Balian - Low Theorem and Regularity of Gabor Systems

John J. Benedetto
Wojciech Czaja
Przemysław Gadziński
Alexander M. Powell

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Abstract

For any positive real numbers A, B, and d satisfying the conditions $\frac{1}{A} + \frac{1}{B} = 1$, $d > 2$, we construct a Gabor orthonormal basis for $L^2(\mathbb{R})$, such that the generating function $g \in L^2(\mathbb{R})$ satisfies the condition: $\int_{\mathbb{R}} |g(x)|^2(1 + |x|^A)/\log^d(2 + |x|)\, dx < \infty$ and $\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2(1 + |\xi|^B)/\log^d(2 + |\xi|)\, d\xi < \infty$.

1 Introduction

The harmonic analysis version of *Heisenberg's uncertainty principle* states that a function f and its Fourier transform $\hat{f}$ cannot be well localized simultaneously. It can be written in the form of the following inequality:

$$\|f\|_2^4 \leq 16\pi^2 \left(\int_{\mathbb{R}} |f(x)|^2|x|^2 dx\right) \left(\int_{\widehat{\mathbb{R}}} |\hat{f}(\xi)|^2|\xi|^2 d\xi\right), \tag{1.1}$$

where for $\xi \in \widehat{\mathbb{R}}$ the Fourier transform $\hat{f}$ of a function f is formally defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The uncertainty principle has many generalizations, for example those of Beurling–Hörmander, Benedetto–Heinig, and Cowling–Price, which involve more general weight functions and different norms, among other possible extensions. We refer the reader to the work of Bonami, Demange, and Jaming [7] and Wilczok [25], as well as [5], for more on this subject. Let us also mention the imposing survey by Folland and Sitaram [9] and the treatise by Havin and Jöricke [15] for extensive treatments of uncertainty principle inequalities in harmonic analysis.

While the right side of (1.1) is one particular way of studying the joint localization of f and $\hat{f}$, there are many other approaches. For example, the continuous Gabor transform is a very natural tool for trying to understand f and $\hat{f}$ simultaneously. Given $f, g \in L^2(\mathbb{R})$, the *continuous Gabor transform* $S_g(f)$ is defined by

$$S_g(f)(x, \xi) = \int_{\mathbb{R}} f(t) e^{2\pi i t \cdot \xi} \overline{g(t-x)} dt,$$

for some function $g \in L^2(\mathbb{R})$. $S_g(f)$ is also called the *short-time Fourier transform* or *windowed Fourier transform* with *window* g. It turns out that the connection between the uncertainty principle and Gabor transform is deep, as one can see, for example, in the work of Gröchenig [13].

For a given function $g \in L^2(\mathbb{R})$, a discrete version of the short-time Fourier transform is the collection of functions $g_{m,n}$ defined by

$$g_{m,n}(x) = e^{-2\pi i x n} g(x - m),$$

for $m, n \in \mathbb{Z}$. This collection of functions is called the *Gabor system* generated by g. When $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, we shall refer to it as a *Gabor orthonormal basis.*

The following result is a version of the uncertainty principle for Gabor systems. It was first proved independently by Balian [3] and Low [22]. For more references see, e.g., [5], [6].

Theorem 1.1 (Balian–Low theorem) *Let $g \in L^2(\mathbb{R})$ have the property that $\{g_{m,n} : m,n \in \mathbb{Z}\}$ is a Gabor orthonormal basis for $L^2(\mathbb{R})$. Then either*

$$\int_{\mathbb{R}} |g(x)|^2 |x|^2 dx = \infty \quad \text{or} \quad \int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 |\xi|^2 d\xi = \infty. \tag{1.2}$$

It is natural to ask if this result is sharp. That is, can one replace the weights $|x|^2$ and $|\xi|^2$ with something smaller? We shall show in the next sections that one cannot expect to have the Balian–Low "phenomenon" with powers of $|x|$ and $|\xi|$ less than 2 in (1.2). In fact, as a special case of our main theorem, Theorem 3.11, we shall obtain the following result.

Theorem 1.2 *Let $\epsilon > 0$. There is a constructible $g \in L^2(\mathbb{R})$ with the property that $\{g_{m,n} : m,n \in \mathbb{Z}\}$ is a Gabor orthonormal basis for $L^2(\mathbb{R})$, and such that*

$$\int_{\mathbb{R}} |g(x)|^2 \frac{1+|x|^2}{\log^{1+\epsilon}(2+|x|)} dx < \infty \tag{1.3}$$

and

$$\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 \frac{1+|\xi|^2}{\log^{2+\epsilon}(2+|\xi|)} d\xi < \infty. \tag{1.4}$$

Before proceeding to the next section and the statement and proof of Theorem 3.11, we define the Zak transform and state the result, Criterion 1.3, which influeces the direction of our proof.

