WEIGHTED FOURIER INEQUALITIES: NEW PROOFS AND GENERALIZATIONS

JOHN J. BENEDETTO AND HANS P. HEINIG

Abstract. Fourier transform inequalities in weighted Lebesgue spaces are proved. The inequalities are generalizations of the Plancherel theorem, they are characterized in terms of uncertainty principle relations between pairs of weights, and they are put in the context of existing weighted Fourier transform inequalities. The proofs are new and relatively elementary, and they give rise to good and explicit constants controlling the continuity of the Fourier transform operator. The smaller the constant is, the more applicable the inequality will be in establishing weighted uncertainty principle or entropy inequalities. There are two essentially different proofs, one depending on operator theory and one depending on Lorentz spaces. The results from these approaches are quantitatively compared, leading to classical questions concerning multipliers and to new questions concerning wavelets.

1. Introduction

1.1. **Background.** The *Fourier transform* of a complex-valued Lebesgue measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ on Euclidean space $\mathbb{R}^n$ is formally defined as

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\gamma}\, dx, \tag{1.1}$$

where $\gamma \in \widehat{\mathbb{R}}^n (= \mathbb{R}^n)$. If f belongs to the Lebesgue space $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then its L^p-norm is designated $\|f\|_p$. It is elementary to see that $\|\hat{f}\|_\infty \le \|f\|_1$ for $f \in L^1(\mathbb{R}^n)$; and if $f \in L^2(\mathbb{R}^n)$ then the Plancherel theorem asserts that $\|\hat{f}\|_2 = \|f\|_2$, see the first paragraph of *Section 2*.

Both of these norm relationships can be viewed as special cases of a weighted Fourier transform norm inequality,

$$\|\hat{f}\|_{L^q_u} \le C\|f\|_{L^p_v}, \tag{1.2}$$

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where the *weights* u, v are non-negative, locally integrable functions on $\widehat{\mathbb{R}}^n$, $\mathbb{R}^n$, respectively, where C is independent of a class of functions f, and where

$$\|f\|_{L^p_v} = \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p} \tag{1.3}$$

and

$$\|\hat{f}\|_{L^q_u} = \left(\int_{\widehat{\mathbb{R}}^n} |\hat{f}(\gamma)|^q u(\gamma)d\gamma\right)^{1/q}$$

for $1 \leq p,\, q < \infty$. The usual adjustment is made in the definition of $\|f\|_{L^\infty_v}$; and in the case of $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we obviously have $\|f\|_p = \|f\|_{L^p_1}$ in terms of the notation (1.3). By definition, $L^p_v(\mathbb{R}^n)$ is the space of complex-valued Lebesgue measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ for which $\|f\|_{L^p_v} < \infty$.

The main problems concerning (1.2) are characterizing the relationship between the weights u and v to ensure the validity of (1.2), and, in this case, of finding the smallest possible constant C so that (1.2) is true for all $f \in L^p_v(\mathbb{R}^n)$. Both problems are related to the uncertainty principle in harmonic analysis [HJ94], [FS97]. In this case of characterization, the uncertainty principle is manifested by conditions such as

$$\sup_{s>0} \left(\int_0^{1/s} u(\gamma)d\gamma\right)^{1/q} \left(\int_0^s v(x)^{-(p'-1)}dx\right)^{1/p'} < \infty. \tag{1.4}$$

This particular condition (1.4) gives rise to (1.2) for $L^p_v(\mathbb{R})$ in the case of even weights u and v in which u is decreasing and v is increasing on $(0, \infty)$, e.g., [Muc83], [BH83], [JS84], [Hei84]. In the case of finding the smallest possible constant C, inequalities such as (1.2) are an essential step in proving weighted uncertainty principle inequalities which generalize those of Heisenberg type such as

$$\|f\|_2^2 \leq 4\pi\|(t - t_0)f(t)\|_2\|(\gamma - \gamma_0)\hat{f}(\gamma)\|_2,$$

e.g., [HS86], [BF94](Chapter 7.6 and 7.8).

In this paper, we shall prove general inequalities (1.2) illustrating the role and limits of operator theory in obtaining them, proving them from the point of view of Lorentz spaces, obtaining explicit constants C, and showing the theoretical obstructions when it is impossible to compute optimal constants.

Besides the inherent mathematical motivation of going beyond the Plancherel theorem in this way, we are motivated to understand and apply general inequalities (1.2) in a manner analogous to recent developments and applications of comparably general notions such as Wiener amalgam spaces and Besov spaces, e.g., see [CDD01], [CDD02], [Jaf01].

1.2. **Results and outline.** The results by Muckenhoupt [Muc83] and Jurkat and Sampson [JS84], as well as those in [BH83] and [Hei84], all cited in Subsection 1.1, also deal with general (non-even, non-monotone on $(0, \infty)$) weights u and v on $\widehat{\mathbb{R}}$ and $\mathbb{R}$, respectively. In this case the sufficient condition (1.4) is replaced by a similar one in terms of equimeasurable decreasing rearrangements of u and $1/v$, denoted by u^* and $(1/v)^*$, respectively, see Subsection 1.4 for definitions. These results were further developed in the 1980s in [BHJ87a], [HS89] and put in the context of related parts of harmonic analysis in [Hei90].

One aspect of the underlying ideas in this development was the characterization in [BHJ87b] by the authors with R. Johnson of a class of inequalities of type (1.2) in terms of so-called $A_p$weights, see *Remark 3.2b*. This work was extended in [HS89] by one of the authors with G. Sinnamon, and the interest in such a characterization is that, heretofore, A_p weights were only considered in terms of characterizations dealing with maximal functions and Hilbert transforms, e.g., [GCRdF85].

It should be pointed out that necessary conditions for the validity of (1.2) in terms of conditions such as (1.4) are sometimes valid, and their proofs, although technical, are essentially easier than proofs of the sufficient conditions, e.g., see *Theorem 4.1.* Also, our conclusions of the form (1.2) are always stated for the range $1 < p,\ q < \infty$ even though they are sometimes valid for $p, q \in \{1, \infty\}$. We have not included the latter cases in order to keep an already long presentation from getting out of hand by the inclusion of new techniques.

Theorem 2.1, which comprises our contribution in *Section 2*, provides sufficient conditions for (1.2) of the type (1.4) for general weights and the index range $1 < p,\ q < \infty$. The proof is operator theoretic, and draws on a sophisticated body of information. In order to gauge the effect of our present approach, we point out our shortcomings from the early 1980s. For example, in [BH83] it was necessary for us to treat the following cases separately: (i) $1 < p < 2,\ p \leq q$; (ii) $2 \leq p \leq q$; (iii) $p = q = 2$. Moreover, in case (i), the constant C in (1.2) became unbounded as $p \longrightarrow 2-$, and, in case (ii), it became unbounded as $q \longrightarrow 2+$. In retrospect, although we were using a powerful weapon due to Calderón [Cal66], we were not able to adapt it to our approach in a way to make reasonable estimates on constants. In any case, in the proof of *Theorem 2.1* herein the constant C remains bounded in all cases; and, in fact, a specific upper bound of C is proved. We also prove (1.2) in the index range $1 < q < p < \infty$.

Section 3 is devoted to an exposition of Lorentz spaces and to the work of Sawyer [Saw90] which we shall use in *Section 4*. We do not use the results of Flett [Fle73] from 1973 on the classical Lorentz spaces $L(p, q)$, nor do we use the comparably beautiful recent results of Sinnamon [Sin01]; on the other hand, their theories do complement our approach, e.g., see the first paragraph of *Section 4*. We close *Section 3* with a remark on Köthe spaces, which can be considered a formulation in topological vector spaces of a natural generalization of Lorentz spaces.

In *Section 4*, *Theorems 4.1* and *4.2* invoke conditions similar to (1.4) to characterize the continuity of the Fourier transform in weighted Lorentz spaces. The weights and index ranges are general, and there are basic examples in *Examples 4.3* and *4.4*. Mapping properties of operators (besides the Fourier transform) on Lorentz spaces with power weights are important in the theory of interpolation [BL76], [Tri78].

Theorem 5.1 of *Section 5* is our peroration in terms of obtaining (1.2) for a general class of weights and for the index range $1 < p,\ q < \infty$. The proof uses *Theorems 4.1* and *4.2*. We quantitatively compare *Theorems 2.1* and *5.1* in *Remark 5.2*; and then in *Example 5.3* we make a study of power weights in this context. This all leads to Pitt's theorem, including its role in the theory of multipliers and in a wavelet theory problem, as well as in the evolution of dealing with (1.2), see *Remark 5.5* and the next subsection.

1.3. **Pitt's theorem.** In 1937, Pitt [Pit37] proved the following theorem for the case of Fourier series. *Let* $1 < p \leq q < \infty$, *choose* $0 < b < 1/p'$, *set* $\beta = 1 - \frac{1}{p} - \frac{1}{q} - b < 0$, *and define* $v(x) = |x|^{bp}$ *for all* $x \in \mathbb{R}$. *There is* $C > 0$ *such that*

$$\left(\int_{\widehat{\mathbb{R}}} |\widehat{f}(\gamma)|^q |\gamma|^{\beta q} d\gamma \right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f(x)|^p |x|^{bp} dx \right)^{1/p} \tag{1.5}$$

for all $f \in L^p_v(\mathbb{R})$. In particular, $\widehat{f}$ is well-defined in this case.

In *Example 5.4* we shall obtain Pitt's theorem on $\mathbb{R}^n$ as a consequence of our general theorems. Even with all of the Fourier inequalities for weights more general than polynomial weights, we have chosen to highlight Pitt's theorem since it has been a catalyst for developing some critical results in 20th century classical harmonic analysis. We close this paper in *Remark 5.5* by tracing some of these results as an attractive and unified body of ideas.

With regard to our comment about weighted uncertainty principle inequalities in Subsection 1.1, we point out that Beckner [Bec95] proved a sharp form of Pitt's theorem on $\mathbb{R}^n$ for the case $p = q = 2$, thereby allowing him to obtain a logarithmic estimate of uncertainty.

1.4. **Mathematical prerequisites. 1.4.1.** The unit sphere S^{n-1} in $\mathbb{R}^n$ is the boundary of the open unit ball $B_n(0,1) \subseteq \mathbb{R}^n$, centered at $0 \in \mathbb{R}^n$ and with radius 1. The volume of $B_n(0,1)$ is denoted by

$$|B_n(0,1)| = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)},$$

and the surface area of S^{n-1} is $\omega_{n-1} \equiv n|B_n(0,1)|$. Recall that if $f \in L^1(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f(x) dx = \int_{S^{n-1}} \int_0^\infty \rho^{n-1} f(\rho\theta) d\rho d\sigma_{n-1}(\theta),$$

where $x = \rho\theta \in \mathbb{R}^n \backslash \{0\}$, $\rho > 0$, $\theta \in S^{n-1}$, and σ_{n-1} is surface measure on S^{n-1}.

1.4.2. Let (X, μ) be a measure space, where $X \subseteq \mathbb{R}^n$; and let f be a complex-valued μ-measurable function on X. The *distribution function* $D_f : [0,\infty) \longrightarrow [0,\infty)$ of f is defined by

$$D_f(s) = \mu\{x \in X : |f(x)| > s\}.$$

Two measurable functions f and g on measure spaces (X,μ) and (Y,ν), respectively, are *equimeasurable* if $D_f = D_g$ on $[0,\infty)$. The *decreasing rearrangement* of f defined on (X,μ) is the function $f^* : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$f^*(t) = \inf\{s \geq 0 : D_f(s) \leq t\}.$$

We use the convention $\inf \emptyset = \infty$, so that if $D_f(s) > t$ for all $s \in [0,\infty)$ then $f^*(t) = \infty$.

For a given μ-measurable f on (X,μ), f^* is a non-negative, decreasing, right continuous function on $[0,\infty)$; and f and f^* are equimeasurable, where f^* is considered as a Lebesgue measurable function on $[0,\infty)$. Furthermore, for any $0 < p < \infty$,

$$\int_X |f(x)|^p d\mu(x) = p\int_0^\infty s^{p-1} D_f(s) ds = \int_0^\infty f^*(t)^p dt.$$

These ideas have had an impact on harmonic analysis for most of the 20th century, e.g., [HL31], [HLP52], [Zyg59], [SW71], [BS88].

1.5. **Notation.** We shall use the standard notation in harmonic analysis as found in [SW71], [GCRdF85], and [Ste93a].

On the other hand, we shall also use the following notational conventions. The conjugate index of a given $p > 1$, is $p' = p/(p-1)$. Our space variables are $x, y \in \mathbb{R}^n$, and our spectral variables are $\lambda, \gamma \in \widehat{\mathbb{R}}^n$. When dealing with the domain $(0,\infty)$ of rearrangements we shall use the variables $s, t \in (0,\infty)$. On the occasion when there are too many integrals or exponents in a formula we shall suppress using a variable, e.g., the conditions (i) and (ii) of *Theorem A* in *Section 2*. We shall write "$\equiv$" when we are defining a constant, e.g., see the same conditions of *Theorem A*. We shall adhere to the convention $0 \cdot \infty = 0$.

Inequalities such as (1.2) are interpreted in the sense that if the right side is finite then so is the left side and the inequality holds. Lest there be any doubt, $\hat{f}^*$ means $(\hat{f})^*$. Also $F^\vee$ denotes the inverse Fourier transform of the function F, χ_E is the characteristic function of the Lebesgue measurable set $E \subseteq \mathbb{R}^n$, and $|E|$ is its Lebesgue measure. All of the functions with which we deal are Lebesgue measurable on either $\mathbb{R}^n$ or $(0,\infty)$, and we usually omit this hypothesis. $L^1_{\text{loc}}(X)$, $X \subseteq \mathbb{R}^n$, is the space of complex-valued locally Lebesgue integrable functions on X. A *weight function* v on X is a non-negative element of $L^1_{\text{loc}}(X)$, where X is $\mathbb{R}^n$ or $(0,\infty)$. Finally,

$$L^1 + L^2 = \{f = f_1 + f_2 : f_1 \in L^1(\mathbb{R}^n) \text{ and } f_2 \in L^2(\mathbb{R}^n)\}.$$

2. Weighted Fourier inequalities - type $(1,\infty)$, $(2,2)$ method

In the Introduction we motivated the relevance of proving weighted Fourier transform norm inequalities by stating $\|\widehat{f}\|_\infty \leq \|f\|_1$ for $f \in L^1(\mathbb{R}^n)$ and $\|\widehat{f}\|_2 = \|f\|_2$ for $f \in L^2(\mathbb{R}^n)$. These results assert that the operator $\mathcal{F}$ defined in (1.1) is bounded from L^1 to L^∞ and from L^2 to L^2. Any bounded linear operator with these properties is said to be of type $(1,\infty)$ and $(2,2)$. The main result, *Theorem 2.1*, in this section is in terms of the Fourier transform operator $\mathcal{F}$. However, it is essentially valid for any bounded linear operator of type $(1,\infty)$ and $(2,2)$, cf., *Remark 5.5* c,d.

