linear span of the translates of $f \in L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ if and only if $|[\widehat{f} = 0]| = 0$, e.g., [Ben97]. This approach is useful in attempting to develop an L^1-theory of MRA, in which case Wiener's L^2-result can be replaced by his Tauberian Theorem, e.g., [Ben75], [Zim94].

4.4 Wavelet Frames from Frame Multiresolution Analyses

Let $\{V_j, \phi\}$ be an FMRA of $L^2(\mathbb{R})$. We shall now investigate when and how a *wavelet* ψ can be constructed such that $\{\tau_k\psi : k \in \mathbb{Z}\}$ is a frame for the orthogonal complement W_0 of V_0 in V_1.

Proposition 4.8 *Suppose $\{V_j, \phi\}$ is an FMRA of $L^2(\mathbb{R})$. Then there is a 1-periodic function $H_0 \in L^\infty(\mathbb{T})$ such that*

$$\widehat{\phi}(2\cdot) = H_0\,\widehat{\phi}. \tag{1.13}$$

Proof: V_1 is closed and invariant under integer shifts, so $V_0 \subseteq V_1$ if and only if $\phi \in V_1$. As $\{D\tau_k\phi : k \in \mathbb{Z}\}$ is a frame for V_1, then by Proposition 4.3 there is a 1-periodic function $H_0 \in L^2(\mathbb{T})$ such that (1.13) holds. Moreover, we may choose $H_0 = 0$ on $[\Phi = 0]$. Periodizing the square modulus of (1.13) yields

$$\Phi = |H_0(\cdot/2)|^2\Phi(\cdot/2) + |H_0(\cdot/2 + 1/2)|^2\Phi(\cdot/2 + 1/2), \tag{1.14}$$

where $\Phi = P(|\widehat{\phi}|^2)$. Since $0 < A \leq \Phi \leq B < \infty$ for some A, B off of $[\Phi = 0]$, we see that $|H_0|^2 + |\tau_{1/2}H_0|^2 \in L^\infty(\mathbb{T})$. Thus, $H_0 \in L^\infty(\mathbb{T})$. $\square$

Given any 1-periodic function $H_1 \in L^2(\mathbb{T})$, define $\psi \in L^2(\mathbb{R})$ by means of the equation

$$\widehat{\psi}(2\cdot) = H_1\, \widehat{\phi}. \tag{1.15}$$

Proposition 4.9 *Let $\{V_j, \phi\}$ be an FMRA of $L^2(\mathbb{R})$ and let $H_0 \in L^\infty(\mathbb{T})$ be the function defined in (1.13). Using equation (1.15) and the definition of W_0 as the orthogonal complement of V_0 in V_1, we have*

$$\psi \in W_0 \quad \textit{if and only if} \quad \overline{H}_0 H_1 \Phi + \tau_{1/2}(\overline{H}_0 H_1 \Phi) = 0. \tag{1.16}$$

Proof: Clearly, by Proposition 4.3 and the definition of ψ, $\psi \in V_1$. Moreover,

$$\begin{aligned}
\psi \in V_0^\perp \quad &\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \langle \psi, \tau_k \phi \rangle = 0 \\
&\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \langle \widehat{\psi}, e_k \widehat{\phi} \rangle = 0 \\
&\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \langle H_1(\cdot/2)\widehat{\phi}(\cdot/2), e_k H_0(\cdot/2)\widehat{\phi}(\cdot/2) \rangle = 0 \\
&\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \langle H_1 \widehat{\phi}, e_{2k} H_0 \widehat{\phi} \rangle = 0 \\
&\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \int_{\mathbb{R}} H_1 \overline{H}_0 |\widehat{\phi}|^2 e_{2k} = 0 \\
&\Leftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad \int_{[0,1/2)} \left(H_1 \overline{H}_0 \Phi + \tau_{1/2}(H_1 \overline{H}_0 \Phi) \right) e_{2k} = 0;
\end{aligned}$$

and the result follows by the uniqueness theorem of Fourier series, e.g., [Ben97]. $\square$

Theorems 5.4 and 5.5 in [BeLi98] combine to yield the following result.

Proposition 4.10 *Let $\{V_j, \phi\}$ be an FMRA of $L^2(\mathbb{R})$, and let H_0, H_1, and ψ be defined as in equations (1.13) and (1.15). Assume $\psi \in W_0$ and $P(|\widehat{\psi}|^2) \in L^\infty(\mathbb{T})$. If there are $G_0, G_1 \in L^\infty(\mathbb{T})$ such that*

$$\begin{aligned}
H_0 G_0 \Phi + H_1 G_1 \Phi \quad &= \quad \Phi & (1.17) \\
\tau_{1/2}(H_0 \Phi) G_0 + \tau_{1/2}(H_1 \Phi) G_1 \quad &= \quad 0, & (1.18)
\end{aligned}$$

then $\{\tau_k \psi : k \in \mathbb{Z}\}$ is a frame for W_0.

We remark that the conclusion of the comparable result in [BeLi98] is that "$\{\tau_k \psi : k \in \mathbb{Z}\}$ generates the Banach space W_0" in the sense that the map $c \mapsto \sum_{k \in \mathbb{Z}} c_k \tau_k \psi$ is a surjection of $l^2(\mathbb{Z})$ onto W_0. Designating this map by K^*, we see that it is the adjoint of the bounded (see Proposition 4.1) Bessel map for $\{\tau_k \psi : k \in \mathbb{Z}\}$. Thus, K^* must be bounded below on $(\mathcal{N}(K^*))^\perp$ by the open mapping theorem, and so, by Proposition 3.4, $\{\tau_k \psi : k \in \mathbb{Z}\}$ is a frame.

Moreover, if g_0 and g_1 are the Fourier coefficient sequences of G_0 and G_1, respectively, then

$$\phi(2\cdot -k) = 2^{-1/2} D\tau_k\phi = \sum_j g_0(2j-k)\tau_j\phi + \sum_j g_1(2j-k)\tau_j\psi \quad (1.19)$$

for all $k \in \mathbb{Z}$, cf., Theorem 5.4 [BeLi98].

Theorem 4.11 *Suppose $\{V_j, \phi\}$ is an FMRA of $L^2(\mathbb{R})$ and let $H_0 \in L^\infty(\mathbb{T})$ be the function defined in (1.13). Set*

$$\Gamma = \{\gamma \in \mathbb{T} : \Phi(2\gamma) = 0, \Phi(\gamma) > 0, \Phi(\gamma + 1/2) > 0\}. \quad (1.20)$$

a. If $|\Gamma| > 0$, there is no $\psi \in L^2(\mathbb{R})$ for which $\{\tau_k\psi : k \in \mathbb{Z}\}$ is a frame for the orthogonal complement W_0 of V_0 in V_1.

b. If $|\Gamma| = 0$, then there exist $H_1, G_0, G_1 \in L^\infty(\mathbb{T})$ such that (1.16), (1.17), and (1.18) hold. Thus, in light of Proposition 4.10, if ψ is defined by (1.15), then $\{\tau_k\psi : k \in \mathbb{Z}\}$ is a frame for W_0. In this case, the wavelet system $\{D^j\tau_k\psi : j, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$.

