PERIODIC WAVELET TRANSFORMS AND PERIODICITY DETECTION

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Abstract. The theory of periodic wavelet transforms presented here was originally developed to deal with the problem of epileptic seizure prediction. A central theorem in the theory is the characterization of wavelets having time and scale periodic wavelet transforms. In fact, we prove that such wavelets are precisely *generalized Haar wavelets* plus a logarithmic term.

It became apparent that the aforementioned theorem could not only be quantified to analyze seizure prediction, but could also provide a technique to address a large class of periodicity detection problems. An essential step in this quantification is the geometric and linear algebra construction of a generalized Haar wavelet associated with a given periodicity. This gives rise to an algorithm for periodicity detection based on the periodicity of wavelet transforms defined by generalized Haar wavelets and implemented by wavelet averaging methods. The algorithm detects periodicities embedded in significant noise.

The algorithm depends on a discretized version $W^p_\psi f(n,m)$ of the continuous wavelet transform. The version defined provides a fast algorithm with which to compute $W^p_\psi f(n,m)$ from $W^p_\psi f(n-1,m)$ or $W^p_\psi f(n,m-1)$. This has led to the theory of non–dyadic wavelet frames in $l^2(\mathbb{Z})$ developed by the second-named author, and which will appear elsewhere.

1. Introduction. Generalized Haar wavelets were introduced in [4]. The theory and some applications of these generalized Haar wavelets will be developed in this paper.

In [3], the authors addressed a component of the problem of predicting epileptic seizures. A satisfactory solution of this problem would provide maximal lead time in which to predict an epileptic seizure [24, 25]. It was shown that spectrograms of electrical potential time series derived from brain activities of patients during seizure episodes exhibit multiple chirps consistent with the relatively simple almost periodic behavior of the observed time-series [6]. In the process, electrocorticogram (ECoG) data was used instead of the more common electroencephalogram (EEG) data. To obtain ECoG data, electrodes are planted directly on the cortex, eliminating some noise. To analyze the periodic components in these time-series, a redundant non–dyadic wavelet analysis was used, which the authors in [3] referred to as wavelet integer scale processing (WISP). The wavelet transform obtained with respect to the Haar wavelet showed, among other things, that the almost periodic behavior in the signal resulted in almost periodic behavior in both time and scale in the wavelet transform [4].

Mathematically, the non–normalized continuous Haar wavelet transform of a periodic signal is periodic in time and in scale. Continuing the work in [3], we realized the origin of this phenomen, and verified the time–scale continuous wavelet transform periodicity of periodic signals for a large class of Haar-type wavelets which we

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call *generalized Haar wavelets.* These elementary observations and calculations are the subject of Section 2. Section 3 is more mathematically substantive. In it we describe all integrable wavelets whose non–normalized continuous wavelet transforms are 1–periodic in both time and scale for all 1–periodic bounded measurable functions. Naturally, these wavelets include the generalized Haar wavelets, but they can also have a well–defined logarithmic term. The main results, which are proved using methods from harmonic analysis, are given on the real line $\mathbb{R}$, but also have a formulation on d–dimensional Euclidean space $\mathbb{R}^d$ [21].

Motivated by the epileptic seizure problem, or more accurately the early detection problem in our approach, and based on the results in Sections 2 and 3, we have developed a method aimed at detecting periodic behavior imbedded in noisy environments. This is the subject of Sections 4 and 5. In Section 4 we construct generalized Haar wavelets that are optimal as far as detecting prescribed periodicities in given data. The construction is geometric and invokes methods from linear algebra. We use these optimal generalized Haar wavelets in Section 5 to design our wavelet periodicity detection algorithm. The algorithm is based on averaging wavelet transforms, and it will give perfect periodicity information, as far as pattern and period, for the case of periodic signals in non–noisy environments. The averaging strategy and use of optimal generalized Haar wavelets optimizes available information for detecting suspected periodicities in noisy data.

The implementation of the algorithm designed in Section 5 requires the computation of discretized versions of the continuous wavelet transform. The redundancy in such discretized versions offers robustness to noise, but more calculations are needed than to compute a dyadic wavelet transform. In Section 6, we shall present a fast algorithm which significantly reduces the number of these calculations in case the analyzing wavelet is a generalized Haar wavelet.

Apropos this description of our paper, the readers who are only interested in applicable periodicity detection techniques need only read the statements of Theorem 3.1 and Theorem 4.2, and then go directly to Section 5, where Theorem 3.1 is invoked, and to Section 6. Our point of view and idea for periodicity detection has to be compared critically with other methods and, in particular, with a variety of spectral estimation techniques, e.g., see [14, 22].

Notation. We employ standard notation from harmonic analysis and wavelet theory, e.g., see [2, 9, 15, 16, 28]. In order to avoid any confusion, we begin by reviewing some of the notation herein in which different choices are sometimes made by others.

The Fourier transform of $f \in L^1(\mathbb{R})$ is $\widehat{f} : \widehat{\mathbb{R}} \to \mathbb{C}$ where $\widehat{f}(\gamma) = \int f(t)e^{-2\pi it\gamma}\, dt$, $\widehat{\mathbb{R}} = \mathbb{R}$ when considered as the domain of the Fourier transform and "$\int$" designates integration over $\mathbb{R}$. $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\widehat{\mathbb{R}})$ is the Fourier transform operator on $L^2(\mathbb{R})$. The Fourier series of a 1–periodic function $\varphi : \widehat{\mathbb{R}} \mapsto \mathbb{C}$ is denoted by $S(\varphi)(\gamma) = \sum \widehat{\varphi}[m]e^{-2\pi im\gamma}$, where $\widehat{\varphi}[m] = \int_{\mathbb{T}} \varphi(\gamma)e^{2\pi im\gamma}\, d\gamma$, $\mathbb{T} = \widehat{\mathbb{R}}/\mathbb{Z}$ is the quotient group, and "$\sum$" designates summation over the integers $\mathbb{Z}$. We shall also deal with $\mathbb{T}_T = \widehat{\mathbb{R}}/T\mathbb{Z}$ for other values of T besides $T = 1$. $A(\widehat{\mathbb{R}})$ and $A(\mathbb{T})$ are the spaces of absolutely convergent Fourier transforms on $\widehat{\mathbb{R}}$ and absolutely convergent Fourier series on $\mathbb{T}$, respectively.

$\mathbb{R}^+$ and $\mathbb{R}^-$ are the sets of positive and negative real numbers in $\mathbb{R}$; and $\widehat{\mathbb{R}}^+$ and $\widehat{\mathbb{R}}^-$ are the sets of positive and negative real numbers in $\widehat{\mathbb{R}}$. $\mathbf{1}_X$ denotes the characteristic function of the set X, and the Heaviside function H on $\mathbb{R}$ is $\mathbf{1}_{(0,\infty)}$. $|\ldots|$ designates absolute value, Lebesgue measure, or cardinality; and the meaning

will be clear from the context.
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2. Generalized Haar wavelets. We shall introduce the notion of *generalized Haar wavelets*. There are two reasons to consider such wavelets. First, they allow a fast computation of a discretized version of the continuous wavelet transform by means of a recursive algorithm. This is the subject of Section 6. The second reason is presented below in Propositions 2.3 and 2.5, where we obtain time–scale periodicity for the non–normalized continuous *generalized Haar wavelet transform* of a periodic signal.

The term *wavelet* does not possess a unique definition. The following definition (part *a*) is appropriate for our needs.

DEFINITION 2.1. *a.* A *wavelet* is a complex valued function $\psi \in L^1(\mathbb{R})$ with one vanishing moment, i.e., ψ has the property that

$$\int \psi(t)\, dt = 0.$$

b. If ψ is a wavelet, the *non–normalized continuous wavelet transform* W_ψ is the mapping

$$W_\psi : L^\infty(\mathbb{R}) \longrightarrow C_b(\mathbb{R}\times\mathbb{R}^+)$$

defined by

$$W_\psi f(b,a) = \int f(t)\psi(\tfrac{t-b}{a})\, dt.$$

$C_b(\mathbb{R}\times\mathbb{R}^+)$ is the space of complex–valued bounded continuous functions on $\mathbb{R}\times\mathbb{R}^+$.

Similarly, if $1 \le p < \infty$, then the $L^p(\mathbb{R})$*–normalized continuous wavelet transform* W^p_ψ is the mapping

$$W^p_\psi : L^\infty(\mathbb{R}) \longrightarrow C(\mathbb{R}\times\mathbb{R}^+)$$

defined by

$$W^p_\psi f(b,a) = a^{-1/p}\int f(t)\psi(\tfrac{t-b}{a})\, dt.$$

$C(\mathbb{R}\times\mathbb{R}^+)$ is the space of complex–valued continuous functions on $\mathbb{R}\times\mathbb{R}^+$.

Fundamental works on wavelet theory are due to Meyer [16], Daubechies [9], and Mallat [15]. This paper is devoted to the theory and usefulness of wavelets described in the following definition.

DEFINITION 2.2. A *generalized Haar wavelet of degree* M is a wavelet with the property that there exist $M \in \mathbb{R}$ and $s_i \in \mathbb{R}$ such that $\psi|_{[s_i,s_{i+1})} = c_i \in \mathbb{C}$ and $Ms_i \in \mathbb{Z}$ for all $i \in \mathbb{Z}$.

Note that generalized Haar wavelets are bounded functions, and that their coefficients are summable, i.e., $\{c_i\}_{i\in\mathbb{Z}} \in l^1(\mathbb{Z})$.

The first observation in our approach to periodicity detection and computation is the following fact.

PROPOSITION 2.3. *Let $f \in L^1(\mathbb{T}_T)$, i.e., f is T–periodic and integrable on $[0,T]$, and let ψ be a generalized Haar wavelet of degree M. Then $W_\psi f(b,a)$ is T–periodic in b and MT–periodic in a.*

This result can be proved by a direct calculation. A similar calculation is carried out in the proof of Proposition 2.5.

EXAMPLE 2.4. We choose $f \in L^1(\mathbb{T}_T)$ to be $f(\cdot) = \sin(2\pi(\gamma\cdot+\theta))$, where $\gamma, \theta \in \mathbb{R}$ are fixed, and $\gamma T \in \mathbb{Z} \setminus \{0\}$. If the generalized Haar wavelet is the centered Haar wavelet $\psi = \mathbf{1}_{[-\frac{1}{2},0)} + \mathbf{1}_{[0,\frac{1}{2})}$, then ψ is of degree 2, and

$$W_\psi f(b,a) = \frac{2}{\pi\gamma}\sin^2(\tfrac{\pi\gamma a}{2})\cos(2\pi(\gamma b+\theta)), \tag{2.1}$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R}^+$. Clearly, the right side of (2.1) is T–periodic in b and $2T$–periodic in a.

The graph of $W_\psi f(b,a)$, for $\theta = 0$ and $\gamma = T = 1$, is illustrated in Figure 2.1.

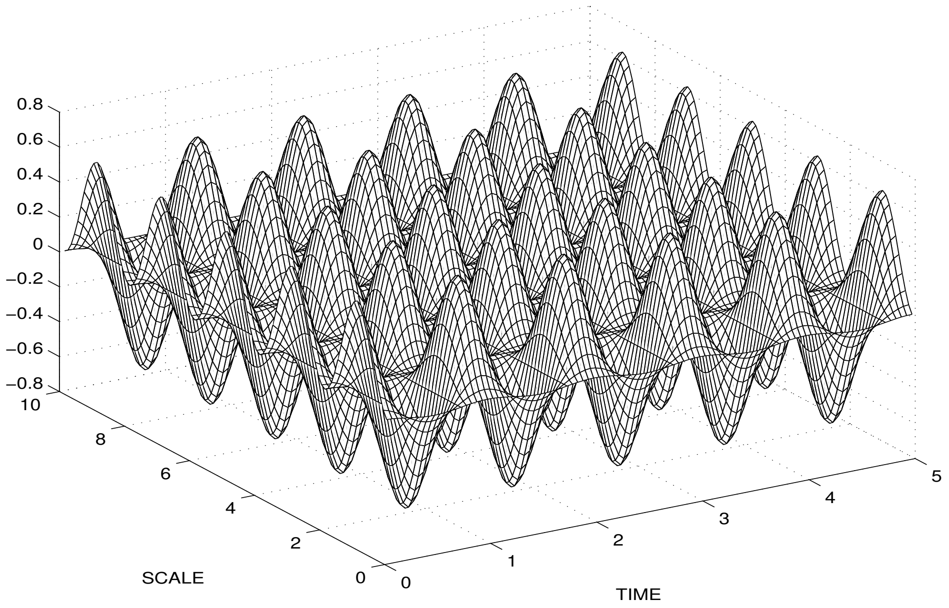

FIG. 2.1. *Non–normalized Haar wavelet transform of a sine function.*

Figure 2.2 illustrates the reason for periodicity in time. In fact, for a fixed scale, moving the wavelet by a full period across time does not change the innerproduct

$$\langle f, \psi(\tfrac{\cdot-b}{a})\rangle = W_\psi f(b,a).$$

Figure 2.3 illustrates the cancellations leading to periodicity in scale. These are due to the fact that $\sum c_i = 0$.

Proposition 2.3 implies that if the signal s has the particular form $s(t) = Af(ct)$ for constants A and c, then the relative maxima of $W_\psi s(b,a)$ form a lattice in time–scale space. The horizontal (time) distance between two neighboring vertices of the

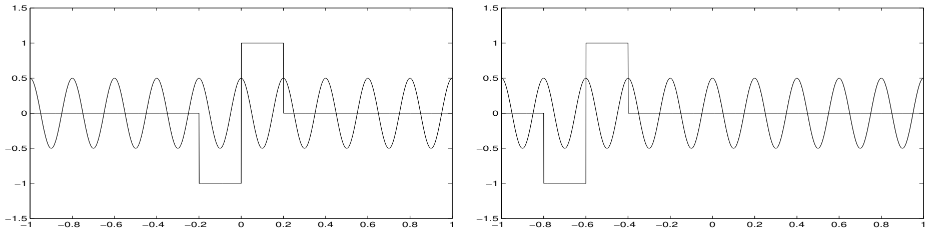

FIG. 2.2. *Periodicity in time.*

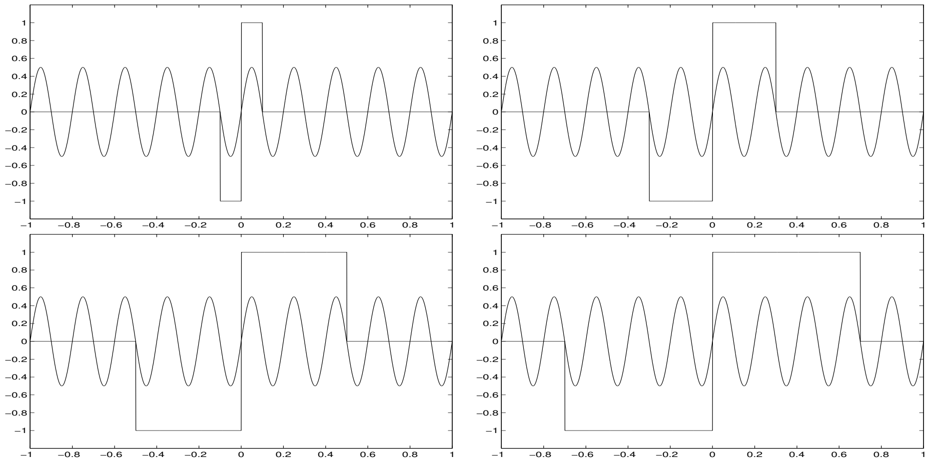

FIG. 2.3. *Periodicity in scale is caused by cancellations in the continuous wavelet transform.*

lattice is $1/c$, and the vertical (scale) distance between two neighboring vertices is M/c. This regularity displays redundancy in the following way: each rectangle of size $1/c \times M/c$ in the wavelet transform contains all of the information in the whole wavelet transform.

Additional structure of ψ can force additional features upon the wavelet transform of periodic functions, as can be seen in the following proposition. This approach will be discussed further in Section 4.3.

PROPOSITION 2.5. *Let $f \in L^1(\mathbb{T}_T)$, and let ψ be a generalized Haar wavelet of degree M.*
a. If ψ is even, i.e., $\psi(-t) = \psi(t)$ for $t \in \mathbb{R}$, then $W_\psi f(b,a) = -W_\psi f(b, MT-a)$ for $0 < a < MT$.
b. If ψ is odd, i.e. $-\psi(-t) = \psi(t)$ for $t \in \mathbb{R}$, then $W_\psi f(b,a) = W_\psi f(b, MT-a)$ for $0 < a < MT$.