The proof of *Theorem 2.1* requires a few well-known facts which we shall now state. The first is a weight characterization of the Hardy operator on weighted Lebesgue spaces, e.g., see [Maz85] (Theorem 2 of Section 1.3) for a proof.

Theorem A *Let u and v be weight functions on $(0,\infty)$ and suppose $1 < p,q < \infty$. There is $C > 0$ such that for all non-negative Lebesgue measurable functions f on $(0,\infty)$ the weighted Hardy inequality*

$$\left(\int_0^\infty \left(\int_0^t f\right)^q u(t)dt\right)^{1/q} \leq C\left(\int_0^\infty f(t)^p v(t)dt\right)^{1/p} \tag{2.1}$$

is satisfied if and only if

(i) for $1 < p \leq q < \infty$,

$$\sup_{s>0}\left(\int_s^\infty u(t)dt\right)^{1/q}\left(\int_0^s v(t)^{1-p'}dt\right)^{1/p'} \equiv A_1 < \infty,$$

and

(ii) for $1 < q < p < \infty$,

$$\left(\int_0^\infty\left(\int_s^\infty u\right)^{r/q}\left(\int_0^s v^{1-p'}\right)^{r/q'} v(s)^{1-p'}ds\right)^{1/r} \equiv A_2 < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Moreover, if C is the best constant in (2.1), then in case (i) we have

$$A_1 \leq C \leq A_1 (q')^{1/p'} q^{1/q},$$

and in case (ii) we have

$$\left(\frac{p-q}{p-1}\right)^{1/q'} q^{1/q} A_2 \leq C \leq (p')^{1/q'} q^{1/q} A_2.$$

Maz'ja's treatment of *Theorem A* in [Maz85] is a little more general than what we have stated and also includes the index values 1 and ∞. For example, in *Theorem A(i)* in the cases $p=1$ or $p=\infty$ we have $C = A_1$. In the case $p=q$, the characterization of (2.1) in *Theorem A* is due to Artola (unpublished), Talenti (1969), and Tomaselli (1969). Muckenhoupt (1972) gave an elegant proof in terms of Schur's lemma. The case $1 < p \leq q < \infty$ was first published by J. S. Bradley (1978), and independently by Kokilašvili (1979) and Andersen and Muckenhoupt (1982). The case $1 < q < p < \infty$ was first published by Maz'ja and Rosin (1980), and independently by Sawyer (1984). The case $0 < q < p$, $p \geq 1$, which is not considered in *Theorem A*, is due to Sinnamon (1987). References for these attributions are found in

[Maz85], [Hei90], and [BH92]; and it should be pointed out that *Theorem A* is also part of an unpublished folklore (for which by definition we can not provide bibliographic references!).

Note that the weights u and v in *Theorem A* do not necessarily have to be locally integrable, but only Lebesgue measurable. Of course, the weight conditions for both cases (i) and (ii) require u to be an element of $L^1(s,\infty)$ for all $s>0$.

Our next ingredient for proving *Theorem 2.1* is a rearrangement estimate for operators of type $(1,\infty)$ and $(2,2)$. It is due to Jodeit and Torchinsky [JT71] (Theorem 4.7).

Theorem B *Let $q\geq 2$. There is $K_q>0$ such that, for all $f\in L^1+L^2$ and for all $s\geq 0$, the inequality*

$$\int_0^s \widehat{f}^*(t)^q dt \leq K_q^q \int_0^s \left(\int_0^{1/t} f^*\right)^q dt \tag{2.2}$$

holds.

Although probably not best possible, in the case $q=2$, the constant K_q can be taken to be $K_q=2$, see the proof of *Theorem 4.6* in [JT71].

Finally, the following two results are needed in our proof of *Theorem 2.1*. They go back to Hardy and Hardy-Littlewood, respectively.

Hardy Lemma *Let ψ and χ be non-negative Lebesgue measurable functions on $(0,\infty)$, and assume*

$$\int_0^s \psi(t)dt \leq \int_0^s \chi(t)dt$$

for all $s>0$. If φ is non-negative and non-decreasing on $(0,\infty)$, then

$$\int_0^\infty \varphi(t)\psi(t)dt \leq \int_0^\infty \varphi(t)\chi(t)dt.$$

For a proof of Hardy's lemma see [BS88](Proposition 3.6 of Chapter 2), cf., [Lux67].

Hardy-Littlewood Rearrangement Inequality *Let f and w be non-negative Lebesgue measurable functions on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} f(x)w(x)dx \leq \int_0^\infty f^*(t)w^*(t)dt \tag{2.3}$$

and

$$\int_0^\infty f^*(t)\frac{1}{(1/w)^*(t)}dt \leq \int_{\mathbb{R}^n} f(x)w(x)dx. \tag{2.4}$$

For a proof and more general formulation of (2.3) see [BS88] (Theorem 2.2 of Chapter 2). The discrete version of the reverse inequality (2.4) is Theorem

368 in Hardy, Littlewood, and Pólya's book [HLP52]. The inequality (2.4) is derived and applied in [Hei84] (Corollary 2.5).

In the statement of *Theorem 2.1* we shall use the constant K from *Theorem B*. In the case $q \geq 2$ considered in *Theorem 2.1* this constant K equals K_q of *Theorem B*. However, in the case $1 < q < 2$, this K is $K_{p'}$. Also, since the method of proof of *Theorem 2.1* also applies to general operators of type $(1, \infty)$ and $(2,2)$, and does not depend on specific properties of the Fourier operator $\mathcal{F}$, we can not expect a sharp converse.

Theorem 2.1. *Let u and v be weight functions on $\mathbb{R}^n$, suppose $1 < p, q < \infty$, and let K be the constant from* Theorem B *associated with the relevant index ≥ 2.*

There is a constant $C > 0$ such that, for all $f \in L^p_v(\mathbb{R}^n)$, the inequality

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma) d\gamma\right)^{1/q} \leq KC \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx\right)^{1/p} \tag{2.5}$$

holds in the following ranges and with the following hypotheses on u and v:

(i) $1 < p \leq q < \infty$ and

$$\sup_{s>0} \left(\int_0^{1/s} u^*(t)dt\right)^{1/q} \left(\int_0^s (1/v)^*(t)^{p'-1} dt\right)^{1/p'} \equiv B_1 < \infty;$$

(ii) for $1 < q < p < \infty$,

$$\left(\int_0^\infty \left(\int_0^{1/s} u^*\right)^{r/q} \left(\int_0^s (1/v)^{*(p'-1)}\right)^{r/q'} (1/v)^*(s)^{p'-1} ds\right)^{1/r} \equiv B_2 < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Moreover, the best constant C in (2.5) satisfies

$$C \leq B_1 \begin{cases} (q')^{1/p'} q^{1/q} & \text{if } \ 1 < p \leq q, q \geq 2, \\ \\ p^{1/q}(p')^{1/p'} & \text{if } \ 1 < p \leq q < 2, \end{cases}$$

and $C \leq B_2 q^{1/q}(p')^{1/p'}$ if $1 < q < p < \infty$.

Proof. a. We shall only prove the theorem for simple functions f. Standard limiting arguments yield the result for $f \in L^p_v(\mathbb{R}^n)$, e.g., see [BHJ87a], [BH92].

Also, we shall first prove the result in part b for $q \geq 2$ in order to apply *Theorem B* directly, and then apply duality arguments in parts c and d to prove the result for the cases $1 < p \leq q < 2$ and $1 < q < p \leq 2$, respectively.

b. The inequality (2.2) from *Theorem B* with $q \geq 2$ and Hardy's lemma with $\psi = \widehat{f}^*, \chi(t) = K_q^q \left(\int_0^{1/t} f^*\right)^q$, and $\varphi = u^*$ allow us to make the estimate

$$\begin{aligned}\left(\int_0^\infty \widehat{f}^*(t)^q u^*(t)dt\right)^{1/q} &\le K_q\left(\int_0^\infty \left(\int_0^{1/t} f^*\right)^q u^*(t)dt\right)^{1/q}\\ &= K_q\left(\int_0^\infty \left(\int_0^s f^*\right)^q \frac{u^*(1/s)}{s^2}ds\right)^{1/q},\end{aligned} \tag{2.6}$$

where the equality follows from the change of variable $t = 1/s$.

Now, by (2.1) of *Theorem A* with u replaced by $u^*(1/s)/s^2$, v by $1/(1/v)^*$, and f by f^*, the right side of (2.6) is less than or equal to

$$K_q C\left(\int_0^\infty f^*(t)^p 1/(1/v)^*(t)dt\right)^{1/p} \tag{2.7}$$

since, with the aid of a change of variables, A_1, respectively, A_2, of *Theorem A* equals B_1, respectively, B_2, for the above replacements of u and v in *Theorem A* in the range $p \le q$, respectively, $q < p$. (Note that although we required $q \ge 2$ to invoke *Theorem B*, in order to obtain (2.7) as a bound of the right side of (2.6) we only required $1 < p \le q < \infty$, respectively, $1 < q < p < \infty$.)

Since $(|f|^p)^* = (f^*)^p$, e.g., [BS88] (page 41), the reverse Hardy-Littlewood inequality (2.4) allows us to bound (2.7) by

$$K_q C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}. \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) we obtain

$$\left(\int_0^\infty \widehat{f}^*(t)^q u^*(t)dt\right)^{1/q} \le K_q C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}. \tag{2.9}$$

An application of the Hardy-Littlewood inequality (2.3), in the case f and w of (2.3) are replaced by $|\widehat{f}|^q$ and u, yields the lower bound

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma)d\gamma\right)^{1/q}$$

of the left side of (2.9). Thus (2.5) is proved for the range $q \ge 2$ and all $p \in (1,\infty)$. Moreover, because $A_1 = B_1$ and $A_2 = B_2$, the assertions in the statement of the theorem about best constant properties of C follow from *Theorem A*.

c. We now consider the case $1 < p \le q \le 2$. By definition of the $L^q_u(\widehat{\mathbb{R}}^n)$ norm and the Hahn-Banach theorem we have

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma)d\gamma\right)^{1/q} = \sup_{\|G\|_{(L^q_u)'}=1}\left|\int_{\widehat{\mathbb{R}}^n} \widehat{f}(\gamma)\overline{G(\gamma)}d\gamma\right|, \tag{2.10}$$

where the sup can be taken over a dense subspace of $\{G : \|G\|_{(L^q_u)'} = 1\}$. The dual space $L^q_u(\widehat{\mathbb{R}}^n)'$ of $L^q_u(\widehat{\mathbb{R}}^n)$ can be identified with $L^{q'}_{u^{-q'/q}}(\widehat{\mathbb{R}}^n)$, e.g., the inclusion $L^{q'}_{u^{-q'/q}}(\widehat{\mathbb{R}}^n) \subseteq L^q_u(\widehat{\mathbb{R}}^n)'$ is a consequence of Hölder's inequality,

$$\left| \int_{\widehat{\mathbb{R}}^n} \widehat{f}(\gamma)u(\gamma)^{1/q}\overline{G(\gamma)}u(\gamma)^{-1/q}\,d\gamma \right| \le \|\widehat{f}\|_{L^q_u}\|G\|_{L^{q'}_{u^{-q'/q}}}, \tag{2.11}$$

and the opposite inclusion is a consequence of the Riesz representation theorem for $L^q(\widehat{\mathbb{R}}^n)$ and the fact that $F \in L^q_u(\widehat{\mathbb{R}}^n)$ if and only if $Fu^{1/q} \in L^q(\widehat{\mathbb{R}}^n)$. We can now invoke the Parseval relation over an appropriate space of test functions G as above, e.g., [BH83], and so the right side of (2.10) is

$$\begin{aligned} &\sup\left|\int_{\mathbb{R}^n} f(x)\overline{G^\vee(x)}\,dx\right| \\ &\le \|f\|_{L^p_v}\sup\left(\int_{\mathbb{R}^n}|G^\vee(x)|^{p'}v(x)^{1-p'}\,dx\right)^{1/p'}, \end{aligned} \tag{2.12}$$

where the inequality (2.12) follows from Hölder's inequality as in (2.11) and where $1-p' = -p'/p$.

Since $p' \ge 2$ and $q' \le p'$ we can use part b in the following way. For clarity, let $Q = p'$, $P = q'$, $U = v^{1-p'}$, and $V = u^{1-q'}$. Then $P \le Q$, and so we shall show that $B_1(P,Q,U,V)$, defined in (i) but in terms of the capitalized indices and weights, is finite. Thus, we shall be able to conclude from part b that

$$\left(\int_{\widehat{\mathbb{R}}^n}|\widehat{f}(\gamma)|^Q U(\gamma)d\gamma\right)^{1/Q} \le K_Q C\left(\int_{\mathbb{R}^n}|f(x)|^P V(x)dx\right)^{1/P}. \tag{2.13}$$

Before proving $B_1(P,Q,U,V) < \infty$, we use (2.13) to bound the right side of (2.12). Hence,

$$\begin{aligned} \left(\int_{\widehat{\mathbb{R}}^n}|\widehat{f}(\gamma)|^q u(\gamma)d\gamma\right)^{1/q} &\le \|f\|_{L^p_v}\sup\left(\int_{\mathbb{R}^n}|G^\vee(x)|^Q U(x)dx\right)^{1/Q} \\ &\le \|f\|_{L^p_v}K_Q C\sup\left(\int_{\widehat{\mathbb{R}}^n}|G(-\gamma)|^P V(\gamma)d\gamma\right)^{1/P} \\ &= K_{p'}C\|f\|_{L^p_v}\sup\left(\int_{\widehat{\mathbb{R}}^n}|G(-\gamma)|^{q'}u(\gamma)^{1-q'}d\gamma\right)^{1/q'} = K_{p'}C\|f\|_{L^p_v}, \end{aligned} \tag{2.14}$$

recalling from part b for this setting of $P \le Q$ that

$$C \le B_1(P,Q,U,V)(Q')^{1/P'}Q^{1/Q} = B_1(P,Q,U,V)p^{1/q}(p')^{1/p'} \tag{2.15}$$

and that the sup in (2.14) is taken over an appropriate dense subspace of $\{G : \|G\|_{(L^q_u)'} = 1\}$.