Proof:

1. Suppose first that $|\Gamma| > 0$, and set $\Gamma_+ = \Gamma \cap [0, 1/2)$ and $\Gamma_- = \Gamma \cap [-1/2, 0)$. By the definition of Γ, neither Γ_+ nor Γ_- has measure zero. Now define functions $F_1, F_2 \in L^\infty(\mathbb{T})$ by the properties that $F_1 = F_2 = 1$ on Γ_+, $F_1 = F_2 = 0$ on $\mathbb{T} \setminus \Gamma$, and $F_1 = 1$ and $F_2 = -1$ on Γ_-. Next, we define functions f_1 and f_2 by the dilation equations

$$\widehat{f_i}(2\cdot) = F_i\,\widehat{\phi} \quad i = 1, 2. \quad (1.21)$$

Note that the f_i, $i = 1, 2$, are not identically zero. In fact, if $f_i = 0$, then from (1.21) we see that $|F_i|^2\Phi = 0$ on $\mathbb{T}$, which contradicts the hypotheses that $|F_i|^2$ and Φ are positive on Γ. Further, equation (1.14) allows us to assert that

$$H_0 = \tau_{1/2} H_0 = 0 \quad \text{on } \Gamma \quad (1.22)$$

by the definition of Γ. Since $H_0 = 0$ on Γ and $F_1 = F_2 = 0$ on $\mathbb{T} \setminus \Gamma$, we have

$$\int_{\mathbb{T}} (F_i H_0 \Phi)\, e_{2k} = 0 \quad (1.23)$$

for all $k \in \mathbb{Z}$. Also, the Parseval-Plancherel Theorem and equation (1.13) can be used to show that

$$\forall\, k \in \mathbb{Z}, \quad \langle f_i, \tau_k\phi\rangle = \langle F_i(\cdot/2)\widehat{\phi}(\cdot/2), H_0(\cdot/2)\widehat{\phi}(\cdot/2)e_k(\cdot)\rangle. \quad (1.24)$$

The left side of (1.23) is the right side of (1.24) and so

$$\forall\, k \in \mathbb{Z}, \quad \langle f_i, \tau_k \phi \rangle = 0, \quad i = 1, 2. \tag{1.25}$$

Therefore, $f_1, f_2 \in W_0$.

Now assume there is a function $\psi \in W_0$ such that $\{\tau_k \psi : k \in \mathbb{Z}\}$ is a frame for W_0. Combining Proposition 4.3 with both this assumption and the hypothesis that $\{\tau_k \phi : k \in \mathbb{Z}\}$ is a frame for V_0, we can assert the existence of 1-periodic functions $H_1 \in L^2(\mathbb{T})$ and $D_i \in L^2(\mathbb{T})$, $i = 1, 2$, for which

$$\widehat{f_i} = D_i\, \widehat{\psi} = D_i\, H_1(\cdot/2)\widehat{\phi}(\cdot/2)$$

where $F_i(\cdot/2) = D_i\, H_1(\cdot/2)$ by (1.21).

If $\gamma/2 \in \Gamma$, then

$$H_1(\gamma/2 + 1/2) = H_1(\gamma/2). \tag{1.26}$$

To see this, note that $F_1(\gamma/2 + 1/2) = F_1(\gamma/2) = 1$ by definition of Γ and F_1, and so

$$D_1(\gamma + 1)H_1(\gamma/2 + 1/2) = F_1(\gamma/2 + 1/2) = 1$$

and

$$D_1(\gamma)H_1(\gamma/2) = F_1(\gamma/2) = 1; \tag{1.27}$$

from which we obtain $D_1(\gamma)(H_1(\gamma/2 + 1/2) - H_1(\gamma/2)) = 0$, since D_1 is 1-periodic. Hence, (1.26) is true since $D_1(\gamma) \neq 0$ by equation (1.27). Similarly, if we replace F_1 by F_2, and if $\gamma/2 \in \Gamma$, then

$$H_1(\gamma/2 + 1/2) = -H_1(\gamma/2). \tag{1.28}$$

Combining (1.26) and (1.28) implies that $H_1(\gamma) = 0$ for all $\gamma \in \Gamma$, and this contradicts (1.27). Therefore, if $|\Gamma| > 0$, then no function ψ, such that $\{\tau_k \psi : k \in \mathbb{Z}\}$ is a frame for W_0, can exist; in fact, $\{\tau_k \psi : k \in \mathbb{Z}\}$ can never span W_0.

2. Now suppose that Γ is a nullset. Define

$$\begin{aligned} E &= [\Phi = 0, \tau_{1/2}\Phi = 0], \\ E_1 &= [\Phi > 0, \tau_{1/2}\Phi = 0], \\ E_2 &= [\Phi = 0, \tau_{1/2}\Phi > 0], \\ F &= [\Phi > 0, \tau_{1/2}\Phi > 0]. \end{aligned}$$

These sets are a partition of $\mathbb{T}$, and they have the further properties that $F = \tau_{1/2}F$, $E = \tau_{1/2}E$, $E_1 = \tau_{1/2}E_2$, and $E_2 = \tau_{1/2}E_1$.

- On E, we can define H_1, G_0, and G_1 to be arbitrary bounded functions (in particular, we may set them equal to 0) for which (1.16), (1.17), and (1.18) hold.
- Equation (1.14) implies that there are constants $a, b > 0$ such that

$$a \leq |H_0|^2 + |\tau_{1/2} H_0|^2 \leq b \quad \text{a.e. on } F.$$

To see this, first note that if $\gamma \notin \Gamma$ and $\gamma \in F$, then $\Phi(2\gamma)) > 0$; and so if $A \leq \Phi \leq B$ off of $[\Phi = 0]$, then (1.14) implies

$$A \leq \Phi(2\gamma) \leq B(|H_0(\gamma)|^2 + |H_0(\gamma + 1/2)|^2) \quad \text{a.e. on } F$$

and

$$A(|H_0(\gamma)|^2 + |H_0(\gamma + 1/2)|^2) \leq \Phi(2\gamma) \leq B \quad \text{a.e. on } F.$$

In particular, we can take $a = A/B$ and $b = B/A$.
Now set $H_1 = e_1 \tau_{1/2}(\overline{H}_0 \Phi)$. Then $H_1 \in L^\infty(\mathbb{T})$, and (1.16) is satisfied on F. Next, the determinant of the system given by (1.17) and (1.18), with G_0 and G_1 regarded as unknowns, is

$$\Delta = \left(H_0 \tau_{1/2} H_1 - H_1 \tau_{1/2} H_0\right) \Phi \, \tau_{1/2} \Phi,$$

so that on F, with the above choice of H_1,

$$\Delta = -e_1 \left(|H_0|^2 \Phi + \tau_{1/2}(|H_0|^2 \Phi)\right) \Phi \, \tau_{1/2} \Phi.$$

Hence $|\Delta| > aA^3$, where A is the lower frame bound of the system $\{\tau_k \phi : k \in \mathbb{Z}\}$ in the FMRA $\{V_j, \phi\}$. By Cramer's rule,

$$G_0 = \frac{\Phi \tau_{1/2}(\Phi \, H_1)}{\Delta} \quad \text{on } F,$$

and so G_0 is bounded on F. A similar argument works for G_1, and so (1.17) and (1.18) are obtained.
- Equation (1.18) always holds on E_1. Also, from (1.14), there are constants $a, b > 0$ such that either $H_0 = 0$ or $a \leq |H_0|^2 \leq b$ on E_1. If $|H_0(\gamma)|^2 = 0$, then (1.16) holds and (1.17) only forces $H_1(\gamma) G_1(\gamma) = 1$, which can be achieved; if $a \leq |H_0(\gamma)|^2 \leq b$, then (1.16) forces $H_1(\gamma) = 0$ so that then (1.17) only requires $H_0(\gamma) G_0(\gamma) = 1$, which can also be accomplished with G_0 bounded on E_1.
- Equation (1.17) always holds on E_2. Also, in accordance with the previous paragraph, and recalling that $E_2 = \tau_{1/2} E_1$, equation (1.14) allows us to assert that there are constants $a, b > 0$ such

that either $0 < a \le |\tau_{1/2}H_0|^2 \le b$ or $\tau_{1/2}H_0 = 0$ on E_2. If $a \le |\tau_{1/2}H_0(\gamma)|^2 \le b$, (1.16) forces only $\tau_{1/2}H_1(\gamma) = 0$, which is compatible with the definition of H_1 on E_1 as in the previous paragraph. Equation (1.18) only requires $\tau_{1/2}H_0(\gamma)G_0(\gamma) = 0$, and so we set $G_0(\gamma) = 0$. If $\tau_{1/2}H_0(\gamma) = 0$, then (1.16) holds; and (1.18) only forces $\tau_{1/2}(H_1(\gamma))G_1(\gamma) = 0$. $\square$