Proof. a. Since ψ is a generalized Haar wavelet of degree M and since ψ is even, there exist $s_i \in \mathbb{R}$ such that $\psi|_{[s_{-(i+1)}, s_{-i})} = \psi|_{[s_i, s_{i+1})} = c_i$, $i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$, and $s_{-i} = -s_i$ for $i \in \mathbb{Z}$.

We compute

$$W_\psi f(b, MT-a) + W_\psi f(b,a) = \int f(t)\psi(\tfrac{t-b}{MT-a})\,dt + \int f(t)\psi(\tfrac{t-b}{a})\,dt$$

$$= \sum_{i\geq 0} c_i \left(\int_{(MT-a)s_i+b}^{(MT-a)s_{i+1}+b} f(t)\,dt + \int_{(MT-a)s_{-(i+1)}+b}^{(MT-a)s_{-i}+b} f(t)\,dt \right.$$
$$\left. + \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{as_{-(i+1)}+b}^{as_{-i}+b} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_{-as_i+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt + \int_{as_{i+1}+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt \right.$$
$$\left. + \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_{-as_{i+1}+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt + \int_{as_i+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_0^{MT(s_{i+1}-s_i)} f(t)\,dt + \int_0^{MT(s_{i+1}-s_i)} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i 2M(s_{i+1}-s_i) \int_0^T f(t)\,dt = M \int \psi(t)\,dt \int_0^T f(t)\,dt = 0,$$

where the last step follows since $\int \psi(t)\,dt = 0$.

b. Since ψ is odd, there exist $s_i \in \mathbb{R}$ such that $-\psi|_{[s_{-(i+1)},s_{-i})} = \psi|_{[s_i,s_{i+1})} = c_i$, $i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$, and $s_{-i} = -s_i$ for $i \in \mathbb{Z}$.

We compute

$$W_\psi f(b, MT-a) - W_\psi f(b,a) = \int f(t)\psi(\tfrac{t-b}{MT-a})\,dt - \int f(t)\psi(\tfrac{t-b}{a})\,dt$$

$$= \sum_{i\geq 0} c_i \left(\int_{(MT-a)s_i+b}^{(MT-a)s_{i+1}+b} f(t)\,dt - \int_{(MT-a)s_{-(i+1)}+b}^{(MT-a)s_{-i}+b} f(t)\,dt \right.$$
$$\left. - \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{as_{-(i+1)}+b}^{as_{-i}+b} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_{MTs_i-as_i+b}^{MTs_{i+1}-as_{i+1}+b} f(t)\,dt - \int_{-MTs_{i+1}+as_{i+1}+b}^{-MTs_i+as_i+b} f(t)\,dt \right.$$
$$\left. - \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_{-as_{i+1}+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt - \int_{as_i+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt \right)$$
$$= \sum_{i\geq 0} c_i \left(\int_0^{MT(s_{i+1}-s_i)} f(t)\,dt - \int_0^{MT(s_{i+1}-s_i)} f(t)\,dt \right) = 0.$$

Note that in part b we did not explicitly use the fact that $\int \psi(t)\,dt = 0$. Nevertheless, $\int \psi(t)\,dt = 0$ since ψ is odd. □

3. Characterization of wavelets with periodic wavelet transforms. The following theorem completely classifies all wavelets which have the property that the non–normalized wavelet transform of any periodic function $f \in L^\infty(\mathbb{R})$ is periodic in scale. Recall that continuous wavelet transforms of periodic functions are always periodic in time.

THEOREM 3.1. *Let $\psi \in L^1(\mathbb{R})$. The following are equivalent:*
i. $W_\psi f(b,a) = \int f(t)\psi(\frac{t-b}{a})\,dt$ *is 1–periodic in a for all $f \in L^\infty(\mathbb{T})$.* (P)
ii. $\widehat{\psi}(0) = 0$ *and ψ has the form*

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbb{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$ *and* $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbb{Z})$.

REMARK 3.2. *a.* Our proof of Theorem 3.1 is classical and detailed to make it readily accessible. Some distributional arguments may have saved a few lines, and some of the lemmas could have been integrated with each other, but we have chosen to exposit the proof as follows to exhibit each of the steps in elementary terms.

b. Theorem 3.1, as well as most results and remarks in this section, has a trivial generalization to S–periodic functions. In fact, if $\psi \in L^1(\mathbb{R})$ has the property that $W_\psi f(b,a)$ is T–periodic in a for all $f \in L^\infty(\mathbb{T}_S)$, then $\tilde{\psi}$ defined by $\tilde{\psi}(t) = \psi(\frac{S}{T}t)$ has property (P). Hence $\tilde{\psi}$ has the form

$$\tilde{\psi}(\cdot) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbb{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$ and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbb{Z})$. Consequently, ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(\frac{Tn}{S},\frac{Tn+T}{S})}(\cdot) + \sum_{n\in\mathbb{Z}} b_n \ln\left|\tfrac{T\,\cdot - Sn}{S}\right|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$ and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbb{Z})$.

To prove Theorem 3.1, we need to establish several lemmas (Lemma 3.3, Lemma 3.5, Lemma 3.6, Lemma 3.7, Lemma 3.10, and Lemma 3.11).

LEMMA 3.3. *Let $\psi \in L^1(\mathbb{R})$. The following are equivalent:*
i. $W_\psi f(b,a) = \int f(t)\psi(\frac{t-b}{a})\,dt$ *is 1–periodic in a for all $f \in L^\infty(\mathbb{T})$.*
ii. $\widehat{\psi}(0) = 0$ *and there exists a continuous function φ on $\mathbb{R}$, 1–periodic on $\widehat{\mathbb{R}}^+$ and 1–periodic on $\widehat{\mathbb{R}}^-$, such that*

$$\forall \gamma \in \widehat{\mathbb{R}} \setminus \{0\}, \quad \widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}.$$

Proof. $i \Longrightarrow ii$. Let us assume $W_\psi f(b,a) = \int f(t)\psi(\frac{t-b}{a})\,dt$ is $T = 1$ periodic in a for all $f \in L^\infty(\mathbb{T})$. Define

$$\forall k \in \mathbb{Z}, \quad S_k(t) = \sum_{|n|\le k} \left(\psi(\tfrac{t-n-b}{a}) - \psi(\tfrac{t-n-b}{a+1})\right).$$

Then, for fixed $b \in \mathbb{R}$ and $a \in \mathbb{R}^+$, we obtain

$$\begin{aligned} 0 &= W_\psi f(b,a) - W_\psi f(b,a+1) \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(t-n) \left(\psi\left(\tfrac{t-n-b}{a}\right) - \psi\left(\tfrac{t-n-b}{a+1}\right)\right) dt = \lim_{k\to\infty} \int_0^1 f(t) S_k(t)\, dt. \end{aligned} \quad (3.1)$$

We can interchange the limit and the integral in (3.1) using the Lebesgue dominated convergence theorem on the partial sums S_k; in fact,

$$\forall k \geq 0, \quad |S_k(t)| \leq \sum_{n\in\mathbb{Z}} \left(|\psi(\tfrac{t-n-b}{a})| + |\psi(\tfrac{t-n-b}{a+1})|\right), \quad (3.2)$$

where the right side of (3.2) is an element of $L^1(\mathbb{T})$ since $\psi \in L^1(\mathbb{R})$.

The calculation in (3.1) shows that if $W_\psi f$ is 1–periodic in scale for all $f \in L^\infty(\mathbb{T})$, then

$$S(t) = \sum_{n\in\mathbb{Z}} \left(\psi(\tfrac{t-n-b}{a}) - \psi(\tfrac{t-n-b}{a+1})\right) = 0.$$

Thus, for all $m \in \mathbb{Z}$, we have

$$\begin{aligned} 0 = \widehat{S}[m] &= \int_0^1 \sum_{n\in\mathbb{Z}} \left(\psi(\tfrac{t-n-b}{a}) - \psi(\tfrac{t-n-b}{a+1})\right) e^{-2\pi imt}\, dt \quad (3.3) \\ &= \sum_{n\in\mathbb{Z}} \left(\int_0^1 \psi(\tfrac{t-n-b}{a}) e^{-2\pi imt}\, dt - \int_0^1 \psi(\tfrac{t-n-b}{a+1}) e^{-2\pi imt}\, dt\right) \\ &= \sum_{n\in\mathbb{Z}} \left(\int_{\frac{0-b-n}{a}}^{\frac{1-b-n}{a}} \psi(u) e^{-2\pi im(au+n+b)} a\, du \right. \\ &\qquad \left. - \int_{\frac{0-b-n}{a+1}}^{\frac{1-b-n}{a+1}} \psi(u) e^{-2\pi im((a+1)u+n+b)} (a+1)\, du\right) \\ &= \int \psi(u) e^{-2\pi im(au+b)} a\, du - \int \psi(u) e^{-2\pi im((a+1)u+b)} (a+1)\, du \\ &= e^{-2\pi imb} (a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1))). \end{aligned}$$

Interchanging the sum and the integral in (3.3) is justified by the Lebesgue dominated convergence theorem.

Because of (3.3), we have

$$a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) = 0$$

for all $a \in \mathbb{R}^+$ and all $m \in \mathbb{Z}$, and in particular $\widehat{\psi}(0) = 0$. If we let

$$\varphi(\gamma) = \gamma\widehat{\psi}(\gamma)$$

for $\gamma \in \widehat{\mathbb{R}}$, and take $m = 1$, we obtain for $\gamma \in \widehat{\mathbb{R}}^+$ that

$$\varphi(\gamma) - \varphi(\gamma+1) = \gamma\widehat{\psi}(\gamma) - (\gamma+1)\widehat{\psi}(\gamma+1) = 0.$$

Hence, $\varphi(\gamma) = \varphi(\gamma+1)$ for all $\gamma \in \widehat{\mathbb{R}}^+$. For $\gamma \in \widehat{\mathbb{R}}^-$ and $a = -\gamma > 0$, we set $m = -1$ and obtain

$$\varphi(\gamma) - \varphi(\gamma-1) = 0.$$

Thus, $\varphi(\gamma) = \varphi(\gamma - 1)$ for all $\gamma \in \widehat{\mathbb{R}}^-$, and *ii* holds.

ii $\Longrightarrow$ *i*. Conversely, if statement *ii* holds, we have for $m > 0$ that

$$\begin{aligned} a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) &= a\frac{\varphi(ma)}{ma} - (a+1)\frac{\varphi(m(a+1))}{m(a+1)} \\ &= \frac{1}{m}(\varphi(ma) - \varphi(ma+m)) = 0, \end{aligned}$$

since $m, am > 0$ and by the 1–periodicity of φ on $\widehat{\mathbb{R}}^+$. Similarly, for $m < 0$,

$$a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) = \frac{1}{m}(\varphi(ma) - \varphi(ma+m)) = 0,$$

since now $m, am < 0$ and the 1–periodicity of φ on $\widehat{\mathbb{R}}^-$ applies. Also, $\widehat{\psi}(0) = 0$ implies $\widehat{S}[0] = 0$.

Hence, $\widehat{S}[m] = 0$ for all $m \in \mathbb{Z}$. By the uniqueness theorem for Fourier transforms we have $S = 0$ in $L^1(\mathbb{T})$. Thus, the periodicity in scale follows, since

$$W_\psi f(b,a) - W_\psi f(b, a+1) = \int_0^1 f(t)S(t)\,dt = 0;$$

and *i* is obtained. ☐

Remark 3.4. *a.* Lemma 3.3 implies that if ψ satisfies property (P), then $\widehat{\psi}$ has the form

$$\widehat{\psi}(\gamma) = \frac{\varphi_1(\gamma)}{\gamma} + \mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma},$$

where φ_1 and φ_2 are 1–periodic on all of $\widehat{\mathbb{R}}$, and H denotes the Heaviside function, i.e., $\mathrm{H} = \mathbf{1}_{(0,\infty)}$.

b. If ψ has property (P), then $\widehat{\psi}(\gamma) = O(\frac{1}{\gamma})$ and $\widehat{\psi}(\gamma) \neq o(\frac{1}{\gamma})$ as $|\gamma| \to \infty$. In particular, if ψ has property (P) then it is not absolutely continuous.

c. Equation (3.1) in the proof of Lemma 3.3 implies that, for fixed $\psi \in L^1(\mathbb{R})$ and fixed $f \in L^\infty(\mathbb{T})$, $W_\psi f$ is 1–periodic in scale if and only if

$$\int_0^1 f(t) \sum_{n\in\mathbb{Z}} \left(\psi(\tfrac{t-n-b}{a}) - \psi(\tfrac{t-n-b}{a+1})\right)\,dt = 0,$$

for all $a \in \mathbb{R}^+$ and all $b \in \mathbb{R}$. This can be helpful if we are interested in picking out one specific periodic component $f \in L^\infty(\mathbb{T})$ in a signal that carries other periodic components besides f. The fact that for periodic $g \neq f \in L^\infty(\mathbb{T})$ the wavelet transform of g might not be periodic implies that the components of g in a signal get blurred in the wavelet transform. This can be helpful to distinguish periodic signals of different shapes.

Lemma 3.5 and Lemma 3.6 will be used to prove Lemma 3.7, and both Lemma 3.6 and 3.7 are used explicitly in the proof of Theorem 3.1

Lemma 3.5. *For all* $0 < \epsilon < 1$ *and* $\gamma \in \widehat{\mathbb{R}}$ *we have*

$$\left| \int_\epsilon^{\frac{1}{\epsilon}} \frac{\sin(2\pi t\gamma)}{t}\,dt \right| \leq 5\pi.$$

The proof is standard and is omitted.

Before stating and proving Lemma 3.6, we shall recall a few facts from harmonic analysis. Let w denote the Fejér kernel defined as

$$\forall t \in \mathbb{R}, \quad w(t) = \frac{1}{2\pi}\left(\frac{\sin\left(\frac{t}{2}\right)}{\frac{t}{2}}\right)^2.$$

Clearly, $w \in L^1(\mathbb{R}) \cap C^{(1)}(\mathbb{R})$ and $w' \in L^1(\mathbb{R})$, since

$$w'(t) = \sin(\tfrac{t}{2})\left[\frac{\cos\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)^2} - \frac{\sin\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)^2}\right],$$

and therefore $w'(t) = O(\frac{1}{t^2})$, $|t| \to \infty$. The Fourier transform $\widehat{w}$ of w is

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{w}(\gamma) = \max\{0, 1 - |\gamma|\}.$$

The de la Vallée–Poussin kernel v is then defined as

$$\forall t \in \mathbb{R}, \quad v(t) = 4w(2t) - w(t).$$

Thus, $v \in L^1(\mathbb{R}) \cap C^{(1)}(\mathbb{R})$ and $v' \in L^1(\mathbb{R})$. Clearly,

$$\widehat{v}(\gamma) = 2\widehat{w}(\tfrac{\gamma}{2}) - \widehat{w}(\gamma),$$

and therefore $\widehat{v}(\gamma) = 1$ for $\gamma \in [-1,1]$ and $\widehat{v}(\gamma) = 0$ for $\gamma \notin [-2,2]$. Note that since $v, v' \in L^1(\mathbb{R})$ and

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{v'}(\gamma) = i\gamma\widehat{v}(\gamma),$$

we have $\gamma\widehat{v}(\gamma) \in A(\widehat{\mathbb{R}})$.

Lemma 3.6. *Let $\psi \in L^1(\mathbb{R})$ be such that for all $\gamma \in \widehat{\mathbb{R}}^+$, resp., for all $\gamma \in \widehat{\mathbb{R}}^-$,*

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$$

with φ 1–periodic on $\widehat{\mathbb{R}}$. Then $\varphi \in A(\mathbb{T})$ and, hence, the Fourier Series $S(\varphi)(\gamma) = \sum_{n\in\mathbb{Z}} b_n e^{-2\pi i n\gamma}$, converges to φ absolutely and uniformly, i.e., $\{b_n\} \in l^1(\mathbb{Z})$.

Proof. Fix $\gamma_0 \in [1,2)$, resp., $\gamma_0 \in [-2,1)$. Define

$$\forall t \in \mathbb{R}, \quad v_{\gamma_0}(t) = \tfrac{1}{8}v(\tfrac{t}{8})e^{2\pi i\gamma_0 t}.$$

Then, clearly,

$$\widehat{v_{\gamma_0}}(\gamma) = \widehat{v}(8(\gamma - \gamma_0)),$$

$\gamma \in \widehat{\mathbb{R}}$, $\widehat{v_{\gamma_0}}(\gamma) = 1$ for $\gamma \in [\gamma_0 - \frac{1}{8}, \gamma_0 + \frac{1}{8}]$, and $\mathrm{supp}(\widehat{v_{\gamma_0}}) \subseteq [\gamma_0 - \frac{1}{4}, \gamma_0 + \frac{1}{4}] \subseteq [\frac{3}{4}, \frac{9}{4}] \subseteq \widehat{\mathbb{R}}^+$, resp., $\mathrm{supp}(\widehat{v_{\gamma_0}}) \subseteq \widehat{\mathbb{R}}^-$. As before, $\gamma\widehat{v_{\gamma_0}}(\gamma) \in A(\widehat{\mathbb{R}})$.