Therefore, (2.14) yields the desired inequality (2.5) for the case $1 < p \leq q \leq 2$ once we prove that $B_1(P, Q, U, V) < \infty$. To this end we compute

$$\begin{aligned}
&\left(\int_0^{1/s} U^*\right)^{1/Q} \left(\int_0^s \left(\frac{1}{V}\right)^{*(P'-1)}\right)^{1/P'} \\
&= \left(\int_0^{1/s} \left(v^{1-p'}\right)^*\right)^{1/p'} \left(\int_0^s \left(\frac{1}{u^{1-q'}}\right)^{*(q-1)}\right)^{1/q} \qquad (2.16)\\
&= \left(\int_0^s u^*\right)^{1/q} \left(\int_0^{1/s} \left(\frac{1}{v}\right)^{*(p'-1)}\right)^{1/p'} \leq B_1,
\end{aligned}$$

since$(q'-1)(q-1) = 1$ and $(|w|^p)^* = (w^*)^p$. The right side of (2.16) is finite by the hypothesis (i). In particular, for this case of $1 < p \leq q \leq 2$, the constant KC in (2.5) is $K_{p'}C$ where $C \leq B_1\, p^{1/q}(p')^{1/p'}$ because of (2.15) and (2.16).

d. Finally, we consider the case $1 < q < p \leq 2$. As in (2.10)-(2.12) of part c we have

$$\begin{aligned}
&\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma) d\gamma\right)^{1/q} \\
&\leq \|f\|_{L^p_v} \sup \left(\int_{\mathbb{R}^n} |G^\vee(x)|^{p'} v(x)^{1-p'}\, dx\right)^{1/p'}, \qquad (2.17)
\end{aligned}$$

where the sup is taken over an appropriate dense subspace of $\{G : \|G\|_{(L^q_u)'} = 1\}$.

Since $p' \geq 2$ and $p' < q'$ we can use part b in following way. For clarity, let $Q = p'$, $P = q'$, $U = v^{1-p'}$, and $V = u^{1-q'}$. Then $Q < P$, and so we shall show that $B_2(P, Q, U, V)$, defined in (ii) but in terms of the capitalized indices and weights, in finite. Thus, we shall be able to conclude from part b that (2.13) is valid. Before proving $B_2(P, Q, U, V) < \infty$, we use (2.13) to bound the right side of (2.17) as in (2.14). We obtain

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma) d\gamma\right)^{1/q} \leq K_{p'} C \|f\|_{L^p_v}, \qquad (2.18)$$

recalling from part b for this setting of $Q < P$ that

$$C \leq B_2(P, Q, U, V) Q^{1/Q}(P')^{1/Q'} = B_2(P, Q, U, V) q^{1/p}(p')^{1/p}.$$

Therefore, (2.18) yields the desired inequality (2.5) for the case $1 < q < p \leq 2$ once we prove that $B_2(P, Q, U, V) < \infty$. To this end, noting that with

$\frac{1}{R} = \frac{1}{Q} - \frac{1}{P}$ we have $R = r$, we compute

$$B_2(P,Q,U,V)^r =$$

$$\int_0^\infty \left(\int_0^{1/s} U^*\right)^{r/Q} \left(\int_0^s \left(\left(\frac{1}{V}\right)^*\right)^{(P'-1)}\right)^{r/Q'} \left(\frac{1}{V}\right)^* (s)^{(P'-1)}\, ds \tag{2.19}$$

$$= \int_0^\infty \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{r/p'} \left(\int_0^s u^*\right)^{r/p} u^*(s)\, ds.$$

Integrating by parts, and using the convention $0 \cdot \infty = 0$ for the boundary term, the right side of (2.19) is

$$-\int_0^\infty \left(\int_0^s u^*\right) \left[\left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{r/p'} \left(\frac{r}{p}\right) \left(\int_0^s u^*\right)^{\frac{r}{p}-1} u^*(s)\right.$$

$$+ \left(\int_0^s u^*\right)^{r/p} \left(\frac{r}{p'}\right) \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{\frac{r}{p'}-1}$$

$$\left. \times \frac{d}{ds} \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)\right] ds$$

$$= -\frac{r}{p} \int_0^\infty \left(\int_0^s u^*\right)^{r/p} \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{r/p'} u^*(s)\, ds$$

$$-\frac{r}{p'} \int_0^\infty \left(\int_0^s u^*\right)^{\frac{r}{p}+1} \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{\frac{r}{p'}-1}$$

$$\times \frac{d}{ds} \left(\int_0^{1/s} \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right) ds.$$

Therefore,

$$\left(1 + \frac{r}{p}\right) B_2(P,Q,U,V)^r =$$

$$\frac{r}{p'} \int_0^\infty \left(\int_0^{1/s} u^*\right)^{r/q} \left(\int_0^s \left(\left(\frac{1}{v}\right)^*\right)^{p'-1}\right)^{r/q'} \left(\frac{1}{v}\right)^* (s)^{p'-1}\, ds \tag{2.20}$$

$$= \frac{r}{p'} B_2^r.$$

Consequently, not only is $B_2(P,Q,U,V)<\infty$ since $B_2<\infty$ by the hypothesis (ii), but, because of (2.20), we compute that

$$\begin{aligned} &B_2(P,Q,U,V)q^{1/p}(p')^{1/p} \\ &= B_2\left(\frac{r/p'}{1+\frac{r}{p}}\right)^{1/r} q^{1/p}(p')^{1/p} \\ &= B_2 q^{1/q}(p')^{1/p'}. \end{aligned}$$

Hence, for this case of $1<q<p\le 2$, the constant KC in (2.5) is $K_{p'}C$ where $C\le B_2q^{1/q}(p')^{1/p'}$. □

3. Weighted Lorentz spaces and Hardy's inequality

In this section we define weighted Lorentz spaces $\Lambda_p(v)$ and the space B_p of weights, and we also state a characterization of $\Lambda_p(v)$ (Theorem C) and a weighted Hardy inequality for non-increasing functions $f\ge 0$ (Theorem D). Theorems C and D are due to Sawyer [Saw90] and depend on his duality principle [Saw90] (Theorem 1), which in turn is an improvement on I. Halperin's expression for the dual norm of $\Lambda_p(v)$ proved in [Hal53], cf., [Lor51] (Theorem 1) for the case $p=1$ and [Lor53] (Theorem 3.6.5). We shall use Theorems C and D in Section 4 in the weight characterization of the boundedness of the Fourier transform operator $\mathcal{F}:\Lambda_p(v)\to\Lambda_q(u)$.

Definition 3.1. Let v be a weight function on $(0,\infty)$ and let $1\le p<\infty$.

a. The *weighted Lorentz space* $\Lambda_p(v)$ is the set of Lebesgue measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ with the property that

$$\rho(f)=\left(\int_0^\infty f^*(t)^p v(t)\,dt\right)^{1/p}<\infty, \tag{3.1}$$

see *Remark 3.2a*.

b. We say that $v\in B_p$ if there is a constant $b_p>0$ such that for all $s>0$,

$$\int_s^\infty \frac{v(t)}{t^p}\,dt\le b_p\frac{1}{s^p}\int_0^s v(t)\,dt, \tag{3.2}$$

see *Remark 3.2b*. It is not difficult to see that if v is non-increasing then $v\in B_p$.

Remark 3.2. Lorentz spaces as Banach spaces

a. G. G. Lorentz defined $\Lambda_p(v)$ in [Lor51] and proved that $\Lambda_p(v)$ *is a normed linear space with* $\|f\|_{\Lambda_p(v)}\equiv\rho(f)$ *if and only if* v *is non-increasing on* $(0,\infty)$. It should be pointed out that Lorentz required $v\in L^1_{\ell oc}(0,\infty)$ for his theory. Further, using a method he developed in [Lor50] for the case $v(t)=\alpha t^{\alpha-1}$, $0<\alpha<1$, Lorentz [Lor51] proved that if v *is non-increasing and* $\int_0^\infty v(t)\,dt=\infty$, *then* $\Lambda_p(v)$ *is a Banach space with norm* $\|f\|_{\Lambda_p(v)}\equiv\rho(f)$, cf., *Remark 3.6* on Köthe spaces.

b. To formulate a Banach space associated with $\Lambda_p(v)$, in the case that v is not necessarily non-increasing, consider the following condition: there is a norm $\|\dots\|$ on $\Lambda_p(v)$ and there are constants $0 < C_1 < C_2 < \infty$ such that for all $f \in \Lambda_p(v)$,

$$C_1\|f\| \le \rho(f) \le C_2\|f\|; \tag{3.3}$$

in particular $\Lambda_p(v)$ is a linear space. Assuming (3.3), and using the classical Riesz-Fischer criterion, the normed linear space $(\Lambda_p(v), \|\dots\|)$ can be shown to be a Banach space by proving that every absolutely summable series is summable.

This is accomplished in the following way. Let $\{f_j\} \subseteq \Lambda_p(v)$ satisfy $\sum_{j=1}^\infty \|f_j\| < \infty$, i.e., an absolutely summable series, and define $g_N = \sum_{j=1}^N |f_j|$ and $g = \sum_{j=1}^\infty |f_j|$. Then $g_N \uparrow g$ so that $g_N^* \uparrow g^*$, and hence $\rho(g_N) \uparrow \rho(g)$ as $N \to \infty$. Further, $\||f_j|\| \le (C_2/C_1)\|f_j\|$ and so

$$\left(\int_0^\infty f_N^*(s)^p v(s)\,ds\right)^{1/p} \le \frac{C_2^2}{C_1}\sum_{j=1}^\infty \|f_j\| < \infty.$$

Combining these facts we have $g \in \Lambda_p(v)$ for otherwise we would obtain a contradiction. Since $g \in \Lambda_p(v)$, a straightforward calculation allows us to verify that $f = \sum_{j=1}^\infty f_j \in \Lambda_p(v)$ (since $|f|^* \le |g|^*$) and that $\|f - \sum_{j=1}^N f_j\| \to 0$ as $N \to \infty$.

c. If, besides assuming $v \in L^1_{\text{loc}}(0,\infty)$ and (3.3), we also assume that $v^{1-p'} \in L^1_{\text{loc}}(0,\infty)$, then we can show that $(\Lambda_p(v), \rho)$ is a Banach function space in the sense of Luxemberg (1955), see Chapter 1 of [BS88], except that the triangle inequality is replaced by $\rho(f+g) \le (C_2/C_1)(\rho(f) + \rho(g))$ for non-negative Lebesgue measurable functions f and g on $\mathbb{R}^n$.

Remark 3.3. A_p **and** B_p **weights**

Let v be a weight function on $\mathbb{R}$ and let $1 < p < \infty$. By definition, $v \in A_p$ if there is a constant $a_p > 0$ such that for each interval $I \subseteq \mathbb{R}$,

$$\left(\frac{1}{|I|^p}\int_I v(t)\,dt\right)^{1/p}\left(\int_I v(t)^{1-p'}\,dt\right)^{1/p'} \le a_p,$$

see [GCRdF85] for the fundamental role of A_p weights vis a vis the maximal function and the Hilbert transform.

Because we are proving weighted Fourier inequalities, it should be pointed out that if v is an even weight function on $\mathbb{R}$ which is non-decreasing on $(0,\infty)$ and if $1 < p \le 2$, then $v \in A_p$ *if and only if there is* $C > 0$ *such that for all* $f \in L^p_v(\mathbb{R})$,

$$\int_{\widehat{\mathbb{R}}} |\widehat{f}(\gamma)||\gamma|^{p-2} v(1/\gamma)\,d\gamma \le C\int_{\mathbb{R}} |f(x)|^p v(x)\,dx, \tag{3.4}$$

see [BHJ87b] for this result and some extensions as well as [HS89] for further generalizations. Besides the perspective afforded by (3.4) we have also defined A_p since Hunt, Muckenhoupt, and Wheeden [HMW73] (Lemma 1) proved that *if* $v \in A_p$ *then* $v \in B_p$. Their result is essential in the proof of (3.4). (Technically, we have only defined A_p on $\mathbb{R}$ and B_p on $(0,\infty)$;

but the extension to $\mathbb{R}^n$ is clear for both concepts, and the result of Hunt, Muckenhoupt, and Wheeden is true for $\mathbb{R}^n$, e.g., [HS89] (Lemma 2.4).)

Example 3.4. A natural generalization of Lorentz' example $v(t) = \alpha t^{\alpha-1}$, $0 < \alpha < 1$, mentioned in *Remark 3.2a*, is to consider the weight function $v(t) = \frac{q}{p} t^{\frac{q}{p}-1}$ on $(0, \infty)$ for any fixed $0 < p, q < \infty$. In this case, $\Lambda_q(v)$ is usually denoted by $L(p, q)$, e.g., [BS88], [Fle73], [SW71]. Clearly, $L(p, p) = L^p(\mathbb{R}^n)$. $L(p, q)$ can be normed in terms of a non-symmetric maximal function so that it becomes a Banach space, e.g., Theorem 3.22 of Chapter V in [SW71].

The weight space B_p is the notion which relates Theorem C, concerning $\Lambda_p(v)$, and Theorem D, dealing with a general weighted Hardy inequality. In fact, Ariño and Muckenhoupt [AM90] established a relationship between B_p and a weighted Hardy inequality by proving that *if* $1 \le p < \infty$ *and* v *is a weight function on* $(0, \infty)$*, then* $v \in B_p$ *if and only if there is a constant* $C > 0$ *such that for all non-increasing, non-negative functions* f *on* $(0, \infty)$ *we have*

$$(3.5) \qquad \left(\int_0^\infty \left(\frac{1}{s} \int_0^s f \right)^p v(s)\, ds \right)^{1/p} \le C \left(\int_0^\infty f(s)^p v(s)\, ds \right)^{1/p},$$

see Theorem D. The relationship between B_p and $\Lambda_p(v)$ is established in the following theorem due to Sawyer [Saw90](Theorem 4).

Theorem C. *Let* v *be a weight function on* $(0, \infty)$ *and let* $1 < p < \infty$*. The following are equivalent:*

(i) $(\Lambda_p(v), \|\ldots\|)$ *is Banach space where* $\|\ldots\|$ *is a norm on* $\Lambda_p(v)$ *satisfying (3.3);*

(ii) $v \in B_p$ *with constant* b_p*;*

(iii) There is a constant C *such that for all* $s > 0$*,*

$$(3.6) \qquad \left(\int_0^s v(t)\, dt \right)^{1/p} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{1-p'} dt \right)^{1/p'} \le C\, s.$$

In *Section 4*, when we assume $v \in B_p$ to use (3.6) we shall designate the C in (3.6) as $C(b_p)$. Sawyer's original treatment of Theorem C proves (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The first implication depends on his duality principle [Saw90] (Theorem 1). The second implication uses the argument in Lemma 2.1 of [AM90], which itself follows from a result (Lemma 21) of Stromberg and Torchinsky [ST89]. The third implication uses (3.5), and the fourth is elementary. A simpler proof of Sawyer's duality principle is due to Stepanov [Ste93b], cf., [GHS96]; and Heinig and Kufner [HK96] adopted Stepanov's method of proof to obtain a more general duality theorem in weighted Orlicz spaces.