Example 4.12

We shall construct FMRAs $\{V_j, \phi\}$ so that the corresponding set Γ in Theorem 4.11 has positive measure, and hence $\{V_j, \phi\}$ does not give rise to an MRA frame. Choose $\delta \in (0, 1/12)$ and set $\alpha = 1/4 + \delta$. The function ϕ defined by $\widehat{\phi} = \mathbf{1}_{[-\alpha,\alpha)}$ defines an FMRA of $L^2(\mathbb{R})$. Let

$$Y = [-1/4 - \delta, -1/4 + \delta) \cup [1/4 - \delta, 1/4 + \delta) \subseteq [-1/2, 1/2),$$

considered as a subset of $\mathbb{T}$. By definition of $\widehat{\phi}$, we see that $Y \subseteq \Gamma$. In particular,

$$2Y = [-1/2, -1/2 + 2\delta) \cup [1/2 - 2\delta, 1/2) \subseteq [-1/2, 1/2), \tag{1.29}$$

considered as a subset of $\mathbb{T}$, and $|\Gamma| > 0$. Let $Z = [-2\alpha, -\alpha) \cup [\alpha, 2\alpha)$ and observe that $W_0 = \left\{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subseteq Z\right\}$. Noting that the following sets (intervals) $[-2\alpha, -2\alpha + 4\delta) \subseteq Z$ and $[2\alpha - 4\delta, 2\alpha) \subseteq Z$ have positive measure and satisfy $\tau_1[-2\alpha, -2\alpha + 4\delta) = [2\alpha - 4\delta, 2\alpha)$, we see that W_0 is undersampled at a Nyquist rate equal to 1. Hence, there can be no function ψ such that $\{\tau_k\psi : k \in \mathbb{Z}\}$ spans W_0.

5 A Generic Example of a Redundant Frame Generated by Integer Translates: Oversampling

For $m > 0$, set

$$\Omega_m = [-(1/2)\, 2^{-m}, (1/2)\, 2^{-m}]$$

and consider $\phi \in L^2(\mathbb{R})$ defined by $\widehat{\phi} = \mathbf{1}_{\Omega_m}$. Then the 1-periodic function $\Phi = P(|\widehat{\phi}|^2)$ equals $\mathbf{1}_{\Omega_m}$ on $[-1/2, 1/2)$. Thus, by Theorem 4.2, the system $\{\tau_k\phi : k \in \mathbb{Z}\}$ is a tight frame with frame bound 1 for its closed span V_0; and, by Proposition 4.3, V_0 is the translation invariant Paley-Wiener space $PW_{\Omega_m} = \left\{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subseteq \Omega_m\right\}$.

Now observe that the set

$$\left\{\sqrt{2^m}\,\tau_{2^m k}\phi : k \in \mathbb{Z}\right\}$$

is the standard Fourier orthonormal basis for PW_{Ω_m} underlying the Classical Sampling Theorem for Ω_m-bandlimited functions, e.g., [Ben97] (Theorem 3.10.10).

The systems

$$\left\{\sqrt{2^m}\tau_{(2^m k+j)}\phi : k \in \mathbb{Z}\right\}, \quad j = 1, \ldots, 2^m,$$

are unitary images of this particular orthonormal basis under translations. Hence, we can interpret expansions for PW_{Ω_m} with respect to the frame $\{\tau_k\phi : k \in \mathbb{Z}\}$ as oversampling the standard Fourier basis by the factor 2^m. This interpretation clarifies the nature of the redundancy in this frame.

Let us now make a simple quantitative study showing how this redundancy gives rise to increased robustness of the signal reconstruction from the decomposition coefficients of a signal, if these coefficients are corrupted with additive mean zero white noise η. This noise could stem from transmission channel imperfections or might be used to model quantization noise, e.g., [Vai93]. To fix ideas, let L be the Bessel map associated with the frame, which could be an ONB, and let f be the signal. Then $L(f)$ is the sequence of decomposition coefficients of f, and the reconstruction is effected by the map

$$\begin{array}{rcl} R : l^2(\mathbb{Z}) & \to & PW_{\Omega_m} \\ c & \mapsto & S^{-1}L^*(c). \end{array}$$

In particular, $L(f) \in l^2(\mathbb{Z})$. By the linearity of R, the reconstruction error due to η is just $R(\eta)$. If R is a multiple of an isometry on $(\mathcal{N}(R))^\perp$, the expected quadratic error ϵ^2 due to the noise η is

$$\epsilon^2 = \mathbf{E}(\|R(\eta)\|^2) = \mathbf{E}(\|R(P_R\eta)\|^2) = \|R\|^2\, \mathbf{E}(\|P_R\eta\|^2), \tag{1.30}$$

where P_R denotes the orthogonal projection onto $(\mathcal{N}(R))^\perp$ and $\mathbf{E}$ is the expectation operator. Recall that for mean zero white noise η with total noise power $W = \sum_{k\in\mathbb{Z}} \mathbf{E}|\eta_k|^2$, the expected spectral energy density is the constant function W on $\mathbb{T}$.

We now compare three different resolutions of identity for PW_{Ω_m}.

1. *Orthogonal decomposition of* PW_{Ω_m}
   Consider the Bessel map L defined by

$$f \mapsto \{\langle f, \tau_{2^m k}\phi\rangle : k \in \mathbb{Z}\},$$

   and reconstruction map R defined by

$$c \mapsto 2^m \sum_{k\in\mathbb{Z}} c_k \tau_{2^m k}\phi. \tag{1.31}$$

By orthogonality of $\left\{\sqrt{2^m}\tau_{2^m k}\phi : k \in \mathbb{Z}\right\}$, $\mathcal{N}(R) = \{0\}$ and $\|R\| = \sqrt{2^m}$. Fix $\sum_{k\in\mathbb{Z}} \mathbf{E}|\eta_k|^2 = W_0$, with some $W_0 > 0$. Then by (1.30), we obtain $\epsilon^2 = 2^m W_0$ for the case of this orthogonal decomposition.

2. *Frame expansions for PW_{Ω_m} by 2^m-fold oversampling*
We shall consider the case of 2^m-fold oversampling. Consider the Bessel map L defined by

$$f \mapsto \{\langle f, \tau_k\phi\rangle : k \in \mathbb{Z}\},$$

and reconstruction map R defined by

$$c \mapsto \sum_{k\in\mathbb{Z}} c_k \tau_k \phi.$$

Since $\{\tau_k\phi : k \in \mathbb{Z}\}$ is a tight frame, R is an isometry on $(\mathcal{N}(R))^{\perp}$ by Proposition 3.4. Notice that $L(f)$ has "2^m times as many" coefficients as in the orthonormal decomposition of case 1; and, in both cases 1 and 2, the individual coefficients, which are all of the form $\langle f, \tau_l\phi\rangle$ for some $l \in \mathbb{Z}$, are scaled equivalently, in the sense that they all arise from integration against shifts of one fixed function ϕ. Thus, to model a fair comparison to case 1, for the total noise power, we now choose $\sum_{k\in\mathbb{Z}} \mathbf{E}|\eta_k|^2 = 2^m W_0$.
Observing that $\mathcal{N}(R) = \left\{c : \sum_{k\in\mathbb{Z}} c_k e_k = 0 \text{ on } \Omega_m\right\}$, we obtain

$$\mathbf{E}(\|P_R\eta\|^2) = |\Omega_m|\, 2^m W_0 = W_0,$$

where $|\Omega_m| = 2^{-m}$ denotes the Lebesgue measure of Ω_m. Thus $\epsilon^2 = W_0$, i.e., in the frame case, there is a gain in robustness over the orthogonal case 1. by precisely the oversampling factor 2^m.