Since $\widehat{\psi} \in A(\widehat{\mathbb{R}})$ we have $\widehat{v_{\gamma_0}}(\gamma)\varphi(\gamma) = \gamma\widehat{v_{\gamma_0}}(\gamma)\widehat{\psi}(\gamma) \in A(\widehat{\mathbb{R}})$. Therefore, by a theorem of Wiener ([30], [2], page 202, [23], page 56), $\widehat{v_{\gamma_0}}\varphi \in A(\mathbb{T})$. Since $\varphi = \widehat{v_{\gamma_0}}\varphi$ in a neighborhood of γ_0, we have $\varphi \in A_{loc(\gamma_0)}(\mathbb{T})$. This result holds for any $\gamma_0 \in [1,2]$, resp., $[-2,-1)$, and hence $\varphi \in A_{loc}(\mathbb{T})$. By Wiener's local membership theorem we have $\varphi \in A(\mathbb{T})$ [2], page 200, [23].

Therefore, we can write $\varphi(\gamma) = \sum_{n\in\mathbb{Z}} b_n e^{-2\pi n\gamma}$ with $\{b_n\} \in l^1(\mathbb{Z})$ and the result is proven. ◻

LEMMA 3.7. *Let $\psi \in L^2(\mathbb{R})$ be such that*

$$\forall \gamma \in \widehat{\mathbb{R}} \setminus \{0\}, \quad \mathcal{F}(\psi)(\gamma) = \frac{\varphi(\gamma)}{\gamma},$$

where $\varphi \in A(\mathbb{T})$. Then ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} \pi i a_n \ \mathrm{sgn}(\cdot - n) = \sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)}(\cdot), \tag{3.4}$$

where $\{a_n\} \in l^1(\mathbb{Z})$ is the sequence of Fourier coefficients of φ and where $\{c_n\} \in l^2(\mathbb{Z})$ is defined by $c_n = 2\pi i \sum_{k\leq n} a_n$. The convergence on the right hand side of (3.4) is pointwise for $t \notin \mathbb{Z}$, as well as in $L^2(\mathbb{R})$.

Thus, for $c_n = 2\pi i \sum_{k\leq n} a_n$, we have the $\mathcal{F}$–pairing,

$$\sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)}(t) \longleftrightarrow \frac{1}{\gamma} \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\gamma}. \tag{3.5}$$

Proof. Clearly, $\widehat{\psi} \in L^2(\widehat{\mathbb{R}})$ since $\psi \in L^2(\mathbb{R})$. Thus, we can apply the L^2–inversion formula

$$\psi(t) = \lim_{N\to\infty} \int_{-N}^{N} \widehat{\psi}(\gamma) e^{2\pi i\gamma t}\, d\gamma,$$

with convergence of this limit in $L^2(\mathbb{R})$. We also have

$$\varphi(\gamma) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\gamma}$$

with $\{a_n\} \in l^1(\mathbb{Z})$ since $\varphi \in A(\mathbb{T})$. We obtain

$$\begin{aligned}
\psi(t) &= \lim_{N\to\infty} \int_{-N}^{N} \widehat{\psi}(\gamma) e^{2\pi i\gamma t} d\gamma = \lim_{N\to\infty} \int_{\frac{1}{N}\leq|\gamma|\leq N} \frac{1}{\gamma} \left(\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\gamma} \right) e^{2\pi i\gamma t}\, d\gamma \\
&= \lim_{N\to\infty} \int_{\frac{1}{N}\leq|\gamma|\leq N} \sum_{n\in\mathbb{Z}} a_n \frac{e^{2\pi i(t-n)\gamma}}{\gamma}\, d\gamma = \lim_{N\to\infty} \sum_{n\in\mathbb{Z}} a_n \int_{\frac{1}{N}\leq|\gamma|\leq N} \frac{e^{2\pi i(t-n)\gamma}}{\gamma}\, d\gamma \\
&= \lim_{N\to\infty} \sum_{n\in\mathbb{Z}} a_n \int_{\frac{1}{N}\leq|\gamma|\leq N} \left(\frac{\cos(2\pi(t-n)\gamma)}{\gamma} + i\, \frac{\sin(2\pi(t-n)\gamma)}{\gamma} \right) d\gamma \\
&= i \lim_{N\to\infty} \sum_{n\in\mathbb{Z}} a_n \int_{\frac{1}{N}\leq|\gamma|\leq N} \frac{\sin(2\pi(t-n)\gamma)}{\gamma}\, d\gamma \\
&= i \sum_{n\in\mathbb{Z}} a_n \lim_{N\to\infty} \int_{\frac{1}{N}\leq|\gamma|\leq N} \frac{\sin(2\pi(t-n)\gamma)}{\gamma}\, d\gamma \\
&= i \sum_{n\in\mathbb{Z}} a_n \left\{ \begin{array}{ccc} \pi & \text{for} & t > n \\ 0 & \text{for} & t = n \\ -\pi & \text{for} & t < n \end{array} \right\},
\end{aligned}$$

where the interchange of integration and summation is true, since $\{a_n\} \in l^1(\mathbb{Z})$ (Lemma 3.6), and where the last equation is a consequence of Lemma 3.5.

Further note that for $t \in (k, k+1)$ we have

$$\begin{aligned}\psi(t) &= \sum_{n\in\mathbb{Z}} \pi i a_n \operatorname{sgn}(t-n) = \sum_{n\leq k} \pi i a_n - \sum_{n\geq k+1} \pi i a_n \\ &= \sum_{n\leq k} \pi i a_n + \sum_{n\in\mathbb{Z}} \pi i a_n - \sum_{n\geq k+1} \pi i a_n = 2\pi i \sum_{n\leq k} a_n = c_k.\end{aligned}$$

Finally, we have $\|\{c_n\}\|_{l^2(\mathbb{Z})} = \|\psi\|_{L^2(\mathbb{R})} < \infty$ and therefore $\{c_n\} \in l^2(\mathbb{Z})$. □

The following theorem is a corollary of Lemmas 3.6 and 3.7.

THEOREM 3.8. *Let $\psi \in L^1(\mathbb{R})$ be defined by*

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma},$$

where φ is 1–periodic on $\widehat{\mathbb{R}}$ and $\widehat{\psi}(0) = 0$. Then ψ is a generalized Haar wavelet of degree 1. In fact,

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} \pi i a_n \operatorname{sgn}(\cdot - n) = \sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)}(\cdot),$$

where

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \varphi(\gamma) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\gamma},$$

and $\{c_n\} \in l^1(\mathbb{Z})$, where

$$c_n = 2\pi i \sum_{k\leq n} a_k.$$

Proof. By our hypotheses, Lemma 3.6 implies that $\varphi \in A(\mathbb{T})$. The result then follows from Lemma 3.7. Since $\psi \in L^1(\mathbb{R})$, we obtain the fact that $\{c_n\} \in l^1(\mathbb{Z}) \subseteq l^2(\mathbb{Z})$ by the calculation $\sum_{n\in\mathbb{Z}} |c_n| = \|\psi\|_{L^1(\mathbb{R})} < \infty$. □

To obtain the main result Theorem 3.1, restated ahead as Theorem 3.12, we need two more lemmas (Lemma 3.10 and Lemma 3.11), the first of which requires a fundamental property of Hilbert transforms stated in Theorem 3.9.

The *Hilbert transform* of a function f is formally defined by

$$\mathcal{H}(f)(t) = \lim_{\epsilon\to 0} \int_{|t-u|\leq\epsilon} \frac{f(u)}{t-u}\, du.$$

For $f \in L^2(\mathbb{R})$ this limit exists for almost every $t \in \mathbb{R}$. Proofs of this fact and the following result can be found in [17, 2].

THEOREM 3.9. *$\mathcal{H} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a well-defined isometry. The Hilbert transform and the Fourier transform are related by the equation,*

$$\forall f \in L^2(\mathbb{R}), \quad \mathcal{H}(f) = \mathcal{F}^{-1}\left(-i \operatorname{sgn}(\cdot) \cdot (\mathcal{F}(f))\right).$$

LEMMA 3.10. *Let $\psi \in L^2(\mathbb{R})$ have the property that*

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \mathcal{F}(\psi)(\gamma) = \mathrm{H}(\gamma)\frac{\varphi(\gamma)}{\gamma},$$

where $\varphi \in A(\mathbb{T})$. Then there exists $\{d_n\} \in l^2(\mathbb{Z})$ and $\{b_n\} \in l^1(\mathbb{Z})$ such that $\sum_{n\in\mathbb{Z}} b_n = 0$ and

$$\psi(\cdot) = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbb{Z}} b_n \ln|\cdot - n|$$

with pointwise convergence for $t \notin \mathbb{Z}$, as well as convergence in $L^2(\mathbb{R})$.

Thus, if $d_n = 2\pi i \sum_{k\leq n} b_k$, we have the $\mathcal{F}$–pairing

$$\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n| \longleftrightarrow \mathrm{H}(\gamma)\frac{1}{\gamma}\sum_{n\in\mathbb{Z}} b_n e^{-2\pi i n\gamma}. \tag{3.6}$$

Proof. Let $\Theta(\gamma) = \frac{\varphi(\gamma)}{\gamma}$, where $\varphi(\gamma) = \sum_{n\in\mathbb{Z}} b_n e^{-2\pi i n\gamma}$. Clearly $\Theta \in L^2(\widehat{\mathbb{R}})$. Lemma 3.7 implies that

$$\mathcal{F}^{-1}(\Theta) = \sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)},$$

where

$$d_n = 2\pi i \sum_{k\leq n} b_k$$

and $\{d_n\} \in l^2(\mathbb{Z})$. Define

$$g = \tfrac{1}{2}\mathcal{F}^{-1}(\Theta) - \tfrac{1}{2i}\mathcal{H}(\mathcal{F}^{-1}(\Theta)).$$

Clearly, $g \in L^2(\mathbb{R})$, and

$$\mathcal{F}(g) = \tfrac{1}{2}\Theta - \tfrac{1}{2i}\mathcal{F}(\mathcal{F}^{-1}(-i \ \mathrm{sgn}\ \mathcal{F}(\mathcal{F}^{-1}(\Theta)))) = \tfrac{1}{2}\Theta + \tfrac{1}{2}\,\mathrm{sgn}\,\Theta = \mathrm{H}\Theta.$$

Hence, $g = \mathcal{F}^{-1}(\mathrm{H}\Theta)$. Further, for $t \notin \mathbb{Z}$, we compute

$$\begin{aligned}
\mathcal{H}(\mathcal{F}^{-1}(\Theta))(t) &= \lim_{\epsilon\to 0}\tfrac{1}{\pi}\int_{|t-u|\geq\epsilon} \frac{\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)}(u)}{t-u}\,du \\
&= -\lim_{\epsilon\to 0}\tfrac{1}{\pi}\int_{|x|\geq\epsilon} \frac{\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(t-n-1,t-n)}(x)}{x}\,dx \\
&= -\tfrac{1}{\pi}\sum_{n\in\mathbb{Z}} d_n \int_{t-n-1}^{t-n} \frac{1}{x}\,dx = -\tfrac{1}{\pi}\sum_{n\in\mathbb{Z}}(d_n - d_{n-1})\ln|t-n| \\
&= -\tfrac{1}{\pi}\sum_{n\in\mathbb{Z}}(2\pi i\sum_{k<n} a_k - 2\pi i \sum_{k<n-1} a_k)\ln|t-n| = -2i\sum_{n\in\mathbb{Z}} b_n \ln|t-n|.
\end{aligned}$$

Therefore, for such $t \notin \mathbb{Z}$,

$$g(t) = \tfrac{1}{2}\mathcal{F}^{-1}(\Theta)(t) - \tfrac{1}{2i}\mathcal{H}(\mathcal{F}^{-1}(\Theta))(t) = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|.$$

□

LEMMA 3.11. *Let $\psi \in L^1(\mathbb{R})$ have the properties that $\widehat{\psi}(0) = 0$ and that for $t \notin \mathbb{Z}$,*

$$\psi(t) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|,$$

where $\{a_n\}$, $\{b_n\} \in l^1(\mathbb{Z})$, and $\{c_n\} \in l^2(\mathbb{Z})$ satisfy the conditions

$$a_n = \tfrac{1}{2\pi i}(e_n - e_{n-1}) - \tfrac{1}{2} b_n,$$

and

$$c_n = e_n - \pi i \sum_{k\le n} b_k.$$

Then $\psi \in L^2(\mathbb{R})$.

Proof. We shall prove this result in four steps.

i. There exists C_1 such that $|e_n| \le C_1$ for all $n \in \mathbb{Z}$.

The hypothesis

$$a_n = \tfrac{1}{2\pi i}(e_n - e_{n-1}) - \tfrac{1}{2} b_n,$$

yields

$$2\pi i \sum_{1\le n\le N} a_n = \sum_{1\le n\le N} (e_n - e_{n-1}) - \pi i \sum_{1\le n\le N} b_n = e_N - e_{1-1} - \pi i \sum_{1\le n\le N} b_n$$

for all $N \ge 1$. Therefore, for all such N, we have

$$\begin{aligned} |e_N - e_0| &= |\pi i \sum_{1\le n\le N} b_n + 2\pi i \sum_{1\le n\le N} a_n| \le \pi(\sum_{1\le n\le N} |b_n| + 2 \sum_{1\le n\le N} |a_n|) \\ &\le \pi(\|\{b_n\}\|_{l^1(\mathbb{Z})} + 2\,\|\{a_n\}\|_{l^1(\mathbb{Z})}), \end{aligned}$$

since $\{a_n\}$ and $\{b_n\} \in l^1(\mathbb{Z})$. Setting

$$C_1 = \pi(\|\{b_n\}\|_{l^1(\mathbb{Z})} + 2\,\|\{a_n\}\|_{l^1(\mathbb{Z})}) + |e_0|,$$

we obtain

$$\forall N \ge 1, \quad |e_N| \le C_1.$$

Clearly, the same bound holds for negative N and $|e_0|$, and step i is complete.

For $n \in \mathbb{Z}$, define $\forall t \in [n - \frac{1}{2}, n + \frac{1}{2}]$,

$$g_n(t) = \sum_{k\ne n} b_k \ln|t-k| = \psi(t) - b_n \ln|t-n| - e_{n-1}\mathbf{1}_{(n-1,n)}(t) - e_n \mathbf{1}_{(n,n+1)}(t).$$

The sum converges pointwise.

ii. There exists C_2 such that $|g_n'| \le C_2$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ for all $n \in \mathbb{Z}$.

Fix $n \in \mathbb{Z}$. For $k \ne n$ let

$$S_k(t) = b_k \ln|t-k|.$$

S_k is continuously differentiable on the interval $[n-\frac{1}{2}, n+\frac{1}{2}]$, and

$$\forall t \in [n-\tfrac{1}{2}, n+\tfrac{1}{2}], \quad |S_k{}'(t)| = |b_k \frac{1}{t-k}| \leq 2|b_k|.$$

Since $\{b_n\} \in l^1(\mathbb{Z})$, we can define

$$h_n(t) = \sum_{k \neq n} b_k \frac{1}{t-k}$$

with uniform convergence on $[n-\frac{1}{2}, n+\frac{1}{2}]$. Hence, g_n is continuously differentiable on $[n-\frac{1}{2}, n+\frac{1}{2}]$ and $g_n' = h_n$. Letting $C_2 = 2\,\|\{b_n\}\|_{l^1(\mathbb{Z})}$ we have

$$|g_n'(t)| \leq C_2$$

for $t \in [n-\frac{1}{2}, n+\frac{1}{2}]$.
iii. There exist $N > 0$ and C_3 such that $|g_n| \leq C_3$ on $[n-\frac{1}{2}, n+\frac{1}{2}]$ for all n for which $|n| \geq N$.

There exists N such that for all $n \geq N$ there is a $t_n \in [n-\frac{1}{2}, n-\frac{1}{4}]$ for which $\psi(t_n) \leq 1$. This statement holds since, if there were infinitely many n_k with $\psi(t) \geq 1$ for all $t \in [n_k - \frac{1}{2}, n_k - \frac{1}{4}]$, we would obtain

$$\int |\psi(t)|\, dt \geq \sum_{k \in \mathbb{N}} \int_{n_k - \frac{1}{2}}^{n_k - \frac{1}{4}} dt = \infty,$$

which contradicts the hypothesis that $\psi \in L^1(\mathbb{R})$.