Theorem D (Theorem 2 in [Saw90]) is a weight characterization for the Hardy operator, defined on non-increasing functions, on weighted L^p-spaces.

Ariño and Muckenhoupt [AM90] proved a single weight, single index version of Theorem D in terms of B_p.

Theorem D. *Let v and w be weight functions on $(0,\infty)$ and suppose $1 < p,q < \infty$. There is $C_H > 0$ such that for all non-increasing, non-negative functions f on $(0,\infty)$ the weighted Hardy inequality*

(3.7)

$$\left(\int_0^\infty \left(\frac{1}{s}\int_0^s f\right)^q w(s)\,ds\right)^{1/q} \le C_H \left(\int_0^\infty f(s)^p v(s)\,ds\right)^{1/p}, \ 1 < p,q < \infty,$$

is satisfied if and only if

(i) for $1 < p \le q < \infty$,

$$\sup_{s>0} \left(\int_0^s w(t)\,dt\right)^{1/q} \left(\int_0^s v(t)\,dt\right)^{-1/p} \equiv C_1 < \infty$$

and

$$\sup_{s>0} \left(\int_s^\infty \frac{w(t)}{t^q}\,dt\right)^{1/q} \left(\int_0^s \left(\frac{1}{t}\int_0^t v\right)^{-p'} v(t)\,dt\right)^{1/p'} \equiv C_2 < \infty$$

and

(ii) for $1 < q < p < \infty$,

$$\left(\int_0^\infty \left[\left(\int_0^s w(t)\,dt\right)^{1/p} \left(\int_0^s v(t)\,dt\right)^{-1/p}\right]^r w(s)\,ds\right)^{1/r} \equiv D_1 < \infty$$

and

$$\left(\int_0^\infty \left[\left(\int_s^\infty \frac{w(t)}{t^q}\,dt\right)^{1/q} \left(\int_0^s \left(\frac{1}{t}\int_0^t v\right)^{-p'} v(t)\,dt\right)^{1/q'}\right]^r \right.$$
$$\left. \times \left(\frac{1}{s}\int_0^s v\right)^{-p'} v(s)\,ds\right)^{1/r} \equiv D_2 < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Moreover, if C_H is the best constant in (3.7), then $C_H \approx C_1 + C_2$, respectively, $D_1 + D_2$.

Remark 3.5. B_p and M_p weights

a. Due to a misprint, the exponents $1/q$ and $1/q'$ in the second integral of *Theorem D* part (ii) were replaced by $1/p$ and $1/p'$, respectively, in the original paper [Saw90] (Theorem 2).

b. Let $1 \le p \le q < \infty$. It is natural to compare *Theorem A* and *Theorem D* in light of the fact that the weighted Hardy inequalities (2.1) of *Theorem A* and (3.7) of *Theorem D* are the same when the weight w of *Theorem D* is $w(t) = u(t)t^q$, where u is from *Theorem A*. On the other hand, with this definition of w, A_1 of *Theorem A* is neither C_1 nor C_2 of *Theorem D* even though the "B_q factor",

$$\left(\int_s^\infty \frac{w(t)}{t^q}\,dt\right)^{1/q},$$

appears in both A_1 and C_2. Of course, the test functions f for *Theorem A* form a larger class than those for *Theorem D*.

c. Now let $1 < p = q < \infty$ and $w(t) = u(t)\,t^q = v(t)$ in *Theorem A*. In this case, we state the condition $A_1 < \infty$ in *Theorem A* by writing $w \in M_p$, i.e.,

$$\forall s > 0, \qquad \left(\int_s^\infty \frac{w(t)}{t^p}\,dt\right)^{1/p}\left(\int_0^s w(t)^{1-p'}\,dt\right)^{1/p'} \le A_1. \tag{3.8}$$

Then, in light of the Ariño and Muckenhoupt theorem (3.5) and the fact that the class of non-increasing, non-negative functions f on $(0,\infty)$ (for (3.5) and *Theorem D*) is a subset of the non-negative functions f on $(0,\infty)$ (for *Theorem A*), we can assert that $M_p \subseteq B_p$. It is not difficult to verify this inclusion directly, see part *d*; and also to show that the inclusion is proper, see part *e*.

Besides this observation that $M_p \subseteq B_p$, we also note that the conditions $C_1 < \infty$ and $C_2 < \infty$ in *Theorem D* can be interpreted as follows for $1 < p = q < \infty$ and $w(t) = u(t)\,t^q = v(t)$: $C_1 < \infty$ is automatically satisfied; and $C_2 < \infty$ is essentially a *mean version* of $A_1 < \infty$, i.e., the factor

$$\left(\int_0^s w(t)^{1-p'}\,dt\right)^{1/p'}$$

in (i) of *Theorem A* is replaced by the factor

$$\left(\int_0^s \left(\frac{1}{t}\int_0^t w\right)^{-p'} w(t)\,dt\right)^{1/p'}$$

in (i) of *Theorem D*. In light of the Ariño and Muckenhoupt theorem (3.5) we can therefore assert that $w \in B_p$ *if and only if*

$$\sup_{s>0}\left(\int_s^\infty \frac{w(t)}{t^p}\,dt\right)^{1/p}\left(\int_0^s \left(\frac{1}{t}\int_0^t w\right)^{-p'} w(t)\,dt\right)^{1/p'} \equiv C_2 < \infty.$$

d. For $1 < p < \infty$ and any weight w on $(0,\infty)$, we have for each $s \in (0,\infty)$ that

$$s = \int_0^s w(t)^{1/p} w(t)^{-1/p}\,dt \le \left(\int_0^s w(t)\,dt\right)^{1/p}\left(\int_0^s w(t)^{1-p'}\,dt\right)^{1/p'},$$

and hence

$$\left(\int_0^s w(t)\,dt\right)^{-1} \le s^{-p}\left(\int_0^s w(t)^{1-p'}\,dt\right)^{p/p'}.$$

Thus, if $w \in M_p$, then

$$\begin{aligned}
&\left(\int_s^\infty \frac{w(t)}{t^p}\,dt\right)\left(\int_0^s w(t)\,dt\right)^{-1} \\
&\leq s^{-p}\left(\int_s^\infty \frac{w(t)}{t^p}\,dt\right)\left(\int_0^s w(t)^{1-p'}\,dt\right)^{p/p'} \\
&\leq {A_1}^p s^{-p},
\end{aligned}$$

and so $w \in B_p$ with constant $b_p = {A_1}^p$.

e. The fact that the inclusion $M_p \subseteq B_p$ is proper is due to Ariño and Muckenhoupt [AM90] (page 728), but perhaps a few details are required to convince the reader. Let

$$w(t) = \begin{cases} 0, & \text{if } 1 < t < 2, \\ t^{-1/2}, & \text{if } 0 < t \leq 1 \text{ or } t \geq 2. \end{cases}$$

Clearly, $w \notin M_p$ by evaluating the product in (3.8) for $s > 1$. On the other hand,

$$\int_s^\infty \frac{w(t)}{t^p}\,dt \leq \frac{1}{p-\frac{1}{2}} s^{-p+\frac{1}{2}} \leq b_p \frac{1}{s^p} \int_0^s w(t)\,dt,$$

where the first inequality is immediate, and the second requires the calculation of $\int_0^s w(t)\,dt$ and separately considering the intervals $0 < s < 2$ and $s \geq 2$. Thus, $w \in B_p$.

Remark 3.6. Köthe spaces

a. Given $\Lambda_p(v)$ and ρ defined by (3.1). Further, let v be non-increasing on $(0,\infty)$ and assume $\int_0^\infty v(t)\,dt = \infty$. It is straightforward to see that

$$(3.9) \qquad \forall f \in \Lambda_p(v), \qquad \rho(f) = \sup_{v_e}\left(\int_0^\infty f^*(t)^p v_e(t)\,dt\right)^{1/p},$$

where the supremum is taken over all non-negative, Lebesgue measurable functions v_e on $(0,\infty)$ which are equimeasurable to v. First, the inequality "$\leq$" is obviously always true. The opposite inequality "$\geq$" follows from the Hardy-Littlewood Rearrangement Inequality (2.3) and the fact that ${v_e}^* = v$ in the case v is non-increasing.

b. Because of (3.9) and the role the right side plays in proving that $\Lambda_p(v)$ is a normed linear space in the case v is non-increasing, it is relevant to mention Köthe spaces. In fact, many of the topological and uniform structure properties of $\Lambda_p(v)$, completeness being an example of the latter, are special cases of results from the theory of Köthe spaces. This observation was made early-on by Lorentz [Lor51] (Section 4), [Lor53] (pages 66-67), where the work in the former citation is possibly independent of Köthe's theory. In light of the Fourier transform inequalities in weighted Lorentz spaces that we shall prove in *Section 4*, we shall now define Köthe spaces in the context of possibly generalizing *Theorems 4.1* and *4.2* beyond Lorentz spaces.

c. In 1934, Köthe and his teacher Toeplitz began the development of the duality theory of a class of topological vector spaces (TVSs), which Köthe called "vollkommene Räume" (*perfect spaces*) and which he developed deeply with his students after World War II, e.g., [Köt69]. There is a generalization of perfect spaces by Dieudonné [Die51] (received November 1950) and an even more general and quite different formulation by Cooper [Coo53] (received November 1951). Dieudonné refers to his generalization as *Köthe spaces*.

Let $\mathcal{D} \subseteq L^1_{loc}(\mathbb{R}^n)$. The *Köthe space* $\mathcal{K}$ *defined by* $\mathcal{D}$ is the linear space

$$\mathcal{K} = \{f \in L^1_{\ell oc}(\mathbb{R}^n) : \forall g \in \mathcal{D},\ fg \in L^1(\mathbb{R}^n)\}.$$

$\mathcal{K}$ does not uniquely define $\mathcal{D}$. Notationally, $\mathcal{K}^*$ is the Köthe space defined by $\mathcal{K}$. Clearly, $\mathcal{D} \subseteq \mathcal{K}^*$ and therefore $\mathcal{K}$ is also defined by $\mathcal{K}^*$. Further, $\mathcal{K}$ and $\mathcal{K}^*$ are in weak duality for the bilinear form $< f, g >= \int_{\mathbb{R}^n} f(x)g(x)\,dx$, and $\mathcal{K}^*$ is the *Köthe dual* of $\mathcal{K}$.

d. A set $\mathcal{N} \subseteq L^1_{\ell oc}(0,\infty)$ is *normal* if $f \in \mathcal{N},\ g \in L^1_{\ell oc}(0,\infty)$, and $|g| \leq |f|$ on $(0,\infty)$ imply $g \in \mathcal{N}$.

Now, for $\Lambda_p(v)$ with v non-increasing on $(0,\infty)$, note that if $f \in \Lambda_p(v)$ then

$$\sup_{h \in \mathcal{N}_e} \left| \int_0^\infty f^*(t)h(t)\,dt \right| \leq \rho(f),$$

where, for a given $v_e \geq 0$ equimeasurable to v, $\mathcal{N}_e$ is the normal set of functions $h = (v_e)^{1/p} g$, $\|g\|_{p'} \leq 1$. In this way, one can establish the relationship between Lorentz Banach spaces and Köthe spaces, see [Lor53] (pages 66-67) and [Die51] (page 101). The details depend on the structure of the bounded sets in Köthe spaces including the fact that the *normal envelope* of every weakly bounded set in a Köthe space is weakly bounded [Köt51] (Theorem 5).

e. Besides their intrinsic relationship with classical ideas such as Lorentz spaces and their original formulation in terms of sequence spaces, Köthe spaces were influential in the development of locally convex TVSs, see [Die81] (pages 217-218). Briefly, Köthe and Toeplitz defined the weak topology $\sigma(\mathcal{K}, \mathcal{K}^*)$ and the associated weakly bounded sets of $\mathcal{K}$. Then, since $\mathcal{K}$ and $\mathcal{K}^*$ are symmetrical, they considered the weak-$*$topology $\sigma(\mathcal{K}^*, \mathcal{K})$ and the corresponding bounded sets B of $\mathcal{K}^*$. These bounded sets give rise to a neighborhood basis $\{V_B\}$ of the origin in $\mathcal{K}$:

$$V_B = \{f \in \mathcal{K} : \forall g \in B,\ |< f, g >| \leq 1\},$$

the polar set of B. Köthe and Toeplitz then proved that the bounded sets in this *strong topology* on $\mathcal{K}$ are the same as for $\sigma(\mathcal{K}, \mathcal{K}^*)$. This theorem is a major example of Mackey's characterization of the range of topologies on a locally convex TVS for which all the bounded sets are the same.

The proof in *Theorem 2.1* of the weighted Fourier transform norm inequality (2.5) depends on using the hypotheses of the theorem to prove (2.6) and

(2.7). For appropriate u and v, (2.6) and (2.7) combine to yield

$$\left(\int_0^\infty (\mathcal{F}f)^*(t)^q u(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty f^*(t)^p v(t)\,dt\right)^{1/p},$$

which reflects the continuity of

$$\mathcal{F} : \Lambda_p(v) \to \Lambda_q(u), \tag{3.10}$$

where the convergence in the Lorentz spaces is defined by (3.1). *Section 4* is devoted to establishing computable relations between weights u and v in order to establish (3.10).

4. Fourier transform inequalities in weighted Lorentz spaces

We now prove Fourier mapping theorems in weighted Lorentz spaces. Because of *Theorem C* and Lorentz' theorem, stated in *Remark 3.2*, these weighted Lorentz spaces are in the Banach space setting. Sinnamon's work [Sin01], referenced earlier, provides different Fourier mapping theorems by foregoing any Banach space structure.

Theorem 4.1. *Let u and v be weight functions on $(0,\infty)$.*

i. Assume u is non-increasing and $v \in B_p$, where $1 < p \le q$ and $q \ge 2$. If

$$\sup_{s>0} s\left(\int_0^{1/s} u\right)^{1/q}\left(\int_0^s v\right)^{-1/p} \equiv C_3 < \infty, \tag{4.1}$$

then there is $C > 0$ such that for all $f \in L^1 + L^2$,

$$\left(\int_0^\infty \hat{f}^*(t)^q u(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty f^*(t)^p v(t)\,dt\right)^{1/p}. \tag{4.2}$$

ii. Conversely, if (4.2) is satisfied for any weight functions u and v on $(0,\infty)$ and for $1 < p, q < \infty$, then (4.1) holds. In fact, the conclusion holds if (4.2) is only assumed to hold over the class of radial characteristic functions $f(x) = \chi_{(0,r)}(|x|)$.