We remark that possibly error-feedback techniques like "noise shaping" through sigma-delta coding at a quantization stage could be used to reduce further the influence of quantization noise, cf., [ASS96] and Proposition 5.3 of [Mal98].

3. *"Averaging" of the orthogonal decomposition of PW_{Ω_m}*
In this case, we shall study redundancy generated by an averaging of the orthogonal decomposition. As in case 1, consider the Bessel map L defined by

$$f \mapsto c = \{\langle f, \tau_{2^m k}\phi\rangle : k \in \mathbb{Z}\}.$$

Now take 2^m identical copies $c^j = c$, $j = 1, \ldots, 2^m$, of this sequence, and reconstruct by averaging these, i.e., just as in (1.31), let R^j be the map defined by

$$\left\{c^1, \ldots, c^{2^m}\right\} \mapsto 2^m \sum_{k\in\mathbb{Z}} c_k^j \tau_{2^m k}\phi,$$

but take

$$R = \frac{1}{2^m} \sum_{j=1,\ldots,2^m} R^j.$$

If zero mean white noise η^j with total power $\sum_{k\in\mathbb{Z}} \mathbf{E}|\eta_k^j|^2 = W_0$ is added to c^j, and η^j and η^l are independent for $j \neq l$, we obtain, analogously to case 1,

$$\epsilon^2 = \left(\frac{1}{2^m}\right)^2 \sum_{j=1,\ldots,2^m} \mathbf{E}(\|R^j\eta^j\|^2) = \left(\frac{1}{2^m}\right)^2 \sum_{j=1,\ldots,2^m} 2^m W_0 = W_0.$$

In other words, 2^m-fold averaging the orthogonal case 1 leads to the same increase in robustness as the 2^m-fold oversampling in the frame decomposition in case 2.

It is interesting to compare these results to Munch's analysis of the interplay between redundancy and robustness in tight Weyl-Heisenberg frames, cf., [Mun92].

For the above choice of ϕ, we now derive a wavelet frame for $L^2(\mathbb{R})$. Taking $H_0 = \mathbf{1}_{\Omega_{m+1}}$, by means of Proposition 4.8, we obtain $V_0 \subseteq V_1$, and $\{V_j, \phi\}$ forms a frame multiresolution analysis. Set $H_1 = \mathbf{1}_{\Omega_m \setminus \Omega_{m+1}}$ and define the wavelet ψ by the dilation equation $\widehat{\psi}(2\cdot) = H_1\, \widehat{\phi}$. Then $P(|\widehat{\psi}|^2) = \mathbf{1}_{\Omega_m \setminus \Omega_{m+1}}$, so that by Theorem 4.2, $\{\tau_k\psi : k \in \mathbb{Z}\}$ is a tight frame with frame bound 1 for the closure of its span. Moreover, setting $G_0 = H_0$ and $G_1 = H_1$, we see that (1.17) and (1.18) of Proposition 4.10 as well as (1.16) are satisfied, and so $\{\tau_k\psi : k \in \mathbb{Z}\}$ generates W_0 and $\left\{D^j\tau_k\psi : j,k \in \mathbb{Z}\right\}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Note that the filters H_0, H_1, G_0, G_1 are all "$2^{-(m+1)}$-band".

Clearly, the 2^m-fold oversampling of $V_0 = PW_{\Omega_m}$ by $\{\tau_k\phi : k \in \mathbb{Z}\}$ observed above carries over to 2^m-fold oversampling of the wavelet space $W_0 = PW_{\Omega_{m-1}\setminus\Omega_m}$ by $\{\tau_k\psi : k \in \mathbb{Z}\}$. Indeed, one can also readily verify directly that $\left\{\sqrt{2^m}\tau_{2^m k}\psi : k \in \mathbb{Z}\right\}$ is an orthonormal basis for W_0. The resulting gain in robustness for signal decompositions with respect to the frame $\left\{D^j\tau_k\psi : j,k \in \mathbb{Z}\right\}$ for $L^2(\mathbb{R})$ is therefore qualitatively and quantitatively identical to that studied before in terms of $\{\tau_k\phi : k \in \mathbb{Z}\}$ and V_0. In particular, if we assume corruption of decomposition coefficients with white noise, this implies there can be no additional gain in robustness arising from noise suppression by the fact that filters of the associated perfect reconstruction multirate system are narrow band.

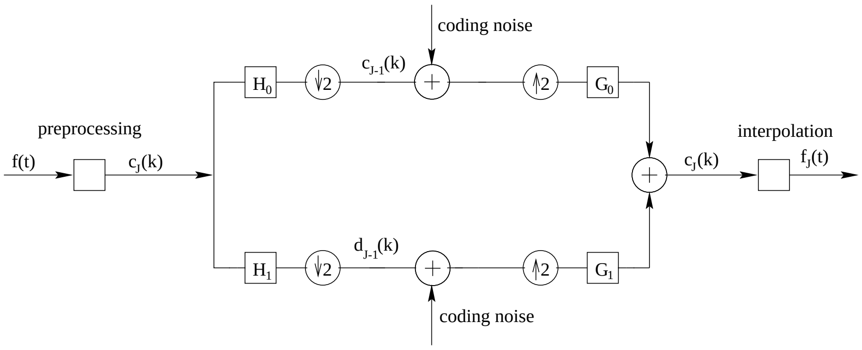

FIGURE 2. Multirate system associated with a Frame Multiresolution Analysis

6 Perfect Reconstruction Multirate Systems from Frame Multiresolution Analyses

Analogous to MRAs, we can associate discrete subband coding schemes with FMRAs in the following way.

Let $\{V_j, \phi\}$ be an FMRA for which there are filters H_0, H_1, G_0, G_1 that satisfy (1.17) and (1.18). Given $f \in L^2(\mathbb{R})$, we can consider the orthogonal projections f_j of f onto V_j; in fact, from Proposition 4.4 and equation (1.11), $f_j = \sum_{k\in\mathbb{Z}} \langle f, D^j \tau_k \phi \rangle D^j \tau_k \theta$. Moreover, as $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$, given $\epsilon > 0$, we can find J such that for all $j > J$, we have $\|f - f_j\| \leq \epsilon$. Thus, with respect to the frame $\left\{D^j \tau_k \theta : k \in \mathbb{Z}\right\}$, f is represented by the digital data $c_j(k) = \langle f, D^j \tau_k \phi \rangle$, and we recall from Remark 4.12 that $\widehat{c}_j$ vanishes on the set $[\Phi = 0]$.

As with MRAs, the dilation equations $\widehat{\phi}(2\cdot) = H_0\,\widehat{\phi}$ and $\widehat{\psi}(2\cdot) = H_1\,\widehat{\phi}$ together with (1.19) induce the following decomposition and reconstruction computations on the discrete data $c_j(k) = \langle f, D^j \tau_k \phi \rangle$:

- Decomposition:

$$\begin{aligned} c_{j-1}(k) &= \langle f, D^{j-1}\tau_k \phi\rangle = 2^{1/2}\sum_n \overline{h}_0(n-2k)c_j(n) \\ d_{j-1}(k) &= \langle f, D^{j-1}\tau_k \psi\rangle = 2^{1/2}\sum_n \overline{h}_1(n-2k)c_j(n) \end{aligned}$$

- Reconstruction:

$$c_j(k) = 2^{-1/2}\sum_n \overline{g}_0(2n-k)c_{j-1}(n) + 2^{-1/2}\sum_n \overline{g}_1(2n-k)d_{j-1}(n).$$

These computations can be interpreted as filtering and down- and upsampling in the multirate system depicted in Figure 2.