Let $|b_n| \leq C_4$ for all $n \in \mathbb{Z}$, and set $C_5 = 1 + C_4 \ln 4 + 2C_1$. We obtain

$$\begin{aligned} |g_n(t_n)| &= |\psi(t_n) - b_n \ln|t_n - n| - e_{n-1}\mathbf{1}_{(n-1,n)}(t_n) - e_n \mathbf{1}_{(n,n+1)}(t_n)| \\ &\leq 1 + |b_n| \ln 4 + |e_{n-1}| + |e_n| \leq 1 + C_4 \ln 4 + 2C_1 = C_5. \end{aligned}$$

Set $C_3 = C_2 + C_5$. Then, for all $t \in [n-\frac{1}{2}, n+\frac{1}{2}]$, there exists $\xi_n \in [\min\{t, t_n\}, \max\{t, t_n\}]$, such that

$$|g_n(t)| \leq |g_n(t) - g_n(t_n)| + |g_n(t_n)| \leq |(t - t_n) g_n'(\xi_n)| + |g_n(t_n)| \leq C_3.$$

iv. $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|^2\, dt\right\} \in l^1(\mathbb{Z})$, and therefore $\psi \in L^2(\mathbb{R})$.

For $n \in \mathbb{Z}$ and $t \in [n-\frac{1}{2}, n+\frac{1}{2}]$, we define

$$\widetilde{g_n}(t) = \sum_{k \neq n} b_k \ln|t-k| + e_{n-1}\mathbf{1}_{(n-1,n)}(t) + e_n \mathbf{1}_{(n,n+1)}(t) = \psi(t) - b_n \ln|t-n|.$$

Hence, for $|n| \geq N$, we obtain

$$\forall t \in [n-\tfrac{1}{2}, n+\tfrac{1}{2}], \quad |\widetilde{g_n}(t)| \leq C_1 + C_3.$$

We shall first show that $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}|\, dt\right\} \in l^1(\mathbb{Z})$.To begin, note that

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\, dt = |b_n| \int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln|t||\, dt = 2|b_n| \int_0^{\frac{1}{2}} |\ln|t||\, dt = |b_n|(\ln 2 + 1),$$

and therefore $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\,dt\right\} \in l^1(\mathbb{Z})$. Since

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|\,dt = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t) - b_n \ln|t-n||\,dt$$
$$\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|\,dt + \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\,dt$$

and $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|\,dt\right\} \in l^1(\mathbb{Z})$, we obtain $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|\,dt\right\} \in l^1(\mathbb{Z})$.

The following calculation concludes the verification of step *iv*:

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|^2\,dt = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t) + b_n \ln|t-n||^2\,dt$$
$$\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|^2 + 2|\widetilde{g_n}(t) b_n \ln|t-n|| + |b_n \ln|t-n||^2\,dt$$
$$\leq (C_1 + C_3) \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|\,dt$$
$$+2(C_1+C_3)\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n|| + C_4|b_n| \int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln|t||^2\,dt.$$

The elements on the right hand side form an $l^1(\mathbb{Z})$ sequence, and therefore $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|^2\,dt\right\} \in l^1(\mathbb{Z})$ and $\psi \in L^2(\mathbb{R})$. □

We can now complete the proof of the characterization theorem on $\mathbb{R}$, which was earlier stated as Theorem 3.1.

THEOREM 3.12. *Let $\psi \in L^1(\mathbb{R})$. The following are equivalent:*
i. $W_\psi f(b,a) = \int f(t)\psi(\frac{t-b}{a})\,dt$ is 1–periodic in a for all $f \in L^\infty(\mathbb{T})$.
ii. $\widehat{\psi}(0) = 0$ and ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbb{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$, and $\{e_n - \pi i \sum_{k\leq n} b_k\} \in l^2(\mathbb{Z})$.

Proof. i $\Longrightarrow$ ii. We apply Lemma 3.3 and obtain

$$\widehat{\psi}(\gamma) = \frac{\varphi_1(\gamma)}{\gamma} + \mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma}$$

and $\widehat{\psi}(0) = 0$, where φ_1 and φ_2 are 1–periodic. By Lemma 3.6 we have $\varphi_1, \varphi_1 + \varphi_2 \in A(\mathbb{T})$, and, hence, $\varphi_1, \varphi_2 \in A(\mathbb{T})$. Let us write

$$\varphi_1(\gamma) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\gamma} \quad \text{and} \quad \varphi_2(\gamma) = \sum_{n\in\mathbb{Z}} b_n e^{-2\pi i n\gamma}.$$

Since $\widehat{\psi}(0) = 0$, we have that $\widehat{\psi}$, $\frac{\varphi_1(\gamma)}{\gamma}$, and $\mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma}$ are bounded on $\widehat{\mathbb{R}}$ and of order $O(\frac{1}{\gamma})$, $|\gamma| \to \infty$, and are therefore in $L^2(\widehat{\mathbb{R}})$. We can calculate $\psi = \mathcal{F}^{-1}(\widehat{\psi})$, using the linearity of $\mathcal{F}^{-1}$, Lemma 3.7, and Lemma 3.10:

$$\begin{aligned}\psi(t) &= \mathcal{F}^{-1}\left(\frac{\varphi_1(\gamma)}{\gamma}\right)(t) + \mathcal{F}^{-1}\left(\mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma}\right)(t)\\ &= \sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} \tfrac{1}{2} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|\\ &= \sum_{n\in\mathbb{Z}} (c_n + \tfrac{1}{2}d_n)\mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|,\end{aligned}$$

where $d_n = 2\pi i \sum_{k\le n} b_k$ and $c_n = 2\pi i \sum_{k\le n} a_k$. Let $e_n = c_n + \frac{1}{2}d_n$.

Since $\{a_n\}, \{b_n\} \in l^1(\mathbb{Z})$, and

$$a_n = \frac{1}{2\pi i}(c_n - c_{n-1}) = \frac{1}{2\pi i}(e_n - \tfrac{1}{2}d_n - e_{n-1} + \tfrac{1}{2}d_{n-1}) = \tfrac{1}{2\pi i}(e_n - e_{n-1}) - \tfrac{1}{2}b_n,$$

we obtain that $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$.

Further, since $\sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)} = \mathcal{F}^{-1}(\frac{\varphi_1(\gamma)}{\gamma}) \in L^2(\mathbb{R})$, we have $\{c_n\} \in l^2(\mathbb{Z})$, where

$$c_n = 2\pi i \sum_{k\le n} a_k = \sum_{k\le n} ((e_k - e_{k-1}) - \pi i b_k) = e_n - \pi i \sum_{k\le n} b_k.$$

Therefore, *ii* holds.

ii $\Longrightarrow$ *i*. Let $\psi \in L^1(\mathbb{R})$ be of the form

$$\psi(t) = \sum_{n\in\mathbb{Z}} e_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|$$

with $\{b_n\}$ and $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$, and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbb{Z})$. For $n \in \mathbb{Z}$, set

$$a_n = \tfrac{1}{2\pi i}(e_n - e_{n-1}) - \tfrac{1}{2}b_n,$$

$c_n = e_n - \pi i \sum_{k\le n} b_k$, and $d_n = 2\pi i \sum_{k\le n} b_k$. Then $c_n = e_n - \frac{1}{2}d_n$ and $a_n = \frac{1}{2\pi i}(c_n - c_{n-1})$. $\psi \in L^2(\mathbb{R})$ by Lemma 3.11. Let

$$\psi_1(t) = \sum_{n\in\mathbb{Z}} c_n \mathbf{1}_{(n,n+1)}(t).$$

Then, $\psi_1 \in L^2(\mathbb{R})$ since $\{c_n\} \in l^2(\mathbb{Z})$ and $\|\psi_1\|_{L^2(\mathbb{R})} = \|\{c_n\}\|_{l^2(\mathbb{Z})}$. Further, let

$$\psi_2(t) = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbb{Z}} b_n \ln|t-n|.$$

Since $\psi \in L^2(\mathbb{R})$ and $\psi_1 \in L^2(\mathbb{R})$, we have $\psi_2 = \psi - \psi_1 \in L^2(\mathbb{R})$.

We can apply Lemma 3.7 to ψ_1 and Lemma 3.10 to ψ_2, and conclude that $\mathcal{F}(\psi_1)$ and $\mathcal{F}(\psi_2)$ are 1–periodic on $\widehat{\mathbb{R}}^+$ and 1–periodic on $\widehat{\mathbb{R}}^-$. Hence,

$$\widehat{\psi} = \mathcal{F}(\psi) = \mathcal{F}(\psi_1 + \psi_2) = \mathcal{F}(\psi_1) + \mathcal{F}(\psi_2)$$

is 1–periodic on $\widehat{\mathbb{R}}^+$ and 1–periodic on $\widehat{\mathbb{R}}^-$. Since $\widehat{\psi}(0) = 0$, we can apply Lemma 3.3, and *i* follows. $\square$

REMARK 3.13. *a.* It is easy to give a formal proof of Theorem 3.12 for the direction $i \Longrightarrow ii$. For this we combine Proposition 2.3 with a comparable calculation for $\psi \in L^1(\mathbb{R})$ defined by

$$\psi(t) = \sum_{n\in\mathbb{Z}} b_n \ln|t-n|,$$

where $\sum_{n\in\mathbb{Z}} b_n = 0$. In fact, noting that

$$\psi\left(\tfrac{t-b}{a}\right) = \sum_{n\in\mathbb{Z}} b_n \ln|t-b-na|\,,$$

we have

$$\begin{aligned}
W_\psi f(b,a+1) &= \int \left(\sum_{n\in\mathbb{Z}} b_n \ln|t-b-n(a+1)|\right) f(t)\,dt\\
&= \sum_{n\in\mathbb{Z}} b_n \int \ln|t-b-n(a+1)|\, f(t)\,dt\\
&= \sum_{n\in\mathbb{Z}} b_n \int \ln|t-b-na|\, f(t+n)\,dt = \sum_{n\in\mathbb{Z}} b_n \int \ln|t-b-na|\, f(t)\,dt\\
&= \int \left(\sum_{n\in\mathbb{Z}} b_n \ln|t-b-na|\right) f(t)\,dt = W_\psi f(b,a).
\end{aligned}$$

b. There is no redundancy in the three conditions *a.* $\{b_n\} \in l^1(\mathbb{Z})$; *b.* $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$; *c.* $\{e_n - \pi i \sum_{k\leq n} b_k\} \in l^2(\mathbb{Z})$.

To see this, first, let $\{e_n\} = \{0\}$ and $\{b_n\} = \{\frac{1}{n^2+1}\}$ for $n \in \mathbb{Z}$. These sequences satisfy *a* and *b* but not *c*. Next, the sequences $\{b_n\}$ and $\{e_n\}$, defined by $\{b_n\} = \{0\}$ and $e_n = \frac{1}{n}$ for n positive and odd and $e_n = 0$ otherwise, fulfill conditions *a* and *c*, but not condition *b*. Finally, the sequences $\{b_n\}$ and $\{e_n\}$, defined by $\{e_n\} = \{0\}$ and $b_n = (-1)^n\frac{1}{n}$ for n positive and $b_n = 0$ otherwise, satisfy *b* and *c*, but not *a*.

In the following examples we shall construct wavelets $\psi \in L^1(\mathbb{R})$ which are not piecewise constant, but which have the property that $W_\psi f$ is 1–periodic in scale for every $f \in L^\infty(\mathbb{T})$.

EXAMPLE 3.14. Consider

$$\forall t \in \mathbb{R}\setminus\{0,1\}, \quad \psi_0(t) = \ln|t| - \ln|t-1| = \ln\left|\tfrac{t}{t-1}\right|.$$

This function clearly satisfies condition *ii* of Theorem 3.12, for, even though $\psi_0 \notin L^1(\mathbb{R})$, we see that $\widehat{\psi}_0(0) = 0$ in the sense of a Cauchy principal value, since $0 = \int_{-N+1}^{N} \psi_0(t)\,dt$ for $N \geq 1$.

Let us now construct $\psi \in L^1(\mathbb{R})$, with $\{b_n\} \neq \{0\}$, for which condition *ii* is satisfied. In fact, set

$$\psi(t) = \sum_{|n|\geq 2} \ln\left|\frac{n}{n+1}\right| \mathbf{1}_{(n,n+1)}(t) + \ln|t| - \ln|t-1|.$$

Then $b_0 = 1$, $b_1 = -1$, $b_n = 0$ for $n \neq 0, 1$, and hence $\{b_n\} \in l^1(\mathbb{Z})$. Further, letting $e_n = \ln\left|\frac{n}{n+1}\right|$ for $|n| \geq 2$, we have

$$a_n = e_n - e_{n-1} = \left(\ln\left|\frac{n}{n+1}\right| - \ln\left|\frac{n-1}{n}\right|\right) = -\ln\left|\frac{(n+1)(n-1)}{n^2}\right| = -\ln\left|1 - \frac{1}{n^2}\right|$$

for $|n| \geq 2$, and, hence, $\{e_n - e_{n-1}\} \in l^1(\mathbb{Z})$. For $|n| \geq 3$, we have

$$e_n - \pi i \sum_{k \leq n} b_k = e_n.$$

In order to show that $\psi \in L^1(\mathbb{R})$ observe that on the positive part of the real axis

$$\int_2^\infty |\psi(t)|\, dt \leq \lim_{N\to\infty} \sum_{n=2}^N \psi(n) = \lim_{N\to\infty} \sum_{n=2}^N \left(\ln \left| \frac{n}{n+1} \right| - \ln \left| \frac{n-1}{n} \right| \right) = \ln 2.$$

A similar calculation holds for the negative part of the real axis and, therefore, $\psi \in L^1(\mathbb{R})$. Finally, using the facts that $\int_{-N+1}^N \psi_0(t)\, dt = 0$ and $\int_{-N}^N \psi(t) - \psi_0(t)\, dt = 0$ for $N \geq 1$, we obtain $\widehat{\psi}(0) = 0$.

Thus, condition *ii* of Theorem 3.12 is satisfied, and so $W_\psi f$ is 1–periodic in scale for all $f \in L^\infty(\mathbb{T})$.

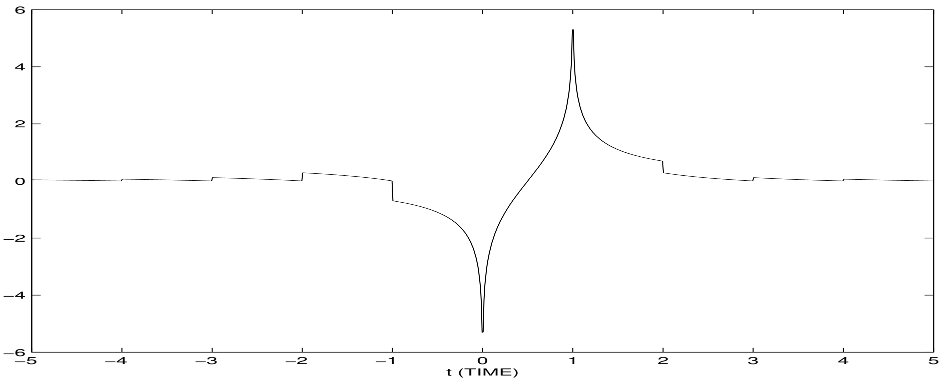

FIG. 3.1. *The wavelet $\psi \in L^1(\mathbb{R})$ of Example 3.14*

EXAMPLE 3.15. We shall construct $\psi \in L^1(\mathbb{R})$ satisfying condition *ii* of Theorem 3.12 and which has the further property that in the representation of part *ii*, $\{e_n\} = \{0\}$, i.e., ψ contains no generalized Haar component.