Proof. i. As in the proof of *Theorem 2.1*, the inequality (2.2) from *Theorem B* (Jodeit-Torchinsky) and Hardy's lemma allow us to assert

$$\left(\int_0^\infty \hat{f}^*(t)^q u(t)\,dt\right)^{1/q} \le K_q\left(\int_0^\infty u(t)\left(\int_0^{1/t} f^*\right)^q dt\right)^{1/q} \tag{4.3}$$

for $f \in L^1 + L^2$. In fact, since $q \ge 2$ we have

$$\int_0^s \hat{f}^*(t)^q\,dt \le K_q^q\left(\int_0^s\left(\int_0^{1/t} f^*\right)^q dt\right) \tag{4.4}$$

for all $s \ge 0$ by (2.2); and so, setting $\psi = \hat{f}^*$, $\varphi = u$, and $\chi(t) = K_q^q\left(\int_0^{1/t} f^*\right)^q$, we obtain (4.3) from (4.4), Hardy's lemma, and the hypothesis that u is non-decreasing on $(0,\infty)$. If we make the change of variable

$t = 1/s$ on the right side of (4.3), then (4.3) becomes

$$\left(\int_0^\infty \hat{f}^*(t)u(t)\,dt\right)^{1/q} \le K_q \left(\int_0^\infty \frac{u(1/s)}{s^{2-q}} \left(\frac{1}{s}\int_0^s f^*\right)^q ds\right)^{1/q}. \tag{4.5}$$

We shall now invoke *Theorem D* with the weight $w(s) = u(1/s)/s^{2-q}$ and constant C_H. Then the right side of (4.5) is bounded by

$$K_q C_H \left(\int_0^\infty f^*(s)^p v(s)\,ds\right)^{1/p},$$

thereby completing the proof of (4.2), if and only if C_1 and C_2 defined in (i) of *Theorem D* are finite. Thus, in order to complete the proof of (4.2) we must verify that

$$\sup_{s>0} \left(\int_0^s \frac{u(1/t)}{t^{2-q}}\,dt\right)^{1/q} \left(\int_0^s v(t)\,dt\right)^{-1/p} \equiv C_1 < \infty \tag{4.6}$$

and

$$\sup_{s>0} \left(\int_s^\infty \frac{u(1/t)}{t^2}\,dt\right)^{1/q} \left(\int_0^s \left(\frac{1}{t}\int_0^t v\right)^{-p'} v(t)\,dt\right)^{1/p'} \equiv C_2 < \infty. \tag{4.7}$$

To show that (4.6) is satisfied first note that $u \in B_q$ since u is non-increasing. Thus.

$$\int_0^s \frac{u(1/t)}{t^{2-q}}\,dt = \int_{1/s}^\infty \frac{u(t)}{t^q}\,dt \le b_q s^q \int_0^{1/s} u(t)\,dt,$$

see (3.6). Hence,

$$\begin{aligned} &\left(\int_0^s \frac{u(1/t)}{t^{2-q}}\,dt\right)^{1/q} \left(\int_0^s v(t)\,dt\right)^{-1/p} \\ &\le b_q^{1/q} s \left(\int_0^{1/s} u(t)\,dt\right)^{1/q} \left(\int_0^s v(t)\,dt\right)^{-1/p} \le b_q^{1/q} C_3; \end{aligned}$$

and so (4.6) is satisfied with $C_1 \le b_q^{1/q} C_3$.

To show that (4.7) is satisfied, observe that the first integral in the product of (4.7) is

$$\left(\int_0^{1/s} u(t)\,dt\right)^{1/q} \tag{4.8}$$

by means of a change of variable. In order to bound the second integral,

$$X \equiv \left(\int_0^s \left(\frac{1}{t}\int_0^t v\right)^{-p'} v(t)\,dt\right)^{1/p'}$$

in this product, we integrate by parts and obtain

$$X = \left\{ p' X^{p'} + s^{p'} \left(\int_0^s v(t)\, dt \right)^{1-p'} - p' \int_0^s \left(\frac{1}{t} \int_0^t v \right)^{1-p'} dt \right\}^{1/p'}.$$

Thus,

$$\begin{aligned}(p'-1)X^{p'} &= p' \int_0^s \left(\frac{1}{t} \int_0^t v \right)^{1-p'} dt - s^{p'} \left(\int_0^s v \right)^{1-p'} \\ &\leq p' \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{1-p'} dt \right),\end{aligned}$$

and so

$$X \leq \left\{ \frac{p'}{p'-1} \int_0^s \left(\frac{1}{t} \int_0^t v \right)^{1-p'} dt \right\}^{1/p'}. \tag{4.9}$$

Since $v \in B_p$, $p > 1$, we can invoke *Theorem C* and the equivalence in terms of (3.6) to bound the right side of (4.9) by

$$C p^{1/p'} s \left(\int_0^s v(t)\, dt \right)^{-1/p}.$$

Hence,

$$\left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{1/p'} \leq C p^{1/p'} s \left(\int_0^s v(t)\, dt \right)^{-1/p}.$$

Combining this estimate with (4.8), we see that (4.7) holds with $C_2 \leq p^{1/p'} C C_3$, where $C = C(b_p)$ is the constant from (3.6) obtained since $v \in B_p$.

ii. If (4.2) is satisfied for $1 < p, q < \infty$ we define

$$\kappa = \kappa(r) = \left(\frac{n}{|B_n(0,1)|} \right)^{1/n} r^{1/n}$$

for fixed $r > 0$. Let $f(x) = \chi_{(0,\kappa)}(|x|)$. The distribution function of f is

$$\begin{aligned}
D_f(s) &\equiv |\{x \in \mathbb{R}^n : \chi_{(0,\kappa)}(|x|) > s\}| \\
&= \begin{cases} 0, & \text{if } \ s \geq 1 \\ \\ \displaystyle\int_{S^{n-1}} \left(\int_0^\kappa \rho^{n-1}\, d\rho \right) d\sigma_{n-1}(\theta), & \text{if } \ 0 < s < 1 \end{cases} \\
&= \begin{cases} 0, & \text{if } \ s \geq 1 \\ \\ \dfrac{|B_n(0,1)|}{n} \kappa^n, & \text{if } \ 0 < s < 1 \end{cases} \\
&= r\ \chi_{(0,1)}(s).
\end{aligned}$$

Hence, $f^*(s) = \chi_{(0,r)}(s), r > 0$. Thus, for this f, the right side of (4.2) is

$$C \left(\int_0^r v(s)\, ds \right)^{1/p}. \tag{4.10}$$

Since $f(x) = \chi_{(0,\kappa)}(|x|)$ is radial, $\hat{f}$ is also radial, and in fact (cf., [Sch61], [Ste93a] (Chapter 9, §6.19)),

$$\hat{f}(|\gamma|) = 2\pi|\gamma|^{(2-n)/2} \int_0^\kappa t^{n/2} J_{(n-2)/2}(2\pi t|\gamma|)\, dt,$$

where the Bessel function is

$$J_{(n-2)/2}(2\pi s) = C_n s^{(n-2)/2} \int_0^{\pi/2} \cos(2\pi s \cos\varphi) \sin^{(n-2)} \varphi d\varphi, \tag{4.11}$$

$C_n = 2\pi^{(n-3)/2}/\Gamma\left(\frac{n-1}{2}\right)$ if $n > 1$, and if $n = 1$ then

$$J_{-1/2}(2\pi s) = \pi^{-1} s^{-1/2} \cos(2\pi s).$$

Note that the integration in (4.11) is usually written from 0 to π, but that in fact the integrals from 0 to $\pi/2$ and $\pi/2$ to π are equal.

We consider here only the case $n > 1$, since the argument for $n = 1$ is essentially the same, and in this latter case the Fourier transform is the sinc function. Let

$$\overline{\kappa} = \overline{\kappa}(r) = \frac{1}{2\pi\kappa} = \frac{1}{2\pi} \left(\frac{|B_n(0,1)|}{n} \right)^{1/n} r^{-1/n},$$

so that, for $t \in (0, \kappa)$ and $\tau \in (0, \overline{\kappa})$, we have

$$\cos(2\pi t\tau \cos\varphi) > \cos 1 > 1/2$$

if $0 < \varphi < \pi/2$. Hence, for such t and τ, (4.11) gives the estimate

$$\begin{aligned} J_{(n-2)/2}(2\pi t\tau) &> \frac{1}{2} C_n (t\tau)^{(n-2)/2} \int_0^{\pi/2} \sin^{(n-2)} \varphi d\varphi \\ &= \frac{1}{2} C_n (t\tau)^{(n-2)/2} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \\ &= \frac{\pi^{(n-2)/2}}{2\Gamma(\frac{n}{2})} (t\tau)^{(n-2)/2}. \end{aligned}$$

Consequently, if $|\gamma| = \tau \in (0, \overline{\kappa})$, then

$$\begin{aligned} \hat{f}(|\gamma|) &> 2\pi |\gamma|^{(2-n)/2} \left[\frac{\pi^{(n-2)/2}}{2\Gamma(\frac{n}{2})} \right] \int_0^{\kappa} t^{n/2} (t|\gamma|)^{(n-2)/2} \, dt \\ &= \tfrac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \textstyle\int_0^{\kappa} t^{n-1} \, dt = \frac{nr}{2}, \end{aligned} \tag{4.12}$$

since the surface area ω_{n-1} of S^{n-1} is $2\pi^{n/2}/\Gamma(\frac{n}{2})$.

Now, for any fixed $r > 0$, we have the estimate

(4.13)

$$\begin{aligned} \left(\int_0^{\infty} \hat{f}^*(t)^q u(t) \, dt \right)^{1/q} &\geq \left(\int_0^{1/r} \hat{f}^*(t)^q u(t) \, dt \right)^{1/q} \\ &= \left(q \int_0^{\infty} s^{q-1} \left[\int_{\{t \in (0,1/r) : \hat{f}^*(t) > s\}} u(t) \, dt \right] ds \right)^{1/q} \\ &= \left(q \int_0^{\infty} s^{q-1} \left[\int_0^{\min(D_{\hat{f}}(s), 1/r)} u(t) \, dt \right] ds \right)^{1/q}, \end{aligned}$$

where the last step follows since

$$\{t : \hat{f}^*(t) > s\} = \{t : t < D_{\hat{f}}(s)\}.$$

Clearly, if $s > 0$, then

$$\begin{aligned} D_{\hat{f}}(s) &= |\{\gamma \in \widehat{\mathbb{R}}^n : |\hat{f}(\gamma)| > s\}| \\ &= \int_{S^{n-1}} \left(\int_{\{\rho > 0 : |\hat{f}(\rho)| > s\}} \rho^{n-1} d\rho \right) d\sigma_{n-1}(\theta). \end{aligned}$$

Further, if $s < nr/2$, then

$$(0, \overline{\kappa}) \subseteq \{\rho > 0 : \hat{f}(\rho) > \frac{nr}{2}\} \subseteq \{\rho > 0 : \hat{f}(\rho) > s\},$$

where the first inclusion follows from (4.12). Thus, in this case,

$$\begin{aligned} D_{\hat{f}}(s) \quad & = \omega_{n-1} \textstyle\int_{\{\rho>0:|\hat{f}(\rho)|>s\}} \rho^{n-1} d\rho \geq \omega_{n-1} \displaystyle\int_0^{\overline{\kappa}} \rho^{n-1}\, d\rho \\ & = \frac{1}{r}\left(\frac{1}{n}\frac{1}{(2\pi)^n}\,|B_n(0,1)|^2\right) = \frac{1}{r}\left(\frac{1}{n}\frac{1}{2^n}\frac{1}{\Gamma(\frac{n+2}{2})^2}\right) \geq \frac{1}{r}, \end{aligned} \tag{4.14}$$

where the second inequality follows from the definition of Γ and the assumption that $n \geq 2$. (Actually, the first integral in (4.14) is infinite.) Hence, $\min(D_{\hat{f}}(s), 1/r) = 1/r$ if $s < nr/2$.

Therefore, combining (4.13) and (4.14), we have

$$\left(\int_0^\infty \hat{f}^*(t)^q u(t)\, dt\right)^{1/q} \geq \left(q\int_0^{nr/2} s^{q-1}\left[\int_0^{1/r} u(t)\, dt\right] ds\right)^{1/q}$$

$$= q^{1/q}\left(\frac{(nr/2)^q}{q}\right)^{1/q}\left(\int_0^{1/r} u(t)\, dt\right)^{1/q} = \frac{nr}{2}\left(\int_0^{1/r} u(t)\, dt\right)^{1/q}.$$

Thus,

$$\begin{aligned} r\left(\int_0^{1/r} u\right)^{1/q}\left(\int_0^r v\right)^{-1/p} & \leq \frac{2}{n}\left(\int_0^\infty \hat{f}^*(t)^q u(t)\, dt\right)\left(\int_0^r v\right)^{-1/p} \\ & \leq \frac{2C}{n}\left(\int_0^r v\right)^{1/p}\left(\int_0^r v\right)^{-1/p} = \frac{2C}{n}, \end{aligned}$$

where the constant C is from the hypothesis (4.2) and where the second inequality is a consequence of the bound (4.10). (4.1) is obtained and the proof is complete. $\square$

The following result provides sufficient conditions for the Fourier transform weighted norm inequality (4.2) of *Theorem 4.1* in the case $2 \leq q < p < \infty$.

Theorem 4.2. *Let u and v be weighted functions on $(0,\infty)$. Assume u is non-increasing and $v \in B_p$, where $2 \leq q < p < \infty$. If*

$$\left(\int_0^\infty \left[\frac{1}{s}\left(\int_0^s u(t)\, dt\right)^{1/p}\left(\int_0^{1/s} v(t)\, dt\right)^{-1/p}\right]^r u(s)\, ds\right)^{1/r} \equiv C_4 < \infty, \tag{4.15}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then there is $C > 0$ such that, for all $f \in L^1 + L^2$, the inequality (4.2) is satisfied.