As with subband coding schemes arising in this way from MRAs, FMRAs are related to filter banks that give perfect reconstruction on their discrete input data. However, different than MRAs, in the case of non-exact FMRAs this input data has vanishing spectrum on the set $[\Phi = 0]$. The filters in a scheme derived from such an FMRA can then be "narrowband", i.e., need not be supported on $[\Phi = 0]$. Of course, in this case the filter bank cannot give perfect reconstruction on discrete input data c that violates the condition $\widehat{c} = 0$ on $[\Phi = 0]$.

As a consequence of Proposition 4.8 and Remark 4.12, we always have that $\widehat{c}_{J-1} = 0$ holds on the set $[P(|\widehat{\phi}|^2) = 0]$. Thus, the subband scheme can be iterated in the low pass branch with filter H_0.

One might think that zero-sets in narrowband reconstruction filters G_0 and G_1 should annihilate the component of broadband coding noise which is supported on these zero-sets, and therefore suppress some noise in the reconstruction. To sharpen intuition on this matter, let us consider the following example.

In the notation of Section 5, suppose that $\widehat{\phi} = \mathbf{1}_{\Omega_m}$, $H_0 = G_0 = \mathbf{1}_{\Omega_{m+1}}$ on $[-1/2, 1/2)$, and $H_1 = G_1 = \mathbf{1}_{\Omega_m \setminus \Omega_{m+1}}$ on $[-1/2, 1/2)$. This filter bank yields perfect reconstruction of discrete data c for which $\operatorname{supp} \widehat{c} \subseteq \Omega_m$ in the case there is no coding noise. True enough, due to the zero-sets of G_0 and G_1, only (the fraction) 2^{-m} of the energy of any flat spectrum broadband noise introduced during coding will corrupt the reconstruction of c.

However, because of the requirement that $\operatorname{supp} \widehat{c} \subseteq \Omega_m$, the input data c can be downsampled 2^m times before aliasing occurs.

Suppose we implement this type of downsampling as "preprocessing" and use conjugate quadrature filters (CQFs). Recall that $H_0, H_1, G_0, G_1 \in L^\infty(\mathbb{T})$ are a system of CQFs if $|H_0|^2 + |\tau_{1/2} H_0|^2 = 1$, $G_0 = H_0$, and $G_1 = H_1 = e_1 \tau_{1/2} \overline{H}_0$, e.g., [Mal98]. In our case, we take $H_0 = G_0 = \mathbf{1}_{[-1/4, 1/4)}$ and $H_1 = G_1 = 1 - H_0$ on $[-1/2, 1/2)$. Then the amount of data to be processed is only (the fraction) 2^{-m} of that in the first scenario where we used narrowband filters. Hence, for a fair comparison of the two processes, the total noise energy injected into the low and high pass branches after downsampling should be reduced accordingly in the CQF process. Thus, although all broadband noise introduced at the coding stage reconstructs in a CQF multirate system, we do not achieve more robustness by processing narrowband input data with a narrowband filter bank when compared to processing downsampled narrowband data with CQF schemes. Moreover, filters that are constant on large sets such as intervals cannot have finite impulse response.

It should be noted that the downsampling type of preprocessing we required in conjunctions with CQFs can only be done in powers of two. Further, CQF type filters are essentially halfband. Therefore, the narrowband filter bank associated with the choice $\widehat{\phi} = \mathbf{1}_{[-3/16, 3/16)}$ could effectively subdivide the band $[-3/16, 3/16)$, whereas processing with CQFs would be problematic.

7 Wavelet Frames for $L^2(\mathbb{R})$ from the Unitary Extension Principle of Ron and Shen

In [RoSh97a], Ron and Shen associate to finitely generated (affine) wavelet systems

$$X = \left\{D^j \tau_k \psi_i : j, k \in \mathbb{Z}, i = 1, \dots, n\right\} \subseteq L^2(\mathbb{R}^d)$$

their *quasi-affine* shift invariant counterparts

$$\begin{aligned} X_q &= \left\{D^j \tau_k \psi_i : j \geq 0, k \in \mathbb{Z}, i = 1, \dots, n\right\} \\ &\quad \cup \left\{2^{j/2} \tau_k D^j \psi_i : j < 0, k \in \mathbb{Z}, i = 1, \dots, n\right\}; \end{aligned}$$

and prove under mild decay conditions on $\widehat{\psi}_i$, that the Bessel maps for X and X_q have identical upper and lower bounds. This is accomplished with the help of "fiberization" techniques introduced in [RoSh95] for the study of shift invariant subspaces of $L^2(\mathbb{R}^d)$. Thus, X and X_q are equivalent as frames. It can be seen from the renormalization in the definition of X_q, that X_q can never be a Riesz basis.

Based upon this equivalence and by investigating the quasi-affine system X_q, Ron and Shen derive a "Unitary Extension Principle", which, if one is willing to have more wavelets than in the orthonormal wavelet constructions coming from MRAs, facilitates the construction of new tight wavelet frames for $L^2(\mathbb{R}^d)$.

We shall state the Extension Principle, illustrate it with one of Ron and Shen's examples for its application, and give a direct and simple proof which also does not need the aforementioned decay condition. For the sake of notational simplicity, we restrict ourselves to the 1-dimensional setting with dyadic dilations, but the proofs carry over to higher dimensions without problems.

7.1 The Framework of the Unitary Extension Principle

Our starting point is a refinable function $\psi_0 \in L^2(\mathbb{R})$ such that

$$\lim_{\gamma \to 0} \widehat{\psi}_0(\gamma) = 1.$$

Here refinability of ψ_0 means the existence of a 1-periodic function H_0, the refinement filter, such that

$$\widehat{\psi}_0(2\cdot) = H_0 \, \widehat{\psi}_0.$$

Observe that $\lim_{\gamma \to 0} H_0(\gamma) = 1$, i.e., H_0 is a lowpass filter.

Furthermore, let H_i, $i = 1, \dots, n$, be 1-periodic functions and define n wavelets ψ_i, $i = 1, \dots, n$, by means of the equations

$$\widehat{\psi}_i(2\cdot) = H_i \, \widehat{\psi}_0. \tag{1.32}$$

Put

$$\mathbf{H} = \begin{pmatrix} H_0 & \tau_{1/2} H_0 \\ H_1 & \tau_{1/2} H_1 \\ \vdots & \vdots \\ H_n & \tau_{1/2} H_n \end{pmatrix}. \tag{1.33}$$

Ron and Shen's Unitary Extension Principle states that *if*

$$\mathbf{H}^* \mathbf{H} = \mathbf{Id} \quad \text{a.e. on the set} \quad [P(|\widehat{\psi}_0|^2) > 0], \tag{1.34}$$

then the wavelet system $\{D^j \tau_k \psi_i\}_{j,k \in \mathbb{Z}, i=1, \dots, n}$, *generated by the* ψ_i, $i = 1, \dots, n$, *is a tight wavelet frame for* $L^2(\mathbb{R})$ *with frame bound 1.* In particular, we have by Theorem 3.7 that

$$\forall\, f \in L^2(\mathbb{R}), \quad f = \sum_{i=1, \dots, n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, D^j \tau_k \psi_i \rangle D^j \tau_k \psi_i.$$

We note that (1.34) clearly forces $|H_i| \le 1, i = 0, \dots, n$, on $[P(|\widehat{\psi}_0|^2) > 0]$, and so the wavelets defined in (1.32) are in $L^2(\mathbb{R})$.