Let

$$\psi(t) = \ln|t+1| - \ln|t+2| + \ln|t-1| - \ln|t-2| = \ln \left| \frac{t^2-1}{t^2-4} \right|.$$

ψ is monotonically decreasing for $|t| \to \infty$. Hence, we can apply the sum criteria to show $\psi \in L^1(\mathbb{R})$. Note that

$$\sum_{3 \leq |n| \leq N} |\psi(n)| = 2\ln 4 - 2\ln\left(\tfrac{N+2}{N-1}\right),$$

and so

$$\sum_{3 \leq |n|} |\psi(n)| = 2\ln 4.$$

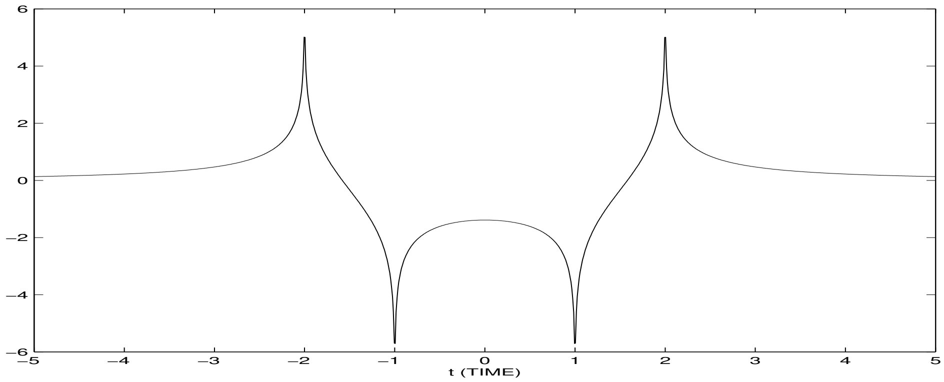

FIG. 3.2. *The wavelet $\psi \in L^1(\mathbb{R})$ of Example 3.15*

4. Optimal generalized Haar wavelets. We shall construct generalized Haar wavelets to detect a specific periodic function f in a noisy signal $s(t) = Af(ct) + N(t)$, $t \in I \subset \mathbb{R}$. In fact, an optimal generalized Haar wavelet ψ is chosen so that we can identify the periodic components of the wavelet transform $W^p_\psi s$.

To apply averaging methods to detect bi–periodic behavior in $W^p_\psi s$ (Section 5), we want $W^p_\psi f$ to be well–localized. This will result in a lattice pattern of relative maxima in time–scale space. For a given periodic signal f, we shall show the existence of an optimal generalized Haar wavelet, which guarantees these relative maxima to be as large as possible.

4.1. Construction of optimal generalized Haar wavelets. Before being more precise with respect to the term *optimal* generalized Haar wavelet, we need to introduce certain restrictions.

We begin by letting M=1 and by fixing $N \in \mathbb{N}$. We consider generalized Haar wavelets ψ^c with compact support and with the form

$$\psi^c|_{[n,n+1)} = c_n \text{ for } n = 0, \ldots, N-1, \quad c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbb{C}^N. \tag{4.1}$$

Additionally, we require

$$0 = \int \psi^c(t)\, dt = \sum_{n=0}^{N-1} c_n, \tag{4.2}$$

and we normalize ψ^c so that

$$\|\psi^c\|_{L^2(\mathbb{R})} = \|c\|_{l^2(\mathbb{C}^N)} = 1. \tag{4.3}$$

Equation (4.2) allows us to achieve the periodicity properties asserted in Proposition 2.3. Note that (4.2) is equivalent to the condition that

$$c \in H = \{x \in \mathbb{C}^N : \sum_{n=0}^{N-1} x_n = \langle x, (1, 1, \ldots, 1, 1)\rangle = 0\}.$$

H is an $N-1$ dimensional subspace, i.e., a hyperplane. Equation (4.3) is a standard normalization constraint in constructing wavelets. For ψ^c it can be expressed as

$$c \in S^{2N-1} = \{x \in \mathbb{C}^N : \|x\|_{l^2(\mathbb{C}^N)} = 1\}.$$

We shall design a wavelet which has a clear single peak in the $(0,T] \times (0,MT] = (0,T] \times (0,T]$ cell of the wavelet transform. Theorem 4.2 shows how to achieve a maximal peak. We need the following adaptation of the Cauchy–Schwarz inequality in order to prove Theorem 4.2.

LEMMA 4.1. *Let U be a k-dimensional subspace of $\mathbb{C}^N$. Let $v \in \mathbb{C}^N$ and let P_U be the orthogonal projection of $\mathbb{C}^N$ onto U. Then*

$$|\langle u, v\rangle| \leq \langle \frac{P_U(v)}{\|P_U(v)\|_{l^2(\mathbb{C}^N)}}, v\rangle$$

for all $u \in U \cap S^{2N-1}$.

THEOREM 4.2. *Let $p > 1$ and $f \in L^\infty(\mathbb{R})$, or let $p \geq 1$, $f \in L^1(\mathbb{T}_T)$, and suppose each $x \in \mathbb{R}$ is a Lebesgue point of f. Let $N \in \mathbb{N}$.*
a. There exists $(b_0, a_0) \in \mathbb{R} \times \mathbb{R}^+$ such that

$$a_0^{-\frac{1}{p}} \|P_H(k_{b_0,a_0})\|_{l^2(\mathbb{C}^N)} = \max_{(b,a)\in\mathbb{R}\times\mathbb{R}^+} a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)},$$

where $k_{b,a} = (k_{b,a,0}, \ldots k_{b,a,N-1}) \in \mathbb{C}^N$ is defined by

$$k_{b,a,n} = \int_{na+b}^{(n+1)a+b} f(t)\, dt$$

and P_H is the orthogonal projection of $\mathbb{C}^N$ onto the hyperplane H.
b. For this (b_0, a_0) we set

$$c_0 = \frac{P_H(k_{b_0,a_0})}{\|P_H(k_{b_0,a_0})\|_{l^2(\mathbb{C}^N)}}.$$

The generalized Haar wavelet ψ^{c_0} satisfies (4.1),(4.2), and (4.3), and

$$|W^p_{\psi^{c_0}} f(b_0, a_0)| \geq |W^p_{\psi^c} f(b,a)| \tag{4.4}$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R}^+$ and all ψ^c satisfying (4.1),(4.2),(4.3).

Proof. i. Let us first fix $(b,a) \in (0,T] \times (0,T]$. We want to construct $c_{b,a} \in \mathbb{C}^N$ such that

$$|W^p_{\psi^{c_{b,a}}} f(b,a)| \geq |W^p_{\psi^c} f(b,a)| \tag{4.5}$$

for all ψ^c satisfying conditions (4.1),(4.2),(4.3). After finding $c_{b,a}$ we shall choose the "optimal" (b_0, a_0) and let $c = c_{b_0,a_0}$.

For $c \in \mathbb{C}^N$ we have

$$W^p_{\psi^c} f(b,a) = a^{-\frac{1}{p}} \sum_{n=0}^{N-1} c_n \int_{na+b}^{(n+1)a+b} f(t)\, dt.$$

Setting

$$k_{b,a,n} = \int_{na+b}^{(n+1)a+b} f(t)\, dt$$

and $k_{b,a} = (k_{b,a,0}, \dots k_{b,a,N-1})$, we obtain

$$W^p_{\psi^c} f(b,a) = a^{-\frac{1}{p}} \sum_{n=0}^{N-1} c_n k_{b,a,n} = a^{-\frac{1}{p}} \langle c, k_{b,a} \rangle. \tag{4.6}$$

Note that conditions (4.2) and (4.3) on ψ^c are equivalent to the following restriction on c:

$$c \in \{x \in \mathbb{C}^N : \sum x_n = 0, \|x\|_{l^2(\mathbb{C}^N)} = 1,\} = H \cap S^{2N-1}.$$

Given the vector $k_{b,a}$ we can optimize (4.6) by projecting $k_{b,a}$ onto the hyperplane H and normalizing the result (Lemma 4.1), i.e., letting $P_H : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be the orthogonal projection of $\mathbb{C}^N$ onto H, we obtain

$$c_{b,a} = \frac{P_H(k_{b,a})}{\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}}$$

as the best choice of $c_{b,a}$, and $\psi^{c_{b,a}}$ fulfills (4.5).

Explicitly, we have

$$P_H(k_{b,a}) = k_{b,a} - \frac{1}{N} \int_b^{Na+b} f(t)\, dt \quad (1, 1, \dots, 1),$$

and therefore

$$c_{b,a,n} = \frac{k_{b,a,n} - \frac{1}{N} \int_b^{Na+b} f(t)\, dt}{\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}},$$

$n = 0, \dots, N-1$, are the optimal choices of values for the generalized Haar wavelet in the case that b and a are fixed.

ii. It remains to show the existence of (b_0, a_0) such that

$$|W^p_{\psi^{c_{b_0,a_0}}} f(b_0, a_0)| \geq |W^p_{\psi^{c_{b,a}}} f(b,a)| \tag{4.7}$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R}^+$, where c_{b_0,a_0} and $c_{b,a}$ are chosen as above. This, together with (4.5), will conclude the proof, see (4.9).

Since $c_{b,a} \in H$, we have

$$\begin{aligned} |W^p_{\psi^{c_{b,a}}} f(b,a)| &= |a^{-\frac{1}{p}} \langle c_{b,a}, k_{b,a} \rangle| = a^{-\frac{1}{p}} |\langle c_{b,a}, P_H(k_{b,a}) \rangle| \\ &= a^{-\frac{1}{p}} \left| \langle \frac{P_H(k_{b,a})}{\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}}, P_H(k_{b,a}) \rangle \right| = a^{-\frac{1}{p}} \, \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}, \end{aligned}$$

and, hence, we need to show the existence of (b_0, a_0) such that

$$a_0^{-\frac{1}{p}} \, \|P_H(k_{b_0,a_0})\|_{l^2(\mathbb{C}^N)} \geq a^{-\frac{1}{p}} \, \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R}^+$.

To see this, first observe that $\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ is T periodic in b. This is the case since $k_{b,a}$ is T periodic in b, i.e., for $n = 0, \dots, N-1$, we have

$$k_{b+T,a,n} = \int_{na+b+T}^{(n+1)a+b+T} f(t)\, dt = \int_{na+b}^{(n+1)a+b} f(u-T) du = \int_{na+b}^{(n+1)a+b} f(u) du = k_{b,a,n}.$$

$\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ is also T–periodic in a. In fact, we compute

$$\begin{aligned}\|P_H(k_{b,a+T})\|^2_{l^2(\mathbb{C}^N)} &= \sum_{n=0}^{N-1}\left(\int_{n(a+T)+b}^{(n+1)(a+T)+b} f(t)\,dt - \frac{1}{N}\int_b^{N(a+T)+b} f(t)\,dt\right)^2\\ &= \sum_{n=0}^{N-1}\left(\int_{na+b}^{(n+1)a+T+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt - \frac{1}{N}N\int_0^T f(t)\,dt\right)^2\\ &= \sum_{n=0}^{N-1}\left(\int_{na+b}^{(n+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)^2 = \|P_H(k_{b,a})\|^2_{l^2(\mathbb{C}^N)}\,.\end{aligned}$$

Since $a^{-1/p}$ is monotonely decreasing for $a \to \infty$ and by the periodicity of $\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ in time and scale, it suffices to show the existence of $(b_0, a_0) \in [0,T] \times (0,T]$ such that

$$a_0^{-\frac{1}{p}}\,\|P_H(k_{b_0,a_0})\|_{l^2(\mathbb{C}^N)} \geq a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$$

for all $(b,a) \in [0,T] \times (0,T]$.

Note that $a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ is continuous on $[0,T] \times (0,T]$. We shall show that, if $p > 1$ and $f \in L^\infty(\mathbb{R})$, or if $p \geq 1$, $f \in L^1(\mathbb{T}_T)$, and each $x \in \mathbb{R}$ is a Lebesgue point of f, then $a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ has a continuous extension to $[0,T] \times [0,T]$, and therefore it obtains a maximum on $[0,T] \times [0,T]$. We shall further show that this maximum is obtained at some $(b_0, a_0) \in [0,T] \times (0,T]$. In fact, we shall verify that for all $b \in \mathbb{R}$

$$\lim_{a \to 0^+} a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)} = 0. \tag{4.8}$$

The proof of (4.8) is divided into two cases. Recall that the n^{th} entry in the vector $P_H(k_{b,a})$ is given by $a^{-\frac{1}{p}}(k_{b,a,n} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)$.

For the first case, let $p > 1$ and $f \in L^\infty(\mathbb{R})$. For $b \in \mathbb{R}$, we compute

$$\begin{aligned}0 &\leq \lim_{a \to 0^+}\left|a^{-\frac{1}{p}}(k_{b,a,n} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)\right|\\ &= \lim_{a \to 0^+} a^{1-\frac{1}{p}}\left|\frac{1}{a}\int_{na+b}^{(n+1)a+b} f(t)\,dt - \frac{1}{aN}\int_b^{Na+b} f(t)\,dt\right|\\ &\leq \lim_{a \to 0^+} a^{1-\frac{1}{p}}(1+N)\,\|f\|_{L^\infty(\mathbb{R})} = 0.\end{aligned}$$

For the second case, let $p \geq 1$, and let $f \in L^1(\mathbb{T}_T)$ have the property that each

$x \in \mathbb{R}$ is a Lebesgue point of f. For $p = 1$ and for all $b \in \mathbb{R}$ we note that

$$\begin{aligned}
&\lim_{a\to 0^+} a^{-1}\left(k_{b,a,n} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)\\
&= \lim_{a\to 0^+} a^{-1}\left(\int_{na+b}^{(n+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)\\
&= (n+1)\lim_{a\to 0^+}\frac{1}{(n+1)a}\int_b^{(n+1)a+b} f(t)\,dt - n\lim_{a\to 0^+}\frac{1}{na}\int_b^{na+b} f(t)\,dt\\
&\qquad - \lim_{a\to 0^+}\frac{1}{N}\int_b^{Na+b} f(t)\,dt\\
&= (n+1)\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt - n\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt - \lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt = 0.
\end{aligned}$$

Using the addition property of limits in the third step of this calculation is a priori valid for almost every b. This is the case since $f \in L^1(\mathbb{T}_T)$, and therefore the limit $\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt$ exists for all Lebesgue points $b \in \mathbb{R}$. Therefore, by hypothesis, the limit exists everywhere.

For $p > 1$ in this second case, we have $a^{1-\frac{1}{p}} \to 0$ as $a \to 0^+$, and, hence,

$$\lim_{a\to 0^+} a^{1-\frac{1}{p}}\left|a^{-1}\left(k_{b,a,n} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)\right| = 0.$$

In both cases, the componentwise convergence of $\lim_{a\to 0^+} a^{-1}P_H(k_{b,a})$, together with the continuity of norms and the fact that$\|av\| = |a|\,\|v\|$, give (4.8).

Let $(b,a) \in \mathbb{R}\times\mathbb{R}^+$ and let ψ^c satisfy (4.1),(4.2),(4.3). Using (4.5) and (4.7) we obtain

$$\begin{aligned}
|W^p_{\psi^{c_{b_0,a_0}}} f(b_0,a_0)| &= a_0^{-\frac{1}{p}}\,\|P_H(k_{b_0,a_0})\|_{l^2(\mathbb{C}^N)}\\
&\geq a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)} = |W^p_{\psi^{c_{b,a}}} f(b,a)| \geq |W^p_{\psi^c} f(b,a)|. \quad (4.9)
\end{aligned}$$

$\square$

Remark 4.3. Theorem 4.2 leads to the following construction algorithm for optimal generalized Haar wavelets. First find b and a such that

$$\begin{aligned}
a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|^2_{l^2(\mathbb{C}^N)} &= a^{-\frac{1}{p}}\sum_{n=0}^{N-1}\left(k_{b,a,n} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)^2\\
&= a^{-\frac{1}{p}}\sum_{n=0}^{N-1}\left(\int_{na+b}^{(n+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\right)^2
\end{aligned}$$

is maximal. Then let

$$c_n = c_{b,a,n} = \frac{k_{b,a,n} - \frac{1}{N}\sum_{n=0}^{N-1} k_{b,a,n}}{\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}} = \frac{\int_{na+b}^{(n+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt}{\|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}}.$$

If $a_0 = T/N$, $\psi^{c_{b,T/N}}_{b,T/N}$ fills out exactly one period of f. In this special case we have

$$\frac{1}{N}\int_b^{NT/N+b} f(t)\,dt = \frac{1}{N}\int_b^{T+b} f(t)\,dt = \frac{1}{N}\int_0^T f(t)\,dt,$$

which is independent of b.

Note that the optimization process depends on the choice of p.

4.2. Examples of optimal generalized Haar wavelets. Example 4.4 and Example 4.5 illustrate how to apply Theorem 4.2.

EXAMPLE 4.4. Figure 4.1. A shows the 1–periodic signal $f(x) = \sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, sampled at 20 samples per unit. Fixing $N = 8$, we calculate $k(b,a) = \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ for this signal. The result is displayed in Figure 4.1.B.