Proof. As in the proof of *Theorem 4.1*(i) we obtain

$$\begin{aligned} \left(\int_0^\infty \hat{f}^*(t)^q u(t)\,dt\right)^{1/q} &\le K_q\left(\int_0^\infty \frac{u(1/s)}{s^{2-q}}\left(\frac{1}{s}\int_0^s f^*\right)^q ds\right)^{1/q} \\ &\le K_q C_H\left(\int_0^\infty f^*(s)^p\, v(s)\,ds\right)^{1/p} \end{aligned} \tag{4.16}$$

by means of *Theorem B* (requiring $q \ge 2$ and giving rise to the constant K_q), Hardy's lemma, and *Theorem D* (with constant C_H). However, in the index range $q < p < \infty$, the second inequality of (4) results from the validity of conditions (ii) of *Theorem D* for the weight $w(s) = u(1/s)/s^{2-q}$. Out hypothesis (4.15) will yield these conditions in the following way.

To show that D_1 in *Theorem D* is finite we make the computation

$$\begin{aligned} D_1^r &\equiv \int_0^\infty \left[\left(\int_0^s \frac{u(1/t)}{t^{2-q}}\,dt\right)^{1/p}\left(\int_0^s v\right)^{-1/p}\right]^r \frac{u(1/s)}{s^{2-q}}\,ds \\ &= \int_0^\infty \left[\left(\int_{1/s}^\infty \frac{u(t)}{t^q}\,dt\right)^{1/p}\left(\int_0^s v\right)^{-1/p}\right]^r \frac{u(1/s)}{s^{2-q}}\,ds \\ &= \int_0^\infty \left[\left(\int_s^\infty \frac{u(t)}{t^q}\,dt\right)^{1/p}\left(\int_0^{1/s} v\right)^{-1/p}\right]^r \frac{u(s)}{s^q}\,ds \end{aligned} \tag{4.17}$$

by means of a change of variable. Since u is decreasing we know that $u \in B_q$, and, hence, using the fact that $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, we see that the last term of (4.17) is bounded by

$$b_q^{r/p}\int_0^\infty \left[\frac{1}{s}\left(\int_0^s u\right)^{1/p}\left(\int_0^{1/s} v\right)^{-1/p}\right]^r u(s)\,ds.$$

Consequently, from our hypothesis (4.15), we have

$$D_1 \le b_q^{1/p} C_4 < \infty.$$

To show that D_2 in *Theorem D* is finite we make the computation

$$D_2^r \equiv \int_s^\infty \left[\left(\int_0^\infty \frac{(1/t)}{t^2}\, dt \right)^{1/q} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{1/q'} \right]^r$$

$$\times \left(\frac{1}{s} \int_0^s v \right)^{-p'} v(s)\, ds$$

$$= \int_0^\infty \left(\int_0^{1/s} u(t) dt \right)^{r/q} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{r/q'}$$

$$\times\, d \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)$$

$$= \frac{\left(\int_0^{1/s} u \right)^{r/q} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{\frac{r}{q'}+1}}{\frac{r}{q'}+1} \Bigg|_0^\infty \tag{4.18}$$

$$+ \frac{\frac{r}{q}}{r(\frac{1}{q'}+\frac{1}{r})} \int_0^\infty \left(\int_0^{1/s} u \right)^{\frac{r}{q}-1} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{r(\frac{1}{q'}+\frac{1}{r})}$$

$$\times \frac{u(1/s)}{s^2}\, ds$$

$$= \frac{p'}{q} \int_0^\infty \left(\int_0^{1/s} u \right)^{r/p} \left(\int_0^s \left(\frac{1}{t} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{r/p'} \frac{u(1/s)}{s^2}\, ds$$

by means of a change of variable and an integration by parts. Since $v \in B_p$, the calculation at the end of part i in the proof of *Theorem 4.1* allows us to make the same assertion now as we did there, viz.,

$$\left(\int_0^s \left(\frac{1}{s} \int_0^t v \right)^{-p'} v(t)\, dt \right)^{1/p'} \leq C p^{1/p'} s \left(\int_0^s v(t)\, dt \right)^{-1/p},$$

where $C = C(b_p)$ is the constant from inequality (3.6) in *Theorem C*. Thus the right side of (4) is bounded by

$$\frac{p'}{q} \int_0^\infty \left(\int_0^{1/s} u \right)^{r/p} \left[p^{1/p'} C(b_p) s \left(\int_0^s v \right)^{-1/p} \right]^r \frac{u(1/s)}{s^2}\, ds$$

$$= \frac{p'}{q} \left(p^{1/p'} C(b_p) \right)^r \int_0^\infty \left(\int_0^t u \right)^{r/p} \left[\frac{1}{t} \left(\int_0^{1/t} v \right)^{-1/p} \right]^r u(t)\, dt.$$

Consequently, from our hypothesis (4.15), we have

$$D_2 \leq \left(\frac{p'}{q} \right)^{1/r} p^{1/p'} C(b_p) C_4 < \infty.$$

The proof of the theorem is complete. $\square$

Example 4.3. a. We give examples of weights satisfying the hypotheses of *Theorem 4.1*. Let $1 < p \le q$ and let $q \ge 2$. We shall construct weight functions u and v on $(0,\infty)$ so that u is non-increasing, $v \in B_p$, and (4.1) is satisfied, i.e.,

$$\sup_{s>0} s \left(\int_0^{1/s} u\right)^{1/q} \left(\int_0^s v\right)^{-1/p} < \infty. \tag{4.1}$$

Let $u(t) = t^\alpha$, $-1 < \alpha \le 0$, and let $v(t) = t^a$, $-1 < a < p-1$. These conditions on p, q, a, α, along with (4.19) below, ensure that u and v satisfy the hypotheses of *Theorem 4.1*.

In fact, u is clearly non-increasing, and $v \in B_p$ since

$$\int_s^\infty \frac{v(t)}{t^p}\,dt = \frac{s^{a-p+1}}{p-a-1} = \left(\frac{a+1}{p-a-1}\right)\frac{1}{s^p}\int_0^s v(t)\,dt;$$

and the product in (4.1) is

$$s\left(\int_0^{1/s} t^\alpha\,dt\right)^{1/q}\left(\int_0^s t^a\,dt\right)^{-1/p} = \frac{(a+1)^{1/p}}{(\alpha+1)^{1/q}} s^{-(\alpha+1)/q} s^{-(a+1)/p} s,$$

which is uniformly bounded for all $s > 0$ if

$$\frac{\alpha+1}{q} + \frac{a+1}{p} = 1. \tag{4.19}$$

We illustrate these weights in two specific cases, parts b and c.

b. Let $1 < p \le 2$ and $q = p'$. Then (4.19) is equivalent to $\alpha = -a(p'/p) = -a/(p-1)$. Suppose $-1 < \alpha \le 0$ so that u is non-increasing. We need only check that $a = -\alpha(p/p') \in (-1, p-1)$. In fact, this formula for a implies that $0 \le a < p-1$. Consequently, if $1 < p \le 2$, $q = p'$, $-1 < \alpha \le 0$, and $a = -\alpha(p/p')$, then there is $C > 0$ such that for all $f \in L^1 + L^2$,

$$\left(\int_0^\infty \hat{f}^*(t)^{p'} t^\alpha\,dt\right)^{1/p'} \le C\left(\int_0^\infty \hat{f}^*(t)^p t^a\,dt\right)^{1/p}.$$

c. In light of the conventional intuition, e.g., [BH83], from which the general theory has been built, it is natural to ask if u and v can both be non-trivally non-increasing in the case of *Theorem 4.1*. The answer is "yes". Let $-1 < \alpha, a < 0$ so that u and v are non-trivally non-increasing, and, in particular, $v \in B_p$ for any $p > 1$. For the case of *Theorem 4.1* we must find p, q, a, α so that $1 < p \le q$, $q \ge 2$, and (4.19) is satisfied.

If $1 < p < 2$ and $a = 1 - p$ then $-1 < a < 0$. We make this choice of p and a, and shall momentarily further restrict the value of p. In this case, (4.19) is equivalent to

$$\alpha = 2\frac{q}{p'} - 1. \tag{4.20}$$

Thus, $-1 < \alpha < 0$ if and only if $q < p'/2$. Since we also require $2 \le q$ we restrict values of p to the interval $(1, 4/3)$. Therefore, for such p and for $2 \le q < p'/2$ we set $a = 1 - p$ and define α by (4.20). Hence, *Theorem 4.1* applies and we obtain (4.2).

Example 4.4. We give examples of weights satisfying the hypotheses of *Theorem 4.2*. Let $2 \le q < p < \infty$. We shall construct weight functions u and v on $(0, \infty)$ so that u is non-increasing, $v \in B_p$, and (4.15) is satisfied, i.e.,

$$\int_0^\infty \left[\frac{1}{s} \left(\int_0^s u(t)\,dt \right)^{1/p} \left(\int_0^{1/s} v(t)\,dt \right)^{-1/p} \right]^r u(s)\,ds < \infty. \tag{4.15}$$

Note that $q > p$ implies $r > 0$. Let $-1 < \alpha \le 0$ and $\beta < -1$, and define

$$u(t) = \begin{cases} t^\alpha, & 0 < t < 1, \\ t^\beta, & t \ge 1 \end{cases}$$

and $v(t) = t^{a+p}$ where $-(p+1) < a < -1$.

Clearly, u is non-increasing, and $v \in B_p$ since

$$\int_s^\infty \frac{v(t)}{t^p}\,dt = -\left(\frac{a+p+1}{a+1} \right) \frac{1}{s^p} \int_0^s v(t)\,dt,$$

as in *Example 4.3a*.

In order to check (4.15) we write the integral there as

$$\int_0^\infty = \int_0^1 + \int_1^\infty \equiv I_1 + I_2.$$

A direct calculation shows that

$$I_1 = \left(\frac{a+p+1}{\alpha+1} \right)^{r/p} \int_0^1 t^{-r+\alpha+(\alpha+1)r/p+(a+p+1)r/p}\,dt.$$

Since the exponent in the integrand is $(\alpha+1)r/q + (a+1)r/p - 1$, we see that $I_1 < \infty$ if and only if

$$\frac{\alpha+1}{q} + \frac{a+1}{p} > 0, \tag{4.21}$$

noting that $\alpha + 1 > 0$, $a + 1 < 0$, and $r > 0$, cf., (4.19).

Another direct calculation shows that

$$I_2 = (a+p+1)^{r/p} \int_1^\infty t^{-r+\beta+(a+p+1)r/p} \left(\frac{1}{\alpha+1} + \left(\frac{-1}{\beta+1} \right) \left(1 - t^{\beta+1} \right) \right)^{r/p} dt.$$

Thus, $\alpha + 1 > 0$ and $-(\beta+1) > 0$ allow us to make the estimate,

$$0 \le I_2 < (a+p+1)^{r/p} \left(\frac{\beta - \alpha}{(\alpha+1)(\beta+1)} \right)^{r/p} \int_1^\infty t^{-r+\beta+(a+p+1)r/p}\,dt;$$

and the right side is finite if $-r + \beta + (a+p+1)r/p + 1 \equiv X < 0$, which is clearly the case since $a < -1$, $\beta < -1$, and $X = \beta + 1 + (a+1)r/p < 0$.

It remains to verify that the condition (4.21) is not vacuous. In fact, we can choose a and α so that $a+1=-(\alpha+1)$. In this case, taking $-1<\alpha\le 0$, we see that $a=-1-(\alpha+1)<-1$ and $a=-2-\alpha>-p-1$. Thus, u and v can be defined compatible with the constraint $a+1=-(\alpha+1)$. Further, (4.21) is obtained in this case because the left side of (4.21) is

$$(\alpha+1)\left(\frac{1}{q}-\frac{1}{p}\right)>0$$

since $\alpha>-1$ and $q<p$.

One might expect that a duality argument as in the weighted Lebesgue case of *Theorem 2.1* would show that *Theorems 4.1* and *4.2* are also valid in the index ranges $1<p<q<2$ and $1<q<p<\infty$, $q<2$. However, the dual of $\Lambda_p(v)$ is not $\Lambda_{p'}(w)$ for some w, see [Saw90] for Sawyer's characterization of the dual of $\Lambda_p(v)$ as well as the fundamental work of Lorentz [Lor50], [Lor51], [Lor53] and Halperin [Hal53]. On the other hand, *Theorems 4.1* and *4.2* can be used to prove Fourier inequalities in weighted Lebesgue spaces for the complete index range $1<p,q<\infty$. This is the subject of *Section 5*.

5. Weighted Fourier inequalities - Lorentz space method

We now apply *Theorems 4.1* and *4.2* to obtain a result similar in content to but different in proof than *Theorem 2.1*, see *Remark 5.2* for the manner in which *Theorem 5.1* can be considered a generalization of *Theorem 2.1*.

Theorem 5.1. *Let u and v be weight functions on $\mathbb{R}^n$ and suppose $1<p,q<\infty$. There is a constant $C>0$ such that for all $f\in L^1+L^2$ the inequality*

$$\left(\int_{\widehat{\mathbb{R}}^n}|\widehat{f}(\gamma)|^q u(\gamma)d\gamma\right)^{1/q}\le C\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)dx\right)^{1/p} \tag{5.1}$$

holds in the following ranges and with the following hypotheses on u and v:

(i) $((1/v)^*)^{-1}\in B_p$, $1<p\le q$, $q\ge 2$, *and*

$$\sup_{s>0} s\left(\int_0^{1/s}u^*(t)dt\right)^{1/q}\left(\int_0^s(1/v)^*(t)^{-1}dt\right)^{-1/p}<\infty; \tag{5.2}$$

(ii) $((1/v)^*)^{-1}\in B_p$, $2\le q<p$, $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, *and*

$$\int_0^\infty\left[\frac{1}{s}\left(\int_0^s u^*(t)\,dt\right)^{1/p}\left(\int_0^{1/s}(1/v)^*(t)^{-1}\,dt\right)^{-1/p}\right]^r u^*(s)\,ds<\infty. \tag{5.3}$$

(iii) $(u^*)^{1-q'}\in B_{q'}$, $1<p\le q<2$, *and*

$$\sup_{s>0}\frac{1}{s}\left(\int_0^{1/s}u^*(t)^{1-q'}\,dt\right)^{-1/q'}\left(\int_0^s(1/v)^*(t)^{p'-1}\,dt\right)^{1/p'}<\infty; \tag{5.4}$$

(iv) $(u^*)^{1-q'} \in B_{q'}$, $1 < q < p$, $q < 2$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *and*

(5.5)
$$\int_0^\infty \left[\frac{1}{s} \left(\int_0^s u^*(t)^{1-q'}\, dt \right)^{1/q'} \left(\int_0^{1/s} (1/v)^*(t)^{p'-1}\, dt \right)^{-1/q'} \right]^r (1/v)^*(s)^{p'-1}\, ds < \infty.$$

Proof. For part (i) we apply *Theorem 4.1* with u and v there replaced by u^* and $1/(1/v)^*$, respectively. Similarly, for part (ii), we apply *Theorem 4.2* with u and v there replaced by u^* and $1/(1/v)^*$, respectively. These two cases deal with $q \geq 2$, and with $1 < p \leq q$ for part (i) and with $q < p$ for part (ii).