Remark 7.1

It was not explicitly stated in [RoSh97a] that $\mathbf{H}^* \mathbf{H} = \mathbf{Id}$ need only hold on $[P(|\widehat{\psi}_0|^2) > 0]$, but it is interesting to note this fact in light of Theorem 4.2 and its role in the FMRA context and Li's theorem to generalize the FMRA approach [Li98]. However, we no longer require ψ_0 be a "good" generator for $\overline{\text{span}}\,\{\tau_k \psi_0 : k \in \mathbb{Z}\}$ in the sense that $\{\tau_k \psi_0 : k \in \mathbb{Z}\}$ is a basis or a (pseudo-)frame for $\overline{\text{span}}\,\{\tau_k \psi_0 : k \in \mathbb{Z}\}$, cf., [Li98]. Also, although

$$\overline{\text{span}}\,\{\tau_k \psi_0 : k \in \mathbb{Z}\} \subseteq \overline{\text{span}}\,\{D \tau_k \psi_0 : k \in \mathbb{Z}\},$$

we do not impose that

$$\overline{\text{span}}\,\{\tau_k \psi_0 : k \in \mathbb{Z}\}$$

and

$$\overline{\text{span}}\,\{\tau_k \psi_i : k \in \mathbb{Z}, i = 1, \dots, n\}$$

split

$$\overline{\text{span}}\,\{D \tau_k \psi_0 : k \in \mathbb{Z}\}$$

as a direct (or even orthogonal) sum.

7.2 Example: Compactly Supported Tight Spline Frames

For illustration and completeness, we briefly and almost verbatim recall a simple example of an application of the Unitary Extension Principle, cf., [RoSh97a]. For a higher dimensional construction, see [RoSh97b].

Let m be a positive integer, and define the 1-periodic function $H_0(\gamma) = \cos^{2m}(\pi\gamma)$. This H_0 is the refinement filter of the centered B-spline ψ_0 of order $2m$ defined by its Fourier transform as

$$\widehat{\psi}_0(\gamma) = \frac{\sin^{2m}(\pi\gamma)}{(\pi\gamma)^{2m}}.$$

Note that $\lim_{\gamma\to 0}\widehat{\psi}_0(\gamma) = 1$. Define $2m$ 1-periodic wavelet filters H_j, $j = 1, \dots, 2m$, by

$$H_j(\gamma) = \sqrt{\binom{2m}{j}}\sin^j(\pi\gamma)\cos^{2m-j}(\pi\gamma).$$

Moreover, it is easy to verify that then the filter matrix $\mathbf{H}$ has orthonormal columns everywhere on $\mathbb{T}$, so by the Unitary Extension Principle, the $2m$ wavelets given by

$$\widehat{\psi}_j(\gamma) = i^j\sqrt{\binom{2m}{j}}\frac{\cos^{2m-j}(\pi\gamma/2)\sin^{2m+j}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}}, \quad j = 1, \dots, 2m$$

generate a tight frame with frame bound 1 for $L^2(\mathbb{R})$. Note that each of the wavelets is a real valued symmetric (or anti-symmetric) function supported in $[-m, m] = \operatorname{supp}\psi_0$, and is a spline of degree $2m-1$, smoothness class C^{2m-2}, and knots at $\mathbb{Z}/2$.

If $m = 1$, the wavelets ψ_1 and ψ_2 are illustrated in Figures 3 and 4, respectively.

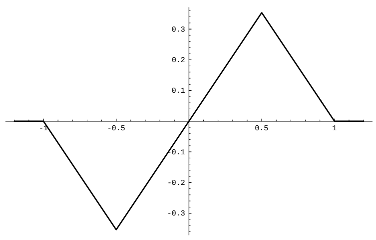

FIGURE 3. $\psi_1 = 2^{-3/2}(\tau_{1/2} - \tau_{1/2})(\mathbf{1}_{[-1/4,1/4]} * \mathbf{1}_{[-1/4,1/4]})$

It is also easy to verify that if $m = 1$, then

$$\det\begin{pmatrix} H_1(\gamma) & \tau_{1/2}H_1(\gamma) \\ H_2(\gamma) & \tau_{1/2}H_2(\gamma) \end{pmatrix}$$

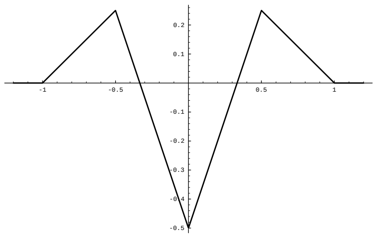

FIGURE 4. $\psi_2 = 4^{-1}(\tau_{1/2} + \tau_{1/2} - 2\tau_0)(\mathbf{1}_{[-1/4,1/4]} * \mathbf{1}_{[-1/4,1/4]})$

vanishes for only finitely many $\gamma \in [-1/2, 1/2)$. In view of Lemma 7.4, this implies that if $\langle f, D^{-1}\tau_k\psi_i\rangle = 0$ for $i = 1, 2$ and all $k \in \mathbb{Z}$, then $\langle f, \tau_k\psi_0\rangle = 0$ for all $k \in \mathbb{Z}$. Consequently,

$$\overline{\text{span}}\,\left\{D^{-1}\tau_k\psi_i\right\}_{k\in\mathbb{Z},\, i=1,\dots,n} = \overline{\text{span}}\,\{\tau_k\psi_0 : k \in \mathbb{Z}\},$$

i.e., at a fixed dilation level, the spaces spanned by the wavelets and the scaling function overlap and are not independent or even orthogonal anymore.

7.3 Multirate Systems from the Unitary Extension Principle

Inspecting Lemma 7.4 below, we see that we can associate a multirate system as in Figure 5 with the refinement equations

$$\widehat{\psi}_i(2\cdot) = H_i\,\widehat{\psi}_0, \quad i = 0, \dots, n.$$

These "oversampled" filter banks are also described in the engineering literature, e.g., [Böl97].

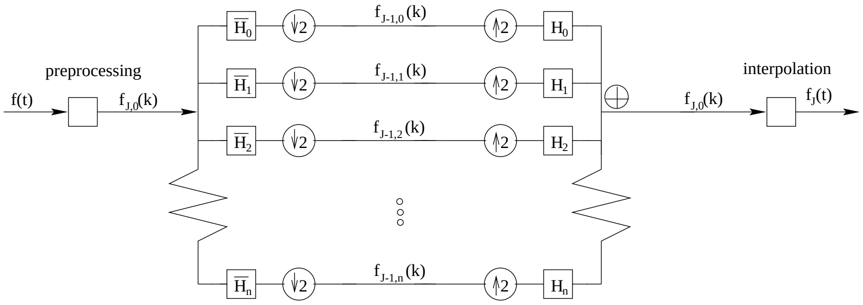

FIGURE 5. Multirate System Related to the Extension Principle

In Figure 5, $f_{j,i}(k) = \langle f, D^j\tau_k\psi_i\rangle$ and $f_j = \sum_{k\in\mathbb{Z}}\langle f, D^j\tau_k\psi_0\rangle D^j\tau_k\psi_0$.

7.4 Proof of the Extension Principle

The idea for the Extension Principle is straightforward. First, for fixed f from a dense set in $L^2(\mathbb{R})$, and given $\epsilon > 0$, we verify that for all sufficiently large j the maps $h \mapsto \{\langle h, D^j \tau_k \psi_0\rangle : k \in \mathbb{Z}\}$ preserve the norm of f up to ϵ. Then, noting that $\mathbf{H}^*\mathbf{H} = \mathbf{Id}$ means that $\mathbf{H}$ is an isometry $\mathbb{C}^2 \to \mathbb{C}^{n+1}$, we observe that for every j, the splitting

$$\{\langle f, D^j \tau_k \psi_0\rangle : k \in \mathbb{Z}\} \mapsto \{\langle f, D^{j-1} \tau_k \psi_i\rangle : k \in \mathbb{Z}, i = 0, \ldots, n\}$$

is energy-preserving. These two claims combined essentially give that

$$\{D^j \tau_k \psi_i : j, k \in \mathbb{Z}, i = 1, \ldots, n\}$$

is a tight frame. Here are the details.