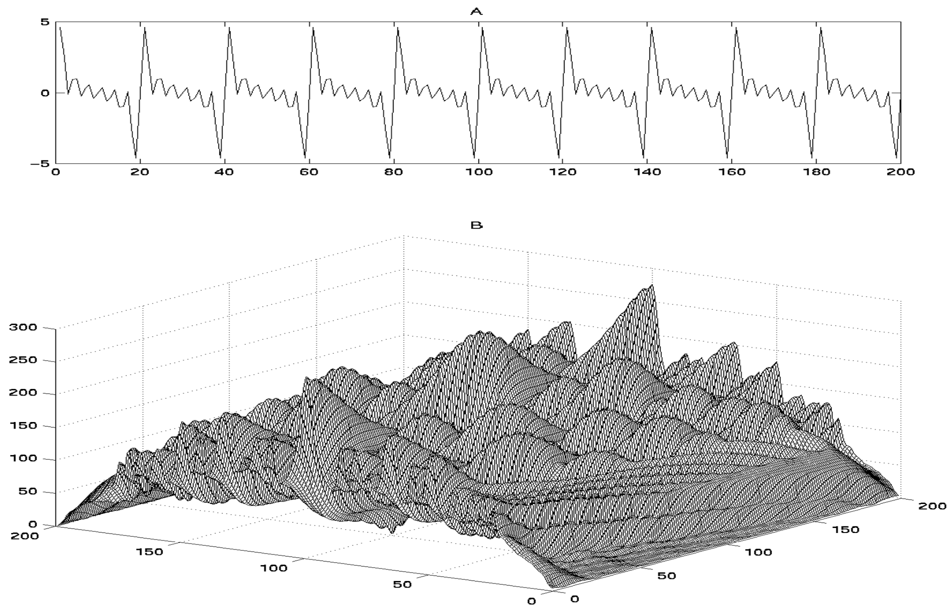

FIG. 4.1. *A: $f(x) = \sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, sampled at 20 samples per unit. B: $k(b,a) = \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$ for $N = 8$.*

Figure 4.2 illustrates the dependence of the generalized Haar wavelet on the choice of the normalization constant p. Figure 4.2.A and Figure 4.2.B display the optimal generalized Haar wavelets for $p = 1$ to $p = 2.4$. For $p > 2.4$ we continue to obtain the same wavelet as for $p = 2.4$. The optimal generalized Haar wavelets for $p = 1$, $p = 1.75$, $p = 2$, and $p = 2.2$ are shown separately below Figure 4.2.A and 4.2.B.

EXAMPLE 4.5. Theorem 4.2 is applied to the epileptic seizure problem in Figure 4.3. Our basic assumption is that the precursors of a seizure will be found in the same periodicities dominant within the seizure itself. Such an assumption implies that what is being detected is a process with the same dynamics (periodicities) as the seizure, but in miniature form, and then difficult to detect within the background electric activity of the brain. After simulating an expected period, in our case the seizure period of an individual patient, we define the periodic function F associated with the seizure period. F is sampled at 130 samples per period for subsequent calculations with the projection P_H. We choose $N = 5$ and calculate $k(b,a) = \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}$. For the normalization constants $p = 1$, $p = 1.35$, and $p = 2$, we obtain distinct optimal generalized Haar wavelets. This particular simulated sample seizure data is designed to mimic full blown "3 per second spike and wave" activity, which is characteristic

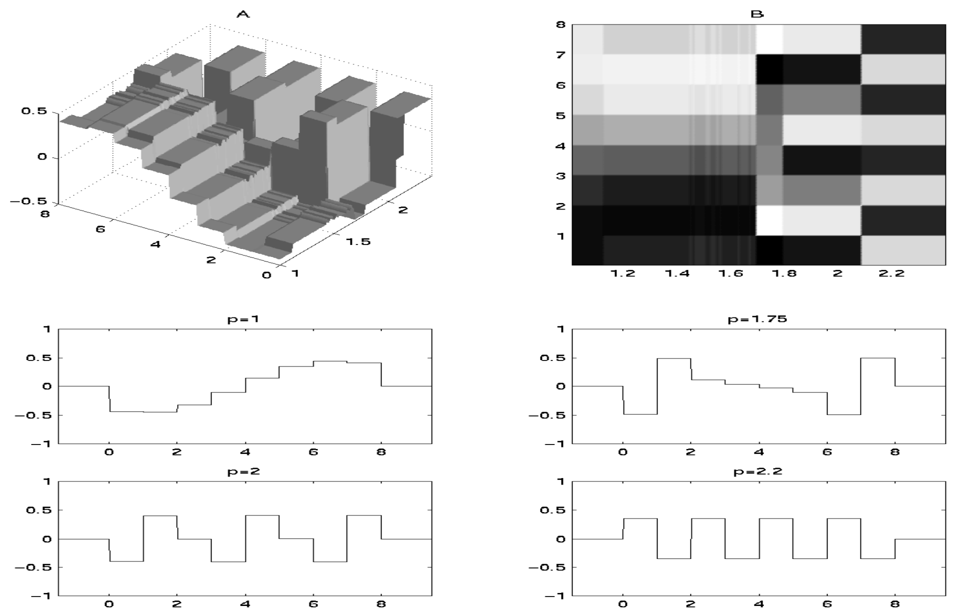

FIG. 4.2. *Optimal generalized Haar wavelets for* $\sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, $p = 1$ *to* $p = 2.4$, $N = 8$.

of petit mal "absence" seizures. Realistically, precursors for actual petit mal absence seizure generally seem to lack this characteristic.

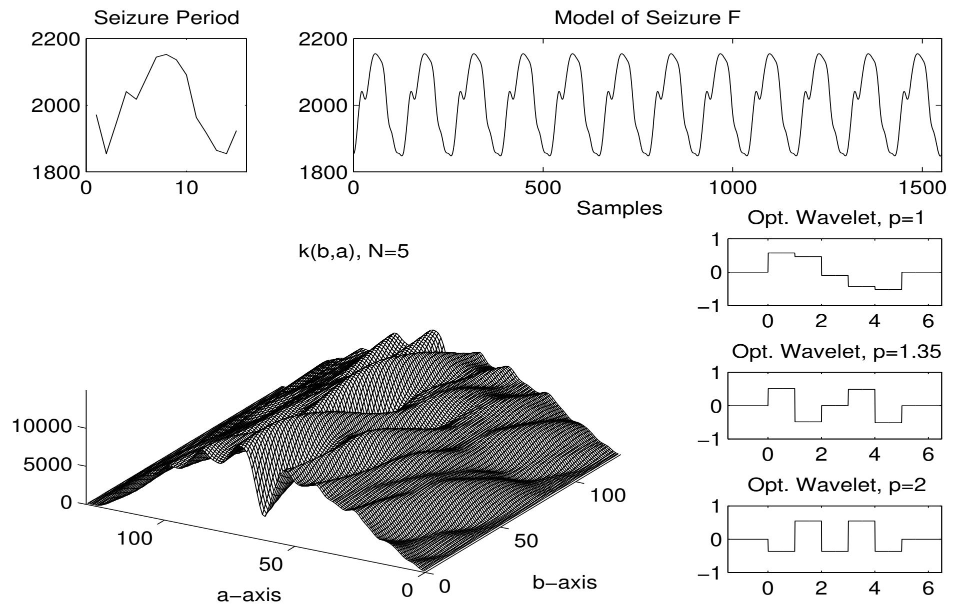

FIG. 4.3. *Construction of the optimal generalized Haar wavelet in the epileptic seizure case.*

4.3. Optimal generalized Haar wavelets with additional properties. In some applications, the signal might have features we want to replicate or highlight when constructing "optimal" generalized Haar wavelets. The following problems arise.

PROBLEM 4.6. Suppose, the periodic signal f is symmetric or almost "symmetric" with respect to a reference point $t_0 \in [0,T]$, i.e., $f(t_0+t) = -f(t_0-t)$ for $t \in \mathbb{R}$ ("odd signal"), or symmetric with respect to a reference axis $t = t_0$, i.e., $f(t_0+t) = f(t_0-t)$ for $t \in \mathbb{R}$ ("even signal"). We would like the constructed wavelet to have the corresponding symmetric form, i.e., we would like to construct an optimal even generalized Haar wavelet or an optimal odd generalized Haar wavelet in order to capitalize on Proposition 2.5.

PROBLEM 4.7. We would like the wavelet transform obtained through the constructed generalized Haar wavelet to be resistant to some specific background behavior in the signal.

PROBLEM 4.8. Our signal might carry two periodic components which we want to analyze separately. Here, the goal is to construct a pair of generalized Haar wavelets which are sensitive in detecting one of the components and overlooking the other.

PROBLEM 4.9. One period of the signal might have parts where it is slowly varying and other parts with high variance. The associated wavelet should focus toward the fast varying part and allow many different values there, while in other parts a few values might be sufficient.

The question of whether we can construct generalized Haar wavelets which take into account a specific feature of a signal has to be answered individually for each such feature. Nevertheless, a small contribution to the general case is made in the remainder of this section. In fact, Theorem 4.10 generalizes Theorem 4.2 and is a tool for solving problems such as those stated above. For example, Proposition 4.11 and Proposition 4.12 use this theorem to give solutions to problems of the kind described in Problem 4.6 and Problem 4.7, respectively. They further illustrate how solutions to some problems can be found. The method is based on Lemma 4.1 and the fact that the optimization process in Theorem 4.2 can be applied if we replace H by any subspace U of $\mathbb{C}^N$ for which $U \subseteq H$.

THEOREM 4.10. *Let $p > 1$ and $f \in L^\infty(\mathbb{R})$, or let $p \geq 1$, $f \in L^1(\mathbb{T}_T)$, and suppose each $x \in \mathbb{R}$ is a Lebesgue point of f. Let $N \in \mathbb{N}$ and let $k_{b,a}$ and H be defined as in Theorem 4.2. If U is a subspace of $\mathbb{C}^N$, then there exists $(b_0,a_0) \in \mathbb{R}\times\mathbb{R}^+$ such that*

$$a_0^{-\frac{1}{p}} \left\|P_{U\cap H}(k_{b_0,a_0})\right\|_{l^2(\mathbb{C}^N)} = \max_{(b,a)\in\mathbb{R}\times\mathbb{R}^+} a^{-\frac{1}{p}} \left\|P_{U\cap H}(k_{b,a})\right\|_{l^2(\mathbb{C}^N)},$$

where $P_{U\cap H}$ is the orthogonal projection of $\mathbb{C}^N$ onto the subspace $U\cap H$. By setting

$$c_0 = \frac{P_{U\cap H}(k_{b_0,a_0})}{\left\|P_{U\cap H}(k_{b_0,a_0})\right\|_{l^2(\mathbb{C}^N)}},$$

we obtain

$$|W^p_{\psi^{c_0}} f(b_0,a_0)| \geq |W^p_{\psi^c} f(b,a)|$$

for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$, $c \in U$, and ψ^c satisfying (4.1), (4.2),(4.3).

Proof. The first steps of the proof of Theorem 4.2 can easily be generalized to the setting of Theorem 4.10 by replacing H by $U\cap H$.

It remains to show that the maximum exists. For this, note that we proved that $k_{b,a}$ is T periodic in b. This implies that $P_{U\cap H}(k_{b,a})$ is T–periodic in b. Essentially, we also showed that $P_H(k_{b,a})$ is T–periodic in a. By the definition of orthogonal projections we have

$$P_{U\cap H}(k_{b,a}) = P_U(P_H(k_{b,a})),$$

and therefore $P_{U\cap H}(k_{b,a})$ is T periodic in a.

We can conclude the existence of the maximum by continuing to follow the proof of Theorem 4.2 and by using the fact that

$$\|P_{U\cap H}(k_{b,a})\|_{l^2(\mathbb{C}^N)} \leq \|P_H(k_{b,a})\|_{l^2(\mathbb{C}^N)}\,.$$

□

Solving a given problem can be approached by defining the subspace U such that $c \in U$ if and only if ψ^c has the desired properties. Of course, such a subspace might not exist.

The problem described in Problem 4.6 can be quantified and resolved in the following way.

PROPOSITION 4.11. *a. For $k = 1,\dots,N$, define $v_k \in \mathbb{C}^{2N}$ by $v_k^i = \delta_{i,k} - \delta_{2N-i+1,k}$ for $i = 1,\dots,2N$. Let $U^e = \operatorname{span}\{v_1,\dots,v_N\}^{\perp}$. Then $c \in U$ if and only if ψ^c is even.*

b. For $k = 1,\dots,N$, define $v_k \in \mathbb{C}^{2N}$ by $v_k^i = \delta_{i,k} + \delta_{2N-i+1,k}$ for $i = 1,\dots,2N$. Let $U^o = \operatorname{span}\{v_1,\dots,v_N\}^{\perp}$. Then $c \in U$ if and only if ψ^c is odd.

Proof. We shall prove part a. The proof of part b is similar.

Clearly, $c \in U^e$ if and only if $c \perp v_k$ for $k = 1,\dots,N$, i.e.,

$$0 = \langle c, v_k\rangle = c_k - c_{2N-k+1} \quad \text{for} \quad k = 1,\dots,N\,.$$

This holds if and only if $c_k = c_{2N-k+1}$, that is, if and only if ψ^c is even. □

Let us discuss further possible features of generalized Haar wavelets. The property $c \in H$ implies that if f is a constant function, then $W^p_{\psi^c} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$. The tophat wavelet ψ^{top} is defined by $top = \frac{1}{\sqrt{6}}(1,-2,1)$, and has the property that if f is a linear function, then $W^p_{\psi^{top}} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$. The Haar wavelet does not possess this property. We are led to the question of whether it is possible to construct a subspace $U \subseteq \mathbb{C}^N$ such that for any $c \in U$ we have the property that $W^p_{\psi^c} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$ and for any polynomial f of degree less or equal some given K. The answer to this question is affirmative as the following proposition shows.

PROPOSITION 4.12. *For $K \leq N-2$, define $v_k = (1^k, 2^k, 3^k, \dots, N^k) \in \mathbb{C}^N$ for $k = 0,\dots,K$ and let $U^K = \operatorname{span}\{v_0,\dots,v_K\}^{\perp}$. Then $c \in U^K$ if and only if ψ^c has the property that for any polynomial f of degree less or equal K $W^p_{\psi^c} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$.*

The proof is omitted.

Using Proposition 4.12 we have the following result.

PROPOSITION 4.13. *The generalized Haar wavelet $\psi^c \neq 0$ has the property that for any polynomial f of degree less or equal K, $W^p_{\psi^c} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$ if and only if $K \leq N-2$ and the polynomial $(x-1)^{K+1}$ divides the polynomial $c_0 + c_1 x + \dots + c_{N-1}x^{N-1}$.*

Proof. Let us first assume that $\psi^c \neq 0$ has the property that for any polynomial f of degree less or equal K, $W^p_{\psi^c} f(b,a) = 0$ for all $(b,a) \in \mathbb{R}\times\mathbb{R}^+$, i.e., $c \perp v_k = (1^k, 2^k, 3^k, \dots, N^k)$ for $k = 0,\dots,K$ (Proposition 4.12). Since $c \neq 0$ and since the family $\{v_k\}_{k=0,\dots,N-1}$ is a basis of $\mathbb{C}^N$ we have $K \leq N-2$.

To show that $(x-1)^{K+1}$ divides $p(x) \equiv c_0 + c_1 x + \ldots + c_{N-1}x^{N-1}$, note that for $s = 0, \ldots, K$, $p^{(s)}(1) = \sum_{n=0}^{N-1} c_n n(n-1)\ldots(n-s+1)$ is a linear combination of $\sum_{n=0}^{N-1} c_n n^k = \langle c, v_k \rangle$. On the other hand $\langle c, v_k \rangle = 0$ for $k = 0, \ldots, K$, and hence $p^{(s)}(1) = 0$. This implies that $(x-1)^{K+1}$ divides $p(x)$.

Conversely, assuming that $K \leq N-2$ and that $(x-1)^{K+1}$ divides $p(x) = c_0 + c_1 x + \ldots + c_{N-1}x^{N-1}$, we have $0 = p^{(s)}(1) = \sum_{n=0}^{N-1} c_n n(n-1)\ldots(n-s+1)$ for $s = 0, \ldots, K$. The fact that $0 = p^{(s)}(1)$ for $s = 0, \ldots, k$ implies $c \perp v_k$, and using Proposition 4.12 we obtain that $\psi^c \neq 0$ has the property that, for any polynomial f of degree less or equal K, $W^p_{\psi^c} f(b,a) = 0$ for $(b,a) \in \mathbb{R} \times \mathbb{R}^+$ □

Proposition 4.13 supplies us with generalized wavelets with $K+1$ vanishing moments and with minimal support. Up to a scalar we can read these wavelets from a modified Pascal triangle.