In these cases, and for these substitutions with u and v, (5.2) becomes (4.1) and (5.3) becomes (4.15), respectively. Further, since u^* is non-increasing and since we are assuming that $1/(1/v)^* \in B_p$ in *Theorem 5.1*, we can apply *Theorems 4.1* and *4.2* for the cases (i) and (ii), respectively. Thus, (4.2) is obtained for both cases, i.e., there is a constant $C > 0$ such that for all $f \in L^1 + L^2$,

$$\left(\int_0^\infty \hat{f}^*(t)^q u^*(t)\, dt \right)^{1/q} \leq C \left(\int_0^\infty f^*(t)^p (1/v)^*(t)^{-1}\, dt \right)^{1/p}. \tag{5.6}$$

Finally, we obtain (5.1) for both cases (i) and (ii) by applying (2.3) to the left side of (5.6) and (2.4) to the right side of (5.6), noting, of course, that $(f^*)^p = (|f|^p)^*$.

For parts (iii) and (iv), and since $1 < q < 2$, we invoke the duality method used in *Theorem 2.1.* In fact, by the same argument used to obtain (2.12) in part c of the proof of *Theorem 2.1*, we have

$$\begin{aligned} \left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q u(\gamma) d\gamma \right)^{1/q} &= \sup_{\|G\|_{(L^q_u)'}=1} \left| \int_{\mathbb{R}^n} f(x) \overline{G^\vee(x)}\, dx \right| \\ &\leq \|f\|_{L^p_v} \sup_{\|G\|_{(L^q_u)'}=1} \left(\int |G^\vee(x)|^{p'} v(x)^{1-p'}\, dx \right)^{1/p'}, \end{aligned} \tag{5.7}$$

where $L^q_u(\widehat{\mathbb{R}}^n)' = L^{q'}_{u^{-q'/q}}(\widehat{\mathbb{R}}^n)$. It should be emphasized that this argument to obtain (5.7) only involves elementary functional analysis and not the general operator theoretic point of view of *Theorem 2.1.*

For part (iii) with $1 < p \leq q < 2$ we have $2 < q' \leq p'$. Thus, we are able to apply part (i) with q (in part (i)) replaced by p', p replaced by q', u replaced by $v^{1-p'}$, and v replaced by $u^{1-q'}$. For clarity, let $Q = p'$, $P = q'$, $U = v^{1-p'}$, and $V = u^{1-q'}$. In particular, $2 < P \leq Q$, and the hypothesis $(u^*)^{1-q'} \in B_{q'}$ becomes $1/(1/V)^* \in B_P$. Further, the hypothesis (5.4) is

$$\sup_{s>0} \frac{1}{s} \left(\int_0^{1/s} (1/V)^*(t)^{-1}\, dt \right)^{-1/P} \left(\int_0^s U^*(t)\, dt \right)^{1/Q} < \infty \tag{5.8}$$

since $(1/V)^* = (u^{q'-1})^* = (u^*)^{q'-1}$ and

$$\left(\left(\frac{1}{v}\right)^*\right)^{p'-1} = \left(\left(\frac{1}{v}\right)^{p'-1}\right)^* = \left(v^{1-p'}\right)^* = U^*.$$

Thus, (5.8) is the capitalized version of (5.2) so that part (i) applies with this notation. Therefore, there is a constant $C > 0$ such that for all $F \in L^P_V(\widehat{\mathbb{R}}^n)$,

$$\left(\int_{\mathbb{R}^n} |F^\vee(x)|^Q U(x)dx\right)^{1/Q} \leq C\left(\int_{\widehat{\mathbb{R}}^n} |F(\gamma)|^P V(\gamma)d\gamma\right)^{1/P}. \tag{5.9}$$

Rewriting (5.9) in its lower-case version, we have

$$\left(\int_{\mathbb{R}^n} |F^\vee(x)|^{p'} v(x)^{1-p'}dx\right)^{1/p'} \leq C\left(\int_{\widehat{\mathbb{R}}^n} |F(\gamma)|^{q'} u(\gamma)^{1-q'}d\gamma\right)^{1/q'} \tag{5.10}$$

for all $F \in L^{q'}_{u^{1-q'}}(\widehat{\mathbb{R}}^n)$. Combining (5.7) and (5.10) we obtain (5.1) in case (iii) since $(L^q_u)' = L^{q'}_{u^{1-q'}}(\mathbb{R}^n)$.

For part (iv) with $1 < q < p$, $q < 2$, we proceed as in part (iii) using (5.7) and invoking part (ii) to obtain the analogue of (5.10) but for this range of indices. Inequality (5.7) and this analogue yield (5.1). $\square$

The constant C in *Theorem 5.1* is a product of the form KC_H, depending on the various case (i) - (iv). The product itself was determined in the proofs of *Theorems 4.1* and *4.2*, and its factors are the constants from *Theorems B* and *D*.

Remark 5.2. Comparison of Theorem 2.1 and Theorem 5.1

It is not apparent that *Theorem 5.1* is a generalization of *Theorem 2.1*. In order to analyze the relationship between *Theorems 2.1* and *5.1* we proceed as follows.

a. Let $1 < p, q < \infty$ and assume weight functions u and v on $\mathbb{R}^n$ satisfy the hypothesis of *Theorem 2.1(i)*, viz.,

$$\sup_{s>0}\left(\int_0^{1/s} u^*(t)dt\right)^{1/q}\left(\int_0^s (1/v)^*(t)^{p'-1}dt\right)^{1/p} \equiv B_1 < \infty. \tag{5.11}$$

Then, Hölder's inequality allows us to make the estimate

$$\begin{aligned}
& s\left(\int_0^{1/s} u^*(t)dt\right)^{1/q}\left(\int_0^s (1/v)^*(t)^{-1}dt\right)^{-1/p} \\
= & \left(\int_0^s (1/v)^*(t)^{-\frac{1}{p}+\frac{1}{p}}\,dt\right)\left(\int_0^{1/s} u^*(t)dt\right)^{1/q}\left(\int_0^s (1/v)^*(t)^{-1}dt\right)^{-1/p} \\
& \le \left(\int_0^s (1/v)^*(t)^{-1}dt\right)^{1/p}\left(\int_0^s (1/v)^*(t)^{p'/p}dt\right)^{1/p'} \\
& \times \left(\int_0^{1/s} u^*(t)dt\right)^{1/q}\left(\int_0^s (1/v)^*(t)^{-1}dt\right)^{-1/p} \le B_1.
\end{aligned}$$

Consequently, if one supposes the hypothesis of *Theorem 2.1*(i), then (5.2) is valid.

b. Again, let $1 < p, q < \infty$ and assume weight functions u and v on $\mathbb{R}^n$ satisfy the hypothesis of *Theorem 2.1*(i), viz., (5.11). Then, Hölder's inequality allows us to make the estimate

$$\begin{aligned}
& \frac{1}{s}\left(\int_0^{1/s} u^*(t)^{1-q'}\,dt\right)^{-1/q'}\left(\int_0^s (1/v)^*(t)^{p'-1}\,dt\right)^{1/p'} \\
= & \left(\int_0^{1/s} u^*(t)^{-\frac{1}{q}+\frac{1}{q}}\,dt\right)\left(\int_0^{1/s} u^*(t)^{1-q'}dt\right)^{-1/q'}\left(\int_0^s (1/v)^*(t)^{p'-1}dt\right)^{1/p'} \\
& \le \left(\int_0^{1/s} u^*(t)dt\right)^{1/q}\left(\int_0^s (1/v)^*(t)^{p'-1}dt\right)^{1/p'} \le B_1,
\end{aligned}$$

since $-q'/q = 1 - q'$. Consequently, if one supposes the hypothesis of *Theorem 2.1*(i), then (5.4) is valid.

c. Next, let $1 < q < p < \infty$ and assume weight functions u and v on $\mathbb{R}^n$ satisfy the hypothesis of *Theorem 2.1*(ii), viz.,

$$\int_0^\infty \left(\int_0^{1/s} u^*\right)^{r/q}\left(\int_0^s (1/v)^{*(p'-1)}\right)^{r/q'}(1/v)^*(s)^{p'-1}ds = B_2^r < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ (and hence $\frac{r}{q'} = \frac{r}{p'} - 1$). Note that $r > 0$ since $q < p$. Then, Hölder's inequality and an integration by parts allow us to make the calculation

$$\int_0^\infty \left[\frac{1}{s}\left(\int_0^s u^*(t)\,dt\right)^{1/p}\left(\int_0^{1/s}(1/v)^*(t)^{-1}\,dt\right)^{-1/p}\right]^r u^*(s)ds$$

$$= \int_0^\infty \left[\left(\int_0^{1/s}((1/v)^*)^{-\frac{1}{p}+\frac{1}{p}}\right)\left(\int_0^s u^*\right)^{1/p}\left(\int_0^{1/s}((1/v)^*)^{-1}\right)^{-1/p}\right]^r u^*(s)ds$$

$$\le \int_0^\infty \left[\left(\int_0^{1/s}((1/v)^*)^{-1}\right)^{1/p}\left(\int_0^{1/s}((1/v)^*)^{p'/p}\right)^{1/p'}\left(\int_0^s u^*\right)^{1/p}\right.$$

$$\times \left(\int_0^{1/s}((1/v)^*)^{-1}\right)^{-1/p^r} u^*(s)ds$$

$$= \int_0^\infty \left[\left(\int_0^{1/s}((1/v)^*)^{p'/p}\right)^{r/p'}\left(\int_0^s u^*\right)^{r/p}\right] d\int_0^s u^*$$

$$= -\int_0^\infty \left[\frac{r}{p}\left(\int_0^s u^*\right)^{r/p}\left(\int_0^{1/s}((1/v)^*)^{p'/p}\right)^{r/p'} u^*(s)\right.$$

$$\left. -\frac{r}{p'}\left(\int_0^s u^*\right)^{\frac{r}{p}+1}\left(\int_0^{1/s}((1/v)^*)^{p'/p}\right)^{\frac{r}{p'}-1}(1/v)^*\left(\frac{1}{s}\right)^{p'/p}\frac{1}{s^2}\right] ds$$

$$\le \frac{r}{p'}\int_0^\infty \left(\int_0^s u^*\right)^{r/q}\left(\int_0^{1/s}((1/v)^*)^{p'/p}\right)^{r/q'}(1/v)^*\left(\frac{1}{s}\right)^{p'/p}\frac{1}{s^2}\,ds,$$

where we have used the convention $0\cdot\infty = 0$ and the fact that there is a negative term in the integration by parts step. By a change of variable $t = 1/s$ on the last term of this calculation, we see that

$$\int_0^\infty \left[\frac{1}{s}\left(\int_0^s u^*\right)^{1/p}\left(\int_0^{1/s}((1/v)^*)^{-1}\right)^{-1/p}\right]^r u^*(s)ds$$

$$\le \frac{r}{p'}\int_0^\infty \left(\int_0^{1/t} u^*\right)^{r/q}\left(\int_0^t((1/v)^*)^{p'/p}\right)^{r/q'}(1/v)^*(t)^{p'/p}\,dt$$

$$\le \frac{r}{p'}B_2^r,$$

since $p'/p = p'-1$. Consequently, if one supposes the hypothesis of *Theorem 2.1*(ii), then (5.3) is valid.

d. A similar calculation in the case $1 < q < p$, $q < 2$, also allows us to conclude that if one supposes the hypothesis of *Theorem 2.1*(ii), then (5.5) is valid.

Remark 5.2 establishes that the hypotheses of *Theorem 2.1* are special cases of the boundedness hypotheses (5.2) - (5.5) of *Theorem 5.1.* This does not prove that *Theorem 5.1* is a generalization of *Theorem 2.1* since, in particular, we have not shown that the hypotheses of *Theorem 2.1* imply the B_p and $B_{q'}$ conditions of *Theorem 5.1.* It is reassuring to know that *Theorems 2.1* and *5.1* are equivalent for power weights and that these theorems are genuine generalizations of Pitt's theorem stated in the Introduction. These two assertions are the content of *Examples 5.3* and *5.4*, respectively.

Example 5.3. a. Let $u(\gamma) = |\gamma|^\alpha$, $v(x) = |x|^a$, where $\gamma \in \widehat{\mathbb{R}}^n$, $x \in \mathbb{R}^n$ and $\alpha < 0$, $a > 0$. It is easy to check that $u^*(t) = C_\alpha t^{\alpha/n}$ and $(1/v)^*(t) = C_a t^{-a/n}$ for all $t > 0$. C_α and C_a are dimensionality constants.

We have chosen $\alpha < 0$, $a > 0$ in light of *Theorem 5.1.* For example, if $\alpha \geq 0$, then $D_u(s) = \infty$ and so $u^* \equiv \infty$ on $(0, \infty)$; and, consequently, if (5.2) is assumed then $v \equiv \infty$ on $\mathbb{R}^n$. In this situation of $\alpha \geq 0$, for the case $1 < p \leq q$, $q \geq 2$, *Theorem 5.1* only guarantees (5.1) for $f \equiv 0$. There is a similar consequence if $a \leq 0$. As such, we assume $\alpha < 0$ and $a > 0$.

b. Now, for these weights, the boundedness hypothesis of *Theorem 2.1*(i) becomes

$$\frac{B_1}{C_a^{1/p} C_\alpha^{1/q}} \geq \left(\int_0^{1/s} t^{\frac{\alpha}{n}} dt \right)^{1/q} \left(\int_0^s t^{-\frac{a}{n}\frac{p'}{p}} dt \right)^{1/p'}$$

$$= \left(\frac{1}{1 + \frac{\alpha}{n}} \right)^{1/q} \left(\frac{1}{1 - \frac{a}{n}\frac{p'}{p}} \right)^{1/p'} s^{-\frac{\alpha}{nq} - \frac{1}{q} + \frac{1}{p'} - \frac{a}{np}};$$

and so the boundedness hypothesis is valid if and only if

$$-n < \alpha \text{ and } a < n(p-1) \tag{5.12}$$

and

$$\frac{1}{n}\left(\frac{a}{p} + \frac{\alpha}{q} \right) = \frac{1}{p'} - \frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{q}, \tag{5.13}$$

where (5.12) ensures local integrability of the weights. Thus, if $\alpha < 0$ and $a > 0$, then (5.12) and (5.13) allow us to use *Theorem 2.1*(i) to conclude that

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q |\gamma|^\alpha d\gamma \right)^{1/q} \leq KC \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^a dx \right)^{1/p}, \tag{5.14}$$

see (2.5) of *Theorem 2.1.*

c. We now turn our attention to *Theorem 5.1* for the weights of part *a*.