Notationally, set

$$Z = \left\{ f \in L^2(\mathbb{R}) : \widehat{f} \text{ is bounded and has compact support} \right\}.$$

Lemma 7.2 *Let $g, \psi_0 \in L^2(\mathbb{R})$. If $P(g\overline{\widehat{\psi}_0}) \in L^2(\mathbb{T})$, then*

$$\sum_{k\in\mathbb{Z}} \langle g, e_k \widehat{\psi}_0\rangle e_k = P(g\overline{\widehat{\psi}_0}).$$

Proof:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \langle g, e_k \widehat{\psi}_0\rangle e_k &= \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} g\overline{\widehat{\psi}_0} e_{-k} \right) e_k \\
&= \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{T}} P(g\overline{\widehat{\psi}_0}) e_{-k} \right) e_k \\
&= P(g\overline{\widehat{\psi}_0}). \quad \square
\end{aligned}$$

This elementary "trick" will be used several times without mention in computations below. Notice that $P(g\overline{\widehat{\psi}_0}) \in L^2(\mathbb{T})$ if g is bounded and has compact support or, by Proposition 4.1, if $P(|\widehat{\psi}_0|^2) \in L^\infty(\mathbb{T})$.

Lemma 7.3 *Suppose that $\psi_0 \in L^2(\mathbb{R})$ satisfies $\lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1$. If $f \in Z$, then, given $\epsilon > 0$, for all sufficiently large j,*

$$\left| \sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0\rangle|^2 - \|f\|^2 \right| < \epsilon \|f\|^2.$$

Proof: By Lemma 7.2 and the Parseval-Plancherel Theorem for the torus,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0\rangle|^2 &= \sum_{k\in\mathbb{Z}} |\langle D^j \widehat{f}, e_k \widehat{\psi}_0\rangle|^2 \\
&= \int_{[-1/2,1/2]} |P((D^j \widehat{f})\overline{\widehat{\psi}_0})|^2.
\end{aligned}$$

Now, by the assumption $\lim_{\gamma\to 0}\widehat{\psi}_0(\gamma)=1$, there is $0<b<1/2$ such that if $|\gamma|<b$, then $1-\epsilon<|\widehat{\psi}_0(\gamma)|^2<1+\epsilon$. Pick J such that for $j>J$,

$$\operatorname{supp}(D^j\widehat{f})\subseteq[-b,b].$$

For these j, we have

$$\int_{[-1/2,1/2)}|P((D^j\widehat{f})\overline{\widehat{\psi}_0})|^2=\int_{[-b,b)}|D^j\widehat{f}|^2|\widehat{\psi}_0|^2,$$

and the lemma follows since $\|D^j\widehat{f}\|=\|f\|$ for all $j\in\mathbb{Z}$. □

Lemma 7.4 *Let ψ_i, H_i, $i=0,\ldots,n$, be as in Section 7.1. For $f\in Z$ define 1-periodic functions*

$$F_{j,i}=\sum_{k\in\mathbb{Z}}\langle f, D^j\tau_k\psi_i\rangle e_k.$$

Then

$$\begin{aligned} F_{j-1,i} &= 2^{-1/2}(\overline{H}_iF_{j,0}+\tau_{1/2}(\overline{H}_iF_{j,0}))(\cdot/2)\\ &= 2^{-1/2}\left[\overline{\mathbf{H}}\begin{pmatrix}F_{j,0}\\ \tau_{1/2}F_{j,0}\end{pmatrix}\right]_i(\cdot/2), \end{aligned}$$

where $[v]_i$ denotes the i^{th} component of the vector v.

Proof: By Lemma 7.2, $F_{j,0}=P((D^j\widehat{f})\overline{\widehat{\psi}_0})$. On the other hand, noting that $\{2^{1/2}e_{2k}:k\in\mathbb{Z}\}$ is an orthonormal basis for $L^2([-1/4,1/4))$, we obtain

$$\begin{aligned} F_{j-1,i} &= \sum_{k\in\mathbb{Z}}\langle f, D^{j-1}\tau_k\psi_i\rangle e_k=\sum_{k\in\mathbb{Z}}\langle f, D^j\tau_{2k}D^{-1}\psi_i\rangle e_k\\ &= \sum_{k\in\mathbb{Z}}\langle D^j\widehat{f}, e_{2k}D\widehat{\psi}_i\rangle e_k=\sum_{k\in\mathbb{Z}}\langle D^j\widehat{f}, e_{2k}2^{1/2}H_i\widehat{\psi}_0\rangle e_k\\ &= 2^{-1/2}\sum_{k\in\mathbb{Z}}\langle (D^j\widehat{f})\overline{H}_i\overline{\widehat{\psi}}_0, 2^{1/2}e_{2k}\rangle 2^{1/2}e_{2k}(\cdot/2)\\ &= 2^{-1/2}\sum_{k\in\mathbb{Z}}\tau_{k/2}((D^j\widehat{f})\overline{H}_i\overline{\widehat{\psi}_0})(\cdot/2)\\ &= 2^{-1/2}\Big(P((D^j\widehat{f})\overline{H}_i\overline{\widehat{\psi}_0})(\cdot/2)+\tau_{1/2}P((D^j\widehat{f})\overline{H}_i\overline{\widehat{\psi}_0})(\cdot/2)\Big)\\ &= 2^{-1/2}\Big(\overline{H}_iP((D^j\widehat{f})\overline{\widehat{\psi}_0})+\tau_{1/2}(\overline{H}_iP((D^j\widehat{f})\overline{\widehat{\psi}_0}))\Big)(\cdot/2)\\ &= 2^{-1/2}\left(\overline{H}_iF_{j,0}+\tau_{1/2}(\overline{H}_i)\tau_{1/2}(F_{j,0})\right)(\cdot/2), \end{aligned}$$

where we used the 1-periodicity of H_i in the penultimate equality. This proves the assertion. □

Lemma 7.5 *Let ψ_i, H_i, $i = 0, \dots, n$, and $\mathbf{H}$ be as in Section 7.1. Assume that $\mathbf{H}^*\mathbf{H} = \mathbf{Id}$ a.e. on the set $[P(|\widehat{\psi}_0|^2) > 0]$. Then, for all $j \in \mathbb{Z}$ and any $f \in Z$,*

$$\sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0 \rangle|^2 = \sum_{i=0,\dots,n} \sum_{k\in\mathbb{Z}} |\langle f, D^{j-1} \tau_k \psi_i \rangle|^2.$$

Proof: By the Parseval-Plancherel Theorem for the torus and Lemma 7.4 we see that

$$\begin{aligned}
\sum_{i=0,\dots,n} \sum_{k\in\mathbb{Z}} |\langle f, D^{j-1}\tau_k\psi_i\rangle|^2 &= \int_{\mathbb{T}} \sum_{i=0,\dots,n} |F_{j-1,i}|^2 \\
&= \int_{\mathbb{T}} 2^{-1} \left\| \overline{\mathbf{H}} \begin{pmatrix} F_{j,0} \\ \tau_{1/2} F_{j,0} \end{pmatrix} \right\|^2_{\mathbb{C}^{n+1}} (\cdot/2) \\
&= \int_{\mathbb{T}} 2^{-1} \left\| \begin{pmatrix} F_{j,0} \\ \tau_{1/2} F_{j,0} \end{pmatrix} \right\|^2_{\mathbb{C}^2} (\cdot/2) \qquad (1.35) \\
&= 2^{-1} \int_{\mathbb{T}} \left(|F_{j,0}|^2 + \tau_{1/2} |F_{j,0}|^2 \right) (\cdot/2) \\
&= \int_{\mathbb{T}} |F_{j,0}|^2 \\
&= \sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0 \rangle|^2,
\end{aligned}$$

where in (1.35) we used the fact that on the set $[P(|\widehat{\psi}_0|^2) > 0]$, the maps $\mathbf{H} : \mathbb{C}^2 \to \mathbb{C}^{n+1}$ are isometries, since, by assumption, $\mathbf{H}^*\mathbf{H} = \mathbf{Id}$ holds almost everywhere there. □