$K=0$							1		−1							
$K=1$						1		−2		1						
$K=2$					1		−3		3		−1					
$K=3$				1		−4		6		−4		1				
$K=4$			1		−5		10		−10		5		−1			
$K=5$		1		−6		15		−20		15		−6		1		

5. Wavelet periodicity detection algorithm.

5.1. Background and idea. We were led to the formulation of the wavelet periodicity detection algorithm, described in Section 5.2, through the study of epileptic seizure prediction problems, see especially [3], cf., [24, 25]. In the case of epileptic seizure prediction, a goal is to detect periodic behavior in EEG data prior to a seizure, where this behavior is indicative of the periodic signature of the seizure itself.

Suppose a periodic seizure signature f is determined from reliable ECoG data obtained by means of a one–time invasive procedure. The following method is a means to detect precursors of f prior to its full blown occurrence during seizure. This detection can be observed from non–invasive EEG data subsequent to the original determination of f. Sufficiently early knowledge of reliable precursors of f allows the use of external treatments to temper the effects of the seizure when it arrives. Our method is composed of three steps.

1. ECoG data of an individual patient are analyzed through spectral and wavelet methods to extract periodic patterns associated with epileptic seizures of a specific patient;

2. Using this knowledge of seizure periodicity, we construct an *optimal generalized Haar wavelet* designed to detect the epileptic periodic patterns of the patient;

3. A fast discretized version of the continuous wavelet transform and wavelet transform averaging techniques are used to detect occurrence and period of the seizure periodicities in the preseizure EEG data of the patient; and the algorithm is formulated to provide real time implementation.

Our method is generally applicable to detect locally periodic components in signals s which can be modeled as

$$s(t) = A(t)f(h(t)) + N(t), \tag{5.1}$$

$t \in I$, where f is a periodic signal defined on the time interval I, A is a non–negative slowly varying function, and h is strictly increasing with h' slowly varying. N denotes background activity. For example, in the case of ECoG data, N is essentially $1/f$

noise. In the case of EEG data and for t in I, N includes noise due to cranial geometry and densities [11, 8]. In both cases N also includes standard low frequency rythms [18].

If F is a trigonometric polynomial, then the signals described in (5.1) have been analyzed by Kronland-Martinet, Seip, Torrésani, et al., to deal with the problem of detecting spectral lines in NMR data [26, 10, 7]. Another technique, that of computing critical frequencies in ECoG seizure data using wavelet transform striations, was described in [3]. These frequencies are related to the *instantaneous frequency* [10] $h'(t)$ of s at t; and, with our periodicity detection and computation problem in mind, $1/h'(t)$ is the *instantaneous period* of s at t.

We shall approach the analysis of (5.1) with a method similar to the aforementioned three step method, e.g., [5]:

1. Non–noisy data are analyzed through spectral and wavelet methods to extract specific periodic patterns of interest, i.e., f;

2. We construct an *optimal generalized Haar wavelet* designed to detect f;

3. Using our discretized version of the continuous wavelet transform and wavelet transform averaging techniques, we detect occurrence and period of these periodicities in real time.

Essentially, we shall describe an algorithm to detect lattice patterns of relative maxima in periodic wavelet transforms. The output of the algorithm is the period of a periodic component in the analyzed signal. The algorithm is based on averaging methods.

5.2. The algorithm. Let f be a T_0–periodic function, and let ψ^c be an even generalized Haar wavelet of degree one, i.e., $\psi^c|_{[n,n+1)} = c_n$ for $n = 0, \dots, N-1$, $c = (c_0, c_1, \dots, c_{N-1}) \in \mathbb{C}^N$, N fixed.

Proposition 2.3 implies that the wavelet transform of the non–normalized wavelet transform $W_{\psi^c} f$ is identical on each cell

$$[b + nT_0, b + (n+1)T_0] \times [jMT_0, (j+1)MT_0]$$

for $n \in \mathbb{Z}$ and $j \in \mathbf{N_0}$. Figure 5.1 shows a non–normalized wavelet transform in topographical form.

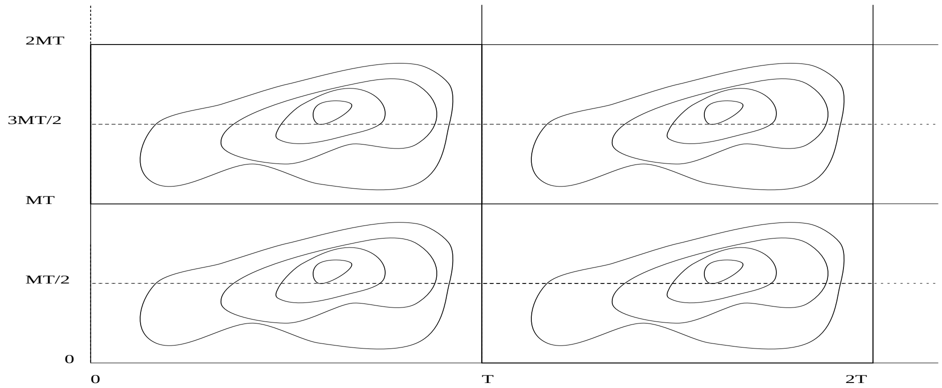

FIG. 5.1. *Time–scale periodicity in topographical form.*

5.2.1. Non–normalized wavelet transform. If $R, Q \in \mathbb{N}$, then the periodicities of the non–normalized wavelet transform imply that

$$W_{\psi^c} f(b,a) = \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_{\psi^c} f(b,a)$$
$$= \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_{\psi^c} f(b+rT_0, a+qT_0). \tag{5.2}$$

Suppose we are given a noisy signal s of the form $s(t) = f(t) + N(t)$ where f is T_0–periodic and N is noise. In order to gain knowledge of the period T_0 of f, we define the average

$$U_{\psi^c}^{Q,R} s(b,a,T) = \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_{\psi^c} s(b+rT, a+qT),$$

where $T \in \mathbb{R}^+, a \in (0,T)$, and $b \in [0,T)$. Clearly, by Proposition 2.3 and Equation (5.2), we have

$$U_{\psi^c}^{Q,R} f(b,a,T_0) = W_{\psi^c} f(b,a)$$

for the periodic signal f. Define

$$Z_{\psi^c}^{Q,R} s(T) = \sup_{a\in(0,T), b\in[0,T)} |U_{\psi^c}^{Q,R} s(b,a,T)|.$$

Therefore,

$$Z_{\psi^c}^{Q,R} f(T_0) = \sup_{a\in(0,T_0), b\in[0,T_0)} |W_{\psi^c} f(b,a)|,$$

which we maximized in Section 4. Further, we expect that $Z_{\psi^c}^{Q,R} f(T)$ is "small" for $T \neq k \cdot T_0, k \in \mathbb{N}$ and Q and R large.

Note that for the noisy signal $s = f + N$, we further expect that

$$Z_{\psi^c}^{Q,R} s(T_0) \approx \sup_{a\in(0,T_0), b\in[0,T_0)} |W_{\psi^c} f(b,a)|$$

and that $Z_{\psi^c}^{Q,R} s(T)$ is small if $T \neq T_0$.

5.2.2. Normalized wavelet transform. In order to analyze an $L^p(\mathbb{R})$ normalized wavelet transform, where $1 \leq p < \infty$, we define

$$v^{p,Q}(a,T) = a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT)^{-\frac{1}{p}},$$

where $a, T \in \mathbb{R}^+$. We compute

$$W_{\psi^c}^p f(b,a) = W_{\psi^c}^p f(b,a) \frac{1}{v^{p,Q}(a,T_0)} a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT_0)^{-\frac{1}{p}}$$
$$= \frac{1}{v^{p,Q}(a,T_0)} \sum_{q=0}^{Q} (a+qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_{\psi^c}^p f(b,a)$$
$$= \frac{1}{v^{p,Q}(a,T_0)} \sum_{q=0}^{Q} W_{\psi^c}^p f(b, a+qT_0)$$
$$= \frac{1}{v^{p,Q}(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_{\psi^c}^p f(b+rT_0, a+qT_0).$$

As in the non–normalized case, we are motivated to define

$$V^{p,Q,R}_{\psi^c}s(b,a,T) = \frac{1}{v^{p,Q}(a,T)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W^p_{\psi^c}s(b+rT, a+qT)$$

for any signal s, where $T \in \mathbb{R}^+, a \in (0,T)$, and $b \in [0,T)$.

For the T_0–periodic signal f we have

$$V^{p,Q,R}_{\psi^c}f(b,a,T_0) = W^p_{\psi^c}f(b,a).$$

Thus, defining

$$Z^{p,Q,R}_{\psi^c}s(T) = \sup_{a\in(0,T), b\in[0,T)} |V^{p,Q,R}_{\psi^c}s(b,a,T)|,$$

for any signal s, we have

$$Z^{p,Q,R}_{\psi^c}f(T_0) = \sup_{a\in(0,T), b\in[0,T)} |W^p_{\psi^c}f(b,a)|.$$

Note that, in this case, the assertion that $Z^{p,Q,R}_{\psi^c}f(T)$ is "small" for $T \neq T_0$ and Q, R large, is supported by the fact that if $a, T \in \mathbb{R}^+$, then

$$\lim_{Q\to\infty} v^{p,Q}(a,T) = \lim_{Q\to\infty} a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT)^{-\frac{1}{p}} = a^{\frac{1}{p}} \lim_{Q\to\infty} \sum_{q=0}^{Q} \left(\frac{1}{a+qT}\right)^{\frac{1}{p}} = \infty.$$

5.3. The algorithm for even or odd generalized Haar wavelets. Let f be a T_0 periodic function, and let ψ^c be either an even generalized Haar wavelet of degree 1, i.e., $\psi^c|_{[n,n+1)} = \psi^c|_{[-n-1,-n)} = c_n$ for $n = 0,\ldots,N-1$, $c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbb{C}^N$, or an odd generalized Haar wavelet of degree 1, i.e., $\psi^c|_{[n,n+1)} = -\psi^c|_{[-n-1,-n)} = c_n$ for $n = 0,\ldots,N-1$, $c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbb{C}^N$, where N is fixed.

Due to Proposition 2.5, the non–normalized wavelet transform $W^p_{\psi^c}f$ is in both cases essentially the same, i.e., the same up to a flip and a sign, on the cells

$$[b+nT_0, b+(n+1)T_0] \times [jMT_0/2, (j+1)MT_0/2]$$

for $n \in \mathbb{Z}$ and $j \in \mathbb{N} \cup \{0\}$. Figure 5.2 shows the resulting wavelet transform.

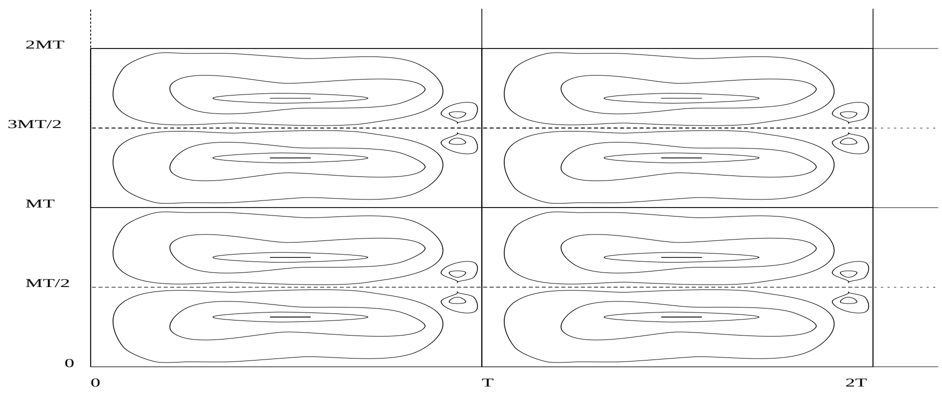

FIG. 5.2. *Time–scale periodicity for odd or even wavelets in topographical form.*

5.3.1. Non–normalized wavelet transform. Now we define the following alternative for averaging:

$$U_{\psi^c}^{Q,R} f(b,a,T) = \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c} f(b+rT, a+qT) \pm W_{\psi^c} f(b+rT, T-a+qT)),$$

where $T \in \mathbb{R}^+, a \in (0, T/2)$, and $b \in [0, T)$. Here, and in the following, $\pm$ denotes $-$ if ψ^c is even and $+$ if ψ^c is odd.

By Proposition 2.5, we have

$$\begin{aligned} U_{\psi^c}^{Q,R} f(b,a,T_0) &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c} f(b+rT_0, a+qT_0) \\ &\quad \pm W_{\psi^c} f(b+rT_0, T_0-a+qT_0)) \\ &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c} f(b,a) \pm W_{\psi^c} f(b, T_0-a)) \\ &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c} f(b,a) + W_{\psi^c} f(b,a)) = W_{\psi^c} f(b,a) \end{aligned}$$

for any T_0 periodic function f.

We proceed as before by defining the test statistic

$$Z_{\psi^c}^{Q,R} s(T) = \sup_{a \in (0,T/2), b \in [0,T)} |U_{\psi^c}^{Q,R} s(b,a,T)|$$

for any signal s.

5.3.2. Normalized wavelet transform. If we are using an $L^p(\mathbb{R})$ normalized wavelet transform for $1 \le p < \infty$, let us define

$$v^{p,Q}(a,T) = a^{\frac{1}{p}} \sum_{q=0}^{Q} ((a+qT)^{-\frac{1}{p}} + (T-a+qT)^{-\frac{1}{p}}), a, T \in \mathbb{R}^+.$$

As before we write

$$V_{\psi^c}^{p,Q,R} s(b,a,T) = \frac{1}{v^{p,Q}(a,T)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c}^p s(b+rT, a+qT) \pm W_{\psi^c}^p f(b+rT, T-a+qT)),$$

with $T \in \mathbb{R}^+$, $a \in (0,T)$ and $b \in [0,T)$.

For $T = T_0$, we get

$$
\begin{aligned}
V_{\psi^c}^{p,Q,R}s(b,a,T_0) &= \frac{1}{v^{p,Q}(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_{\psi^c}^{p} s(b+rT_0, a+qT_0) \\
&\quad \pm W_{\psi^c}^{p} f(b+rT_0, T_0 - a + qT_0)) \\
&= \frac{1}{v^{p,Q}(a,T_0)} \sum_{q=0}^{Q} (W_{\psi^c}^{p} s(b, a+qT_0) \pm W_{\psi^c}^{p} f(b, T_0 - a + qT_0)) \\
&= \frac{1}{v^{p,Q}(a,T_0)} \sum_{q=0}^{Q} ((a+qT_0)^{-\frac{1}{p}} (a+qT_0)^{\frac{1}{p}} W_{\psi^c}^{p} s(b, a+qT_0) \\
&\quad \pm (T_0 - a + qT_0)^{-\frac{1}{p}} (T_0 - a + qT_0)^{\frac{1}{p}} W_{\psi^c}^{p} f(b, T_0 - a + qT_0)) \\
&= \frac{1}{v^{p,Q}(a,T_0)} \sum_{q=0}^{Q} ((a+qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_{\psi^c}^{p} s(b,a)) \\
&\quad \pm (\pm)(T_0 - a + qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_{\psi^c}^{p} f(b,a)) \\
&= W_{\psi^c}^{p} f(b,a) \frac{1}{v^{p,Q}(a,T_0)} a^{\frac{1}{p}} \sum_{q=0}^{Q} ((a+qT_0)^{-\frac{1}{p}} + (T_0 - a + qT_0)^{-\frac{1}{p}}) \\
&= W_{\psi^c}^{p} f(b,a).
\end{aligned}
$$

We can conclude as in Section 5.2.2.

5.4. Examples of the algorithm. In Example 5.1, Example 5.2, and Example 5.3 we apply this method to the signals introduced in Example 4.4 and Example 4.5.

EXAMPLE 5.1. Figure 5.3 A shows the original signal $f(t) = \sin(2\pi t) + \sin(4\pi t) + \sin(6\pi t) + \sin(8\pi t) + \sin(10\pi t) + \sin(12\pi t)$, and Figure 5.3 B represents the absolute value of its Fourier transform. Figure 5.3 C displays the normalized ($p = 1.75$) wavelet transform of this signal, obtained using the optimal generalized Haar wavelet displayed in Figure 4.2 ($N = 8$). $Z_{\psi^c}^{p,Q,R} f(T)$ is then calculated for $T = 1, \dots, 25$ and shown in Figure 5.3 D. The location of the maximum of Z implies the occurence of the periodic signal with period length of 20 samples.

This technique can also be applied successfully to synthesized noisy data as is illustrated in Example 5.2.

EXAMPLE 5.2. In this case white noise is added to the signal in Example 4.4 which is displayed in Figure 4.1. The resulting signal is illustrated in Figure 5.4 A. The same wavelet as in Example 5.1 is applied. The graph in Figure 5.4 D has abscissa T, representing the number of samples per period, and ordinate $Z(T)$, representing $Z_{\psi^c}^{p,Q,R} s(T)$. The graph is automatically generated by the algorithm of Section 5.2. In this case the maximum of $Z(T)$ is clearly observed to be at $T = 20$, even though this underlying de facto periodicity (from Figure 4.1) is by no means apparent from direct observation or analysis of Figure 5.4 A.