With regard to the B_p hypothesis in *Theorem 5.1*(i), it is an easy calculation to check that *if* $n \geq 1$ *and* $p > 1$ *are given, then* $((1/v^*))^{-1} \in B_p$ *if and only if* $0 < a < n(p-1)$, and in this case the lower bound b_p in the definition of B_p in (3.2) is

$$b_p = \frac{n+a}{n(p-1)+a}.$$

In fact, the left side of the B_p inequality (3.2) for the weight $((1/v^*))^{-1}(t) = (1/C_a)t^{a/n}$ is finite and equal to

$$\frac{1}{C_a}\frac{1}{p-1-\frac{a}{n}}s^{-(p-1)+\frac{a}{n}} \tag{5.15}$$

if and only if $a < n(p-1)$; and the right side of the B_p inequality in this case is

$$\frac{b_p}{C_a}\frac{1}{s^p}\int_0^s t^{a/n}\,dt,$$

which is finite and of the same form as (5.15) if and only if $\frac{a}{n}+1>0$, i.e., $a>-n$, which is automatic since we are assuming $a>0$.

In this regard, recall that $A_p \subseteq B_p$, see *Remark 3.2* b, whereas $A_p = B_p$ in the case of power weights.

d. Continuing with *Theorem 5.1* (i) for the weights of parts a, the product in the boundedness condition (5.2) is

$$s\left(\int_0^{1/s} C_\alpha t^{\alpha/n}\,dt\right)^{1/q}\left(\int_0^s C_a^{-1}t^{a/n}\,dt\right)^{-1/p}$$
$$= C_a^{1/p}C_\alpha^{1/q}\left(\frac{1}{1+\frac{\alpha}{n}}\right)^{1/q}\left(1+\frac{a}{n}\right)^{1/p}s^{-\frac{\alpha}{nq}-\frac{1}{q}+\frac{1}{p'}-\frac{a}{np}};$$

and so (5.2) is valid if and only if

$$-n<\alpha<0 \quad\text{and}\quad 0<a \tag{5.16}$$

and

$$\frac{1}{n}\left(\frac{a}{p}+\frac{\alpha}{q}\right)=\frac{1}{p'}-\frac{1}{q}=1-\frac{1}{p}-\frac{1}{q}, \tag{5.17}$$

where (5.16) ensures local integrability of the weights. Note that (5.13) and (5.17) are the same and, from part b, that $((1/v^*))^{-1}\in B_p$ if and only if $0<a<n(p-1)$ (and that $a<n(p-1)$ is required in (5.12)). Thus, in the case $1<p\le q$, $q\ge 2$, conditions (5.16) and (5.17) and $a<n(p-1)$ allow us to use *Theorem 5.1*(i) to conclude that (5.1), i.e., (5.14), is valid for a sufficiently large class of functions f.

e. *In this way*, by considering other cases, *we can verify the equivalence of Theorems 2.1 and 5.1 for power weights.*

For example, in the case $1<p\le q<2$, in order to invoke *Theorem 5.1*(iii), we must verify that $(u^*)^{1-q'}\in B_{q'}$ and

$$s^{-1}\left(\int_0^{1/s}\left(C_\alpha t^{\alpha/n}\right)^{1-q'}dt\right)^{-1/q'}\left(\int_0^s\left(C_a t^{-a/n}\right)^{p'-1}dt\right)^{1/p'}\le C \tag{5.18}$$

independent of $s>0$.

For the claim about $B_{q'}$ we use the approach in part c for the function $t^{\alpha(1-q')/n}$ instead of $t^{a/n}$. Thus, by the analogue of (5.15) for this case we see that $(u^*)^{1-q'}\in B_{q'}$ if $-n<\alpha<0$. In order to verify (5.18) for fixed α,

$-n < \alpha < 0$, we first note that $0 < a < n(p-1)$ is also required to ensure the local integrability necessary for (5.18). Then, by a calculation on the left side of (5.18) analogous to the calculation in part d, we see that for the case $1 < p \leq q < 2$ condition *iii* of *Theorem 5.1* is satisfied if and only if $-n < \alpha < 0$, $0 < a < n(p-1)$, and (5.17) are assumed. These are the same conditions verified in parts c and d for the case $1 < p \leq q$, $q \geq 2$.

Example 5.4. In light of *Example 5.3* we have the following corollary of *Theorem 5.1*: *if* $1 < p \leq q < \infty$, $-n < \alpha < 0$, $0 < a < n(p-1)$, $\frac{1}{n}\left(\frac{a}{p} + \frac{\alpha}{q}\right) = 1 - \frac{1}{p} - \frac{1}{q}$, *and* $v(x) = |x|^a$ *for all* $x \in \mathbb{R}^n$, *then there is* $C > 0$ *such that for all* $f \in L^p_v(\mathbb{R}^n)$,

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{f}(\gamma)|^q |\gamma|^\alpha d\gamma\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^a dx\right)^{1/p}. \tag{5.19}$$

For $n = 1$, (5.19) is precisely Pitt's theorem stated in (1.5), cf., [Roo66]. In fact, given the hypotheses $1 < p \leq q < \infty$, $0 < b < 1/p'$, and $\beta = 1 - \frac{1}{p} - \frac{1}{q} - b < 0$ from Subsection 1.3, and setting $a = bp$ and $\alpha = \beta q$, it is easy to show that $-1 < \alpha < 0$, $0 < a < p-1$, and $\frac{a}{p} + \frac{\alpha}{q} = 1 - \frac{1}{p} - \frac{1}{q}$. These are the hypotheses we used to obtain (5.19).

Remark 5.5. Pitt's theorem, wavelets, and multipliers

a. Pitt's theorem for Fourier series was a generalization of the following inequalities due to Hardy and Littlewood in 1927 [HL27].

i. If $q > 2$, then there is $C(q) > 0$ such that for all $|f(x)|^q x^{q-2} \in L^1(\mathbb{R})$, $\hat{f}$ exists in $L^q(\widehat{\mathbb{R}})$ and

$$\left(\int_{\widehat{\mathbb{R}}} |\hat{f}(\gamma)|^q d\gamma\right)^{1/q} \leq C(q) \left(\int_{\mathbb{R}} |f(x)|^q |x|^{q-2} dx\right)^{1/q}. \tag{5.20}$$

ii. If $1 < p < 2$, then there is $C(p) > 0$ such that for all $f \in L^p(\mathbb{R})$, $\hat{f}$ exists and

$$\left(\int_{\widehat{\mathbb{R}}} |\hat{f}(\gamma)|^p |\gamma|^{p-2} d\gamma\right)^{1/p} \leq C(p) \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}. \tag{5.21}$$

Hardy and Littlewood first proved these inequalities for Fourier series [HL27] (Theorems 2,3,5,and 6); and then were able to state them for Fourier transforms in light of work by Titchmarsh [Tit25], see [Tit67] (Theorems 79 and 80) where Titchmarsh obtains (5.20) and (5.21) by using the original Fourier series inequalities and taking successively longer periodizations. In 1931, Hardy and Littlewood obtained new proofs and finer estimates of the original Fourier series inequalities by means of decreasing rearrangement arguments.

There are natural generalizations of these results in terms of A_p weights. In fact, *if* $1 < p \leq q \leq p' < \infty$ *and* w *is an even weight on* $\mathbb{R}$ *increasing on*

$(0, \infty)$, *then* $w^{q/p} \in A_{1+(q/p')}$ *if and only if*

$$\left(\int_{\widehat{\mathbb{R}}} |\hat{f}(\gamma)|^q |\gamma|^{(q/p')-1} w(1/\gamma)^{q/p} d\gamma\right)^{1/q} \leq C(p,w)\left(\int_{\mathbb{R}} |f(x)|^p w(x) dx\right)^{1/p}, \tag{5.22}$$

see [BHJ87b] as well as [HS89] for extensions to $\mathbb{R}^n$. (5.5) is (5.21) in the case $q = p$ and $w \equiv 1$.

b. In 1931, Paley [Pal31] generalized the inequalities of Hardy and Littlewood in [HL27], [HL31] from trigonometric series to other orthonormal systems, cf., [Lit54] for adulatory comments on Paley's proofs from on high. See also [Zyg59] (Chapter XII, Sections 5 and 6). This direction of generalization was carried on by Littlewood [Lit54], Stein [Ste56], Hirschman [Hir56], and Stein and Weiss [SW58]. The present authors suggest it is natural to resurrect this program in light of wavelet orthonormal bases and related recent decompositions.

c. The essence of this *Remark 5.5* is to comment on fundamental observations by Zygmund [Zyg56] (1956) and Hörmander [Hör60] (1960), cf., a related remark by Hirschman [Hir61] (1961).

To this end, recall that a linear operator,

$$T : L^p_\nu(X) \longrightarrow L^q_\mu(Y),$$

for measure spaces (X, μ) and (Y, ν), is (strong) *type* (p, q) if it is continuous on $L^p_\nu(X)$, see the beginning of *Section 2*. Also, T is *weak type* (p, q) if there is $K > 0$ such that for all $f \in L^p_\nu(X)$ and for all $s > 0$,

$$\mu\{\gamma \in Y : |Tf(\gamma)| > s\} \leq \left(\frac{K}{s}\|f\|_{L^p_\nu}\right)^q. \tag{5.23}$$

The infimum of those $K > 0$ for which (5.23) is valid is referred to as the *weak* (p, q) *norm* of T.

d. In [Zyg56], Zygmund gives a complete proof of the Marcinkiewicz interpolation theorem. He then uses this theorem to give a new proof of the Fourier series version (actually for bounded orthonormal systems) of (5.21), i.e., he gives a new proof of the Hardy-Littlewood-Paley theorem of parts *a* and *b*. His proof makes use of the mapping T defined by $T(f) = \{2\pi i n \hat{f}[n]\}$, when $\{\hat{f}[n]\}$ is the sequence of Fourier coefficients of the 1-periodic function f. T is type (p, p) for $p \in (1, 2]$ but not type $(1, 1)$. Thus, Riesz interpolation is not available to prove (5.21). On the other hand, if we define a measure μ on $\widehat{\mathbb{R}}$ by the property that $\mu(\{n\}) = 1/n^2$, $n \neq 0$, and $\mu(B) = 0$ if $n \notin B$, then the Marcinkiewicz theorem allowed Zygmund to prove $\|Tf\|_{\ell^p_\mu} \leq C\|f\|_{L^p(\mathbb{R}/\mathbb{Z})}$, i.e., the Fourier series analogue of (5.21), by proving that T is weak type $(1, 1)$.

e. In [Hör60] (Theorem 1.10), Hörmander used Zygmund's proof of weak $(1, 1)$ and Zygmund's use of the Marcinkiewicz theorem to generalize (5.21) in the following way. *Let* $1 < p \leq 2$, *let* $u \geq 0$ *be a weight function on* $\widehat{\mathbb{R}}^n$, *and set* $w = u^{1/(2-p)}$. *Assume* w *is in weak-*L^1. *(This means that the distribution function* D_w *is weak type* $(1,1)$, *i.e., there is* $K > 0$ *such that for all* $s > 0$, $D_w(s) \leq K/s$.*) Then there is a constant* $C > 0$ *such that for all* $f \in L^p(\mathbb{R}^n)$,

$$\left(\int_{\widehat{\mathbb{R}}^n} |\hat{f}(\gamma)|^p u(\gamma) d\gamma\right)^{1/p} \leq C\|f\|_{L^p}, \tag{5.24}$$

cf., *Theorem 5.1.* In particular, note that if $u(\gamma) = |\gamma|^{p-2}$ on $\widehat{\mathbb{R}}$ then D_w is weak type $(1,1)$ where $w(\gamma) = u(\gamma)^{1/(2-p)} = |\gamma|^{-1}$ and $K = 2$. In this case, (5.24) reduces to (5.21).

It should also be pointed out that Hörmander's theorem (5.24) is a special case of *Theorem 2.1* with $p = q, 1 < p \leq 2$, and $v = 1$. To verify this assertion we must prove that

$$\sup_{s>0} \left(\int_0^{1/s} u^*(t)\, dt\right)^{1/p} \left(\int_0^s dt\right)^{1/p'} \equiv B_1 < \infty. \tag{5.25}$$

To this end we shall show that $B_1 < \infty$ *if and only if* $D_w(s) \leq K/s$, *where* $u = w^{2-p}$. In one direction, since $u^* = (w^*)^{2-p}$, we see that $B_1 < \infty$ implies

$$\forall s > 0, \qquad \left(w^*\left(\frac{1}{s}\right)\right)^{\frac{2-p}{p}} \leq B_1 s^{\frac{1}{p} - \frac{1}{p'}},$$

i.e., $w^*(s) \leq B^{p/(2-p)}/s$, and hence $D_w(s) \leq K/s$. Conversely, if $w^*(t) \leq K/t$ then $(u^*)^{1/(2-p)}(t) \leq K/t$ and so the product on the left side of (5.25) is bounded by $K^{(2-p)/p}(p-1)^{-1/p}$.

f. Let $L_p^q(\mathbb{R}^n)$ be the space of tempered distributions $T \in S'(\mathbb{R}^n)$ for which there is a constant $C(T)$ such that

$$\forall f \in S(\mathbb{R}^n), \ \|T * f\|_{L^q} \leq C(T)\|f\|_{L^p}.$$

By definition, $M_p^q(\widehat{\mathbb{R}}^n) = L_p^q(\mathbb{R}^n)^\wedge$ is the *space of multipliers of type* (p, q), see [Lar71]. Using an elementary generalization of (5.24) [Hör60] (Corollary 1.6), as well as a straightforward argument [Hör60] (Theorem 1.11), Hörmander proved the following: *Let* $1 < b < \infty$, *let* $f : \mathbb{R}^n \to \mathbb{C}$ *be a measurable function, and assume that there is* $K > 0$ *such that for all* $s > 0$, $D_f(s) \leq K/s^b$. *If* $1 < p \leq 2 \leq q < \infty$ *and* $1/p - 1/q = 1/b$, *then* $f \in M_p^q(\widehat{\mathbb{R}}^n)$. One can expect more general multiplier theorems by using *Theorem 2.1* as we have done in part *e*.

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Department of Mathematics, University of Maryland, College Park, MD 20742, USA

E-mail address: `jjb@math.umd.edu`

URL: `http://www.math.umd.edu/ jjb`

Department of Mathematics, McMaster University, Hamilton, ON L8S-4K1, Canada

E-mail address: `heinig@mcmail.cis.mcmaster.ca`