Lemma 7.6 *Let ψ_i, H_i, $i = 0, \dots, n$, and $\mathbf{H}$ be as in Section 7.1. Assume that $\mathbf{H}^*\mathbf{H} = \mathbf{Id}$ a.e. on the set $[P(|\widehat{\psi}_0|^2) > 0]$. Then*

$$P(|\widehat{\psi}_0|^2) \le 1.$$

Proof: Suppose $f \in Z$. Then from Lemmas 7.3 and 7.5, given $\epsilon > 0$, for all sufficiently large $j > 0$,

$$\sum_{k\in\mathbb{Z}} |\langle f, \tau_k \psi_0 \rangle|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0 \rangle|^2 \le (1+\epsilon)\|f\|^2.$$

Thus, since Z is dense in $L^2(\mathbb{R})$, the Bessel map associated with the system $\{\tau_k \psi_0 : k \in \mathbb{Z}\}$ has norm 1, and the inequality $P(|\widehat{\psi}_0|^2) \le 1$ follows from Proposition 4.1. □

Lemma 7.7 *Let $\psi_0 \in L^2(\mathbb{R})$. If $P(|\widehat{\psi}_0|^2) \leq B^2$ for some B, then for all $f \in L^2(\mathbb{R})$,*

$$\lim_{j\to-\infty} \| \left\{ \langle f, D^j \tau_k \psi_0 \rangle : k \in \mathbb{Z} \right\} \|_{l^2(\mathbb{Z})} = 0.$$

Proof: Suppose first that $|\widehat{f}|$ is bounded by M. The set of these functions is dense in $L^2(\mathbb{R})$. By the Parseval-Plancherel Theorem,

$$\begin{aligned} \| \left\{ \langle f, D^j \tau_k \psi_0 \rangle) : k \in \mathbb{Z} \right\} \|^2_{l^2(\mathbb{Z})} &= \| \left\{ \langle D^j \widehat{f}, e_k \widehat{\psi}_0 \rangle : k \in \mathbb{Z} \right\} \|^2_{l^2(\mathbb{Z})} \\ &= \int_{\mathbb{T}} | \sum_{k\in\mathbb{Z}} \langle D^j \widehat{f}, e_k \widehat{\psi}_0 \rangle e_k |^2 \\ &= \int_{\mathbb{T}} |P((D^j \widehat{f}) \overline{\widehat{\psi}_0})|^2 \leq 2^j M^2 B^2, \end{aligned}$$

and the right side tends to 0 as $j \to -\infty$. Since $\langle f, D^j \tau_k \psi_0 \rangle = \langle D^{-j} f, \tau_k \psi_0 \rangle$ and the Bessel map associated with the system $\{\tau_k \psi_0 : k \in \mathbb{Z}\}$ is bounded by Proposition 4.1, the assertion holds on all of $L^2(\mathbb{R})$. $\square$

We now have all the ingredients for proving the Unitary Extension Principle.

Theorem 7.8 *Let ψ_i, H_i, $i = 0, \ldots, n$, be as in Section 7.1, and assume that $\mathbf{H}^*\mathbf{H} = \mathbf{Id}$ a.e. on the set $[P(|\widehat{\psi}_0|^2) > 0]$. The wavelet system $\left\{ D^j \tau_k \psi_i : j, k \in \mathbb{Z}, i = 1, \ldots, n \right\}$ is a tight frame with frame bound 1 for $L^2(\mathbb{R})$.*

Proof: Let $\epsilon > 0$, and take any $f \in Z$. By Lemma 7.3, for all sufficiently large j,

$$(1-\epsilon)\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, D^j \tau_k \psi_0 \rangle|^2 \leq (1+\epsilon)\|f\|^2.$$

Repeated application of Lemma 7.5 gives

$$\begin{aligned} \forall\, m < j, \quad (1-\epsilon)\|f\|^2 &\leq \sum_{k\in\mathbb{Z}} |\langle f, D^m \tau_k \psi_0 \rangle|^2 \\ &\quad + \sum_{m<l\leq j} \sum_{k\in\mathbb{Z}} |\langle f, D^l \tau_k \psi_i \rangle|^2 \leq (1+\epsilon)\|f\|^2. \end{aligned}$$

Because of Lemma 7.6, Lemma 7.7 applies and shows that

$$\lim_{m\to-\infty} \sum_{k\in\mathbb{Z}} |\langle f, D^m \tau_k \psi_0 \rangle|^2 = 0.$$

Thus, we have that

$$(1-\epsilon)\|f\|^2 \leq \sum_{-\infty<l\leq j} \sum_{k\in\mathbb{Z}} |\langle f, D^l \tau_k \psi_i \rangle|^2 \leq (1+\epsilon)\|f\|^2.$$

Taking $j \to \infty$ and $\epsilon \to 0$, we see that the map defined by

$$L : f \mapsto \left\{ \langle f, D^j \tau_k \psi_i \rangle : j, k \in \mathbb{Z}, i = 1, \ldots, n \right\}$$

is isometric on the dense subspace Z, and so L is an isometry on all of $L^2(\mathbb{R})$. □

Remark 7.9

It is now also not difficult to use the previous analysis in the following way to derive Meyer's Theorem on orthonormal wavelets. Assume that $\lim_{\gamma \to 0} \widehat{\psi_0}(\gamma) = 1$ and that $\{\tau_k \psi_0 : k \in \mathbb{Z}\}$ is an orthonormal basis for its closed span. By standard periodization arguments, ψ_0's refinement filter H_0 as in (7.1) is a conjugate quadrature filter, that is, $|H_0|^2 + |\tau_{1/2} H_0|^2 = 1$. Taking $H_1 = e_1 \tau_{1/2} \overline{H}_0$, we obtain unitary matrices $\mathbf{H}$ in (1.33). On the other hand, we know from Proposition 3.11 that a tight frame with bound 1 and normalized elements is in fact an orthonormal basis. Thus, $\left\{ D^j \tau_k \psi_1 : j \in \mathbb{Z}; k \in \mathbb{Z} \right\}$, where $2^{-1/2} D\widehat{\psi_1} = H_1 \widehat{\psi_0}$, is an orthonormal basis, since one can easily verify that $\|\psi_1\| = 1$. In fact,

$$\begin{aligned}
\frac{1}{2}\|\psi_1\|^2 &= \int_{\widehat{R}} |H_1 \widehat{\psi}_0|^2 \\
&= \int_{[-1/4,1/4)} |H_1|^2 P(|\widehat{\psi}_0|^2) + \tau_{1/2}|H_1|^2 \tau_{1/2} P(|\widehat{\psi}_0|^2) \\
&= \int_{[-1/4,1/4)} (1 - \tau_{1/2}|H_1|^2) P(|\widehat{\psi}_0|^2) + (1 - |H_1|^2) \tau_{1/2} P(|\widehat{\psi}_0|^2) \\
&= \int_{\widehat{R}} |\widehat{\psi}_0|^2 - \int_{[-1/4,1/4)} |H_0|^2 P(|\widehat{\psi}_0|^2) + \tau_{1/2}|H_0|^2 \tau_{1/2} P(|\widehat{\psi}_0|^2) \\
&= \|\psi_0\|^2 - \frac{1}{2}\|\psi_0\|^2 \\
&= \frac{1}{2},
\end{aligned}$$

where the penultimate equality follows from the refinement relation for ψ_0.

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