EXAMPLE 5.3. The seizure signal F constructed in Example 4.5 and shown in Figure 4.3 has a periodicity characterized by its construction using 13 samples per period. We compute the $p = 1.35$ normalized wavelet transform of F. $Z_{\psi^c}^{p,Q,R} F(T)$ is then calculated for $T = 1, \dots, 20$. The maximum of Z in Figure 5.5 implies the occurrence of the periodic signal with period length of 13 samples.

6. The discretized version of the continuous wavelet transform and implementation. In order to apply the results of the preceding section, we need to

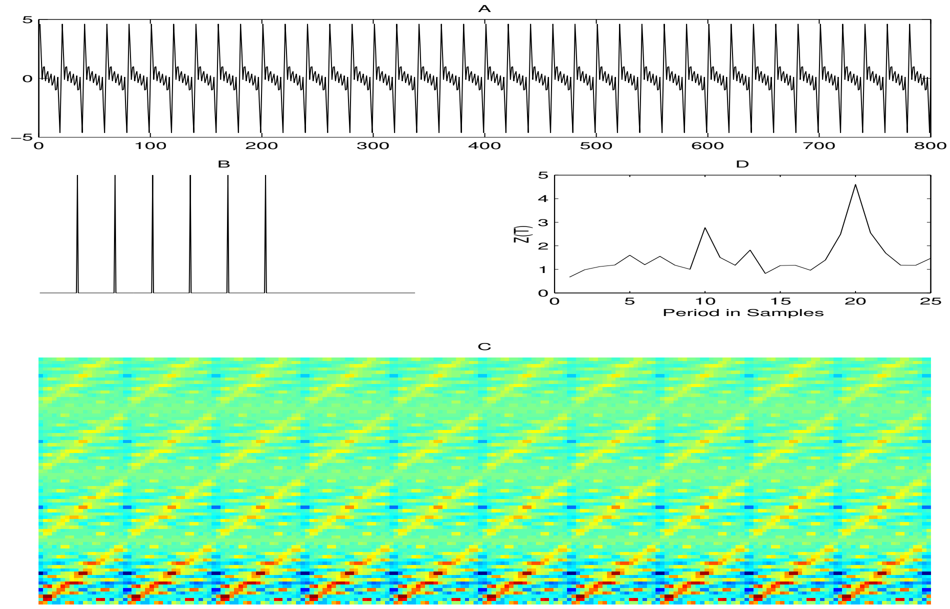

FIG. 5.3. *A:* $f(t) = \sin(2\pi t) + \sin(4\pi t) + \sin(6\pi t) + \sin(8\pi t) + \sin(10\pi t) + \sin(12\pi t)$. *B: Absolute value of its discrete Fourier transform. C: Wavelet transform using the optimal generalized Haar wavelet obtained for* $p = 1.75$ *and* $N = 8$*. D:* $Z_{\psi^c}^{p,Q,R} f(T)$.

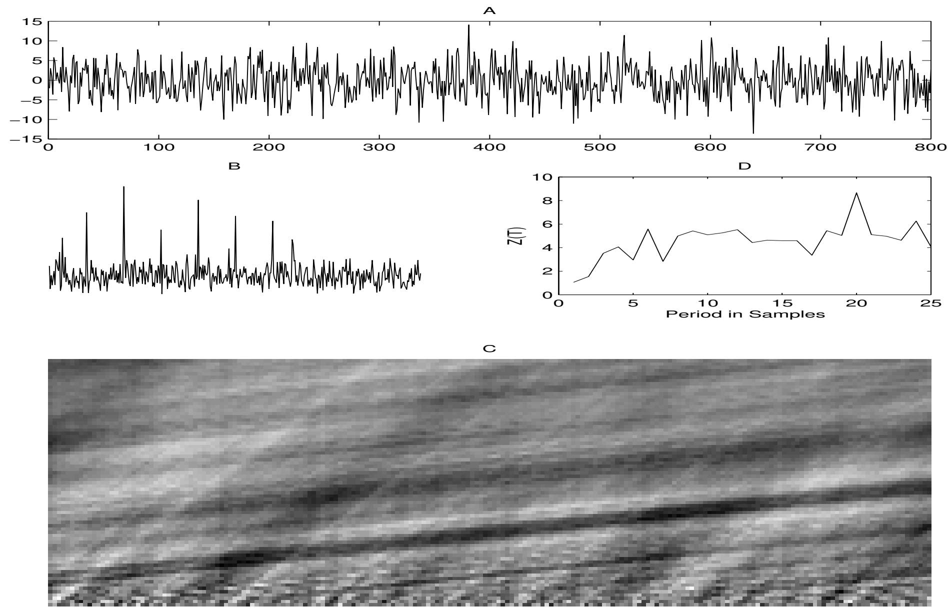

FIG. 5.4. *A:* $s(t) = \sin(2\pi t) + \sin(4\pi t) + \sin(6\pi t) + \sin(8\pi t) + \sin(10\pi t) + \sin(12\pi t) +$ *white noise. B: Absolute value of its discrete Fourier transform. C: Wavelet transform using the optimal generalized Haar wavelet obtained for* $p = 1.75$ *and* $N = 8$ *in Figure 4.2. D:* $Z_{\psi^c}^{p,Q,R} s(T)$.

discretize our results.

Let us assume that we sampled a signal f and obtained the sequence $\{f[n]\}_{n \in \mathbb{Z}}$.

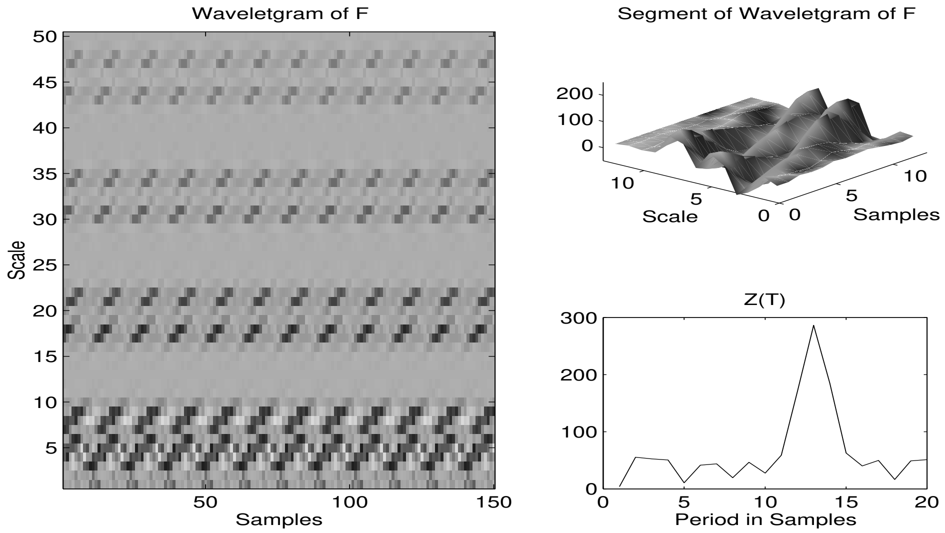

FIG. 5.5. *The wavelet transform of the signal shown in Figure 4.3 sampled at 13 samples per period, and the function* $Z^{p,Q,R}_{\psi^c}F(T)$ *indicating the periodicity of 13 samples.*

Let ψ be a generalized Haar wavelet of degree 1. To avoid ambiguous notation, we let

$$\overline{\psi} = (\ldots, \overline{\psi}\,[-1]\,, \overline{\psi}\,[0]\,, \overline{\psi}\,[1]\,, \ldots)$$

be the vector representing ψ, i.e., $\overline{\psi}\,[k] = \psi(k) = c_k$ for $k \in \mathbb{Z}$.

We shall replace our continuous wavelet transform

$$W^p_\psi f(b,a) = a^{-1/p} \int f(t)\psi(\tfrac{t-b}{a})\,dt$$

with the following discretized version

$$W^p_\psi f[n,m] = m^{-1/p} \sum_{k\in\mathbb{Z}} f[k]\psi(\tfrac{k-n}{m}) = m^{-1/p} \sum_{k\in\mathbb{Z}} f[k]\overline{\psi}\left[\lfloor \tfrac{k-n}{m} \rfloor\right], \tag{6.1}$$

$m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$. $\lfloor x \rfloor$ denotes the largest integer less or equal x. The second equality of (6.1) is a consequence of the fact that ψ is a generalized Haar wavelet of degree 1. Conditions on ψ so that the family $\{\overline{\psi}[\lfloor \frac{\cdot - n}{m} \rfloor]\}_{m\in\mathbb{Z}^+, n\in\mathbb{Z}}$ forms a frame for $l^2(\mathbb{Z})$ are discussed in [19, 20].

We can easily rewrite (6.1) in the more convenient form:

$$\begin{aligned}
W^p_\psi f[n,m] &= m^{-1/p} \sum_{k\in\mathbb{Z}} f[k]\overline{\psi}\left[\lfloor \tfrac{k-n}{m} \rfloor\right] \\
&= m^{-1/p} \begin{pmatrix} & \vdots & \vdots & & \vdots & \vdots \\ + & f[n] & \overline{\psi}\,[0] & + \ldots + & f[n+m-1] & \overline{\psi}\,[0] \\ + & f[n+m] & \overline{\psi}\,[1] & + \ldots + & f[n+2m-1] & \overline{\psi}\,[1] \\ + & f[n+2m] & \overline{\psi}\,[2] & + \ldots + & f[n+3m-1] & \overline{\psi}\,[2] \\ & \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \\
&= m^{-1/p} \sum_{r\in\mathbb{Z}} \left(\sum_{l=0}^{m-1} f[n+mr+l] \right) \overline{\psi}\,[r]\,.
\end{aligned} \tag{6.2}$$

To serve as an example, we shall prove a discrete version of Proposition 2.3.

PROPOSITION 6.1. *Let ψ be a generalized Haar wavelet of degree 1, and let $\{f[n]\}_{n\in\mathbb{Z}}$ be a T–periodic sequence, $T \in \mathbb{Z}^+$, i.e., $f[n+T] = f[n]$ for all $n \in \mathbb{Z}$. Then $m^{1/p}W^p_\psi f[n,m]$ is T–periodic in n and T–periodic in m.*

Proof. The T–periodicity in n follows directly from (6.2) and the fact that $f[n+T] = f[n]$ for all $n \in \mathbb{Z}$. Further, setting $c = \sum_{l=0}^{T-1} f[l]$, we obtain

$$\begin{aligned}(m+T)^{1/p}W^p_\psi f[n,m+T] &= \sum_{r\in\mathbb{Z}}\left(\sum_{l=0}^{m+T-1} f[n+(m+T)r+l]\right)\overline{\psi}[r]\\ &= \sum_{r\in\mathbb{Z}}\left(\sum_{l=0}^{m+T-1} f[n+mr+l]\right)\overline{\psi}[r]\\ &= m^{1/p}W^p_\psi f[n,m] + \sum_{r\in\mathbb{Z}}\left(\sum_{l=m}^{m+T-1} f[n+mr+l]\right)\overline{\psi}[r]\\ &= m^{1/p}W^p_\psi f[n,m] + c\sum_{r\in\mathbb{Z}}\overline{\psi}[r] = m^{1/p}W^p_\psi f[n,m].\end{aligned}$$

□

To analyze a signal through a “continuous” wavelet transform is expensive, since we need to calculate a large number of coefficients $W^p_\psi f[n,m]$. On the other hand the redundancy inherent in such calculation provides stability and robustness to noise; and so it is important to seek fast algorithms. A priori and for large m, the elementary operations needed to calculate $W^p_\psi f[n,m]$ are of order m. The restriction to generalized Haar wavelets gives rise to a recursive procedure to obtain these coefficients. This procedure significantly reduces the number of calculations needed. In fact, if ψ is supported on $[0,N]$, we shall show that to obtain $W^p_\psi f[n,m]$ from $W^p_\psi f[n-1,m]$ or $W^p_\psi f[n,m-1]$ requires only N multiplications, regardless of the size of m and the support of $\overline{\psi}[\lfloor\frac{\cdot-n}{m}\rfloor]$. For convenience, we shall omit the normalization factor $m^{-1/p}$. This factor is certainly independent of both wavelet and signal, and would be multiplied to $W^p_\psi f[n,m]$ in the last step of an implementation.

Let us begin with the trivial case, obtaining $W^p_\psi f[n,m]$ from $W^p_\psi f[n-1,m]$. We have

$$\begin{aligned}W^p_\psi f[n,m] - W^p_\psi f[n-1,m] &= \sum_{r\in\mathbb{Z}}\left(\sum_{l=0}^{m-1} f[n+mr+l] - \sum_{l=0}^{m-1} f[n-1+mr+l]\right)\overline{\psi}[r]\\ &= \sum_{r=0}^{N-1}\left(f[n+mr+m-1] - f[n-1+mr]\right)\overline{\psi}[r]\,.\end{aligned}$$

To obtain $W^p_\psi f[n,m]$ from $W^p_\psi f[n,m-1]$ for scales $m \geq N$ is best understood through Figure 6.1 and Figure 6.2. Again, many products appearing in the summation representing $W^p_\psi f[n,m]$ in (6.2) are already included in $W^p_\psi f[n,m-1]$. In Figure 6.1, we write the part of the signal f that is relevant to obtain $W^p_\psi f[n,m]$ in a rectangular pattern with N rows and m columns. We obtain the non–normalized coefficient $W^p_\psi f[n,m]$ by multiplying the r-th row by $\overline{\psi}[r-1]$ for $r = 1,\ldots,N$ and by adding the results. This is illustrated in Figure 6.1.

In Figure 6.2 we illustrate the contribution of the same segment of f to $W^p_\psi f[n,m-1]$.

$\overline{\psi}[0]$ $f[n\text{-}mN+1]$	$f[n\text{-}mN+2]$					$f[n\text{-}m(N\text{-}1)]$
$\overline{\psi}[1]$ $f[n\text{-}m(N\text{-}1)+1]$	$f[n\text{-}m(N\text{-}1)+2]$					$f[n\text{-}m(N\text{-}2)]$
$\overline{\psi}[N-1]$ $f[n\text{-}m+1]$	$f[n\text{-}m+2]$					$f[n]$

FIG. 6.1. *Contributions of* $f[n-mN+1],\ldots,f[n]$ *to* $W^p_\psi f[n,m]$.

$f[n\text{-}mN+1]$	$f[n\text{-}mN+2]$			$\overline{\psi}[0]$		$f[n\text{-}m(N\text{-}1)]$
$f[n\text{-}m(N\text{-}1)+1]$	$f[n\text{-}m(N\text{-}1)+2]$	$\overline{\psi}[0]$	$\overline{\psi}[1]$			$f[n\text{-}m(N\text{-}2)]$
	$\overline{\psi}[N\text{-}3]$	$\overline{\psi}[N\text{-}2]$				
$\overline{\psi}[N\text{-}2]$ $f[n\text{-}m+1]$	$\overline{\psi}[N\text{-}1]$ $f[n\text{-}m+2]$					$f[n]$

FIG. 6.2. *Contributions of* $f[n-mN+1],\ldots,f[n]$ *to* $W^p_\psi f[n,m-1]$.

The difference $W^p_\psi f[n,m] - W^p_\psi f[n,m-1]$ is easily calculated:

$$\begin{aligned} W^p_\psi f[n,m] - W^p_\psi f[n,m-1] &= \overline{\psi}\,[0] \sum_{l=0}^{N-1} f[n-mN+l] \\ &\quad + (\overline{\psi}\,[1] - \overline{\psi}\,[0]) \sum_{l=1}^{N-1} f[n-m(N-1)+l] \\ &\quad + \ldots + (\overline{\psi}\,[N-1] - \overline{\psi}\,[N-2]) f[n]. \end{aligned}$$

Implementing this procedure, we use the vector $(\overline{\psi}\,[0]\,, \overline{\psi}\,[1] - \overline{\psi}\,[0]\,, \ldots, \overline{\psi}\,[N-1] - \overline{\psi}\,[N-2])$ in order to reduce redundant calculations.

REMARK 6.2. Besides the theory developed by Pfander [19, 20], there have been several other interesting formulations dealing with wavelet frames on discrete groups. These include formulations by Walnut [29], Flornes et al. [12], Steidl [27], Johnston [13], and Antoine et al. [1]. Our calculations in this section show that Pfander's formulation is particularly effective for providing fast algorithms in the case of generalized Haar wavelets.

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