Let $\mathbb{I}$ denote the interval $[0,1]$, and let $\mathbb{T}$ denote the circle group $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$, where the relation $\sim$ identifies 0 and 1. With a function $f \in L^2(\mathbb{R})$ we associate the function $Z(f)$ formally defined by

$$Z(f)(x,\xi) = \sum_{m\in\mathbb{Z}} f(x+m)e^{-2\pi i m\xi} = \mathcal{F}f^x(\xi), \quad (x,\xi) \in \mathbb{R}\times\mathbb{T},$$

where $f^x(k) = f(x+k)$, for $x \in \mathbb{I}$ and $k \in \mathbb{Z}$, and where $\mathcal{F}f^x$ is the Fourier transform of the sequence $\{f^x(k)\}$. It is clear that

$$Z(f)(k+x,\xi) = Z(f)(x,\xi)e^{2\pi i k\xi} \tag{1.5}$$

and

$$Z(f)(x,\xi+k) = Z(f)(x,\xi), \tag{1.6}$$

and so we shall often restrict the function $Z(f)$ to the set $(x,\xi) \in \mathbb{I}\times\mathbb{T}$. The function $Z(f)$ is called the *Zak transform* of the function f. *Z is a unitary map between $L^2(\mathbb{R})$ and $L^2(\mathbb{I}\times\mathbb{T})$*, and it has been extensively used in the study of Gabor systems, e.g., [10], [11].

The Zak transform was first introduced by Gelfand [12]. Later it was generalized to locally compact abelian groups by Weil [24], and it was independently rediscovered by Zak [26], [27]. Other references include the works of Auslander and Tolimieri [1], Brezin [8], Igusa [19], and Janssen [20], [21]. We would also like to refer the reader to the works of Heil and Walnut [16], Hernández and Weiss [17], and the new book by Gröchenig [14]. In abstract harmonic analysis the Zak transform is called the *Weil–Brezin map.*

An elementary consequence of the unitarity of Z is the following criterion for Gabor orthonormal bases, e.g., [4].

Criterion 1.3 *The set of functions $g_{m,n}, m,n \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Z(g)(x,\xi)| = 1$ a.e. for $(x,\xi) \in \mathbb{I}\times\mathbb{T}$.*

We shall use this fact in our construction of a Gabor system generating function, which possesses as much smoothness and decay as possible under the constraints posed by the Balian–Low theorem. In fact, we shall construct a function on $\mathbb{I} \times \mathbb{T}$ which is the Zak transform of a function $g \in L^2(\mathbb{R})$, and which equals 1 in absolute value almost everywhere, cf., [4] for elementary constructions of this type; and, further, g will satisfy the decay and regularity conditions (1.3) and (1.4).

In the next section we shall give some necessary definitions and present the construction of the aforementioned function. In section 3, we shall formulate the main results, and prove the required estimates. Finally, in section 4, we shall state some related questions and remarks.

2 Construction

As mentioned in section 1, our idea for proving Theorem 1.2 is based on Criterion 1.3. As such we shall begin with functions of the form

$$Z(g)(x,\xi) = h(x,\xi) = e^{2\pi i H(x,\xi)}, \tag{2.1}$$

where $H(x,\xi) \in \mathbb{R}$ and $H(1,\xi) = H(0,\xi) + k_\xi$ for some $k_\xi \in \mathbb{Z}$. Notice that the values $H(1,\xi)$ and $H(0,\xi)$ can differ, but the values $e^{2\pi i H(1,\xi)}$ and $e^{2\pi i H(0,\xi)}$ are equal. The variable ξ for the function h can be treated as an element of $\mathbb{T}$, while it must be treated as a number in $\mathbb{I}$ for the function H. (We want to alert the reader to this fact to avoid any confusion.)

There do not exist smooth functions h, as above, on $\mathbb{I} \times \mathbb{T}$ with the property (1.5). Even more is true: there do not exist continuous functions of the form $e^{2\pi i H(x,\xi)}$ on $\mathbb{I} \times \mathbb{T}$, which satisfy (1.5). This assertion follows from a well known result of Janssen [20], and also announced by Zak (1975), that if the Zak transform of a function $f \in L^2(\mathbb{R})$ is continuous, it must have a zero, see, e.g., Theorem 4.3.4 of [16]. This, in turn, follows from Lemma VI.1.7 in [23]. See also [1] and [2].

We shall construct a function g so that $Z(g)$ in (2.1) has the smallest possible *singular support*, i.e., the smallest possible closed set of points where the function is not infinitely differentiable.

In view of our above observations, given a function h, defined by (2.1) on $\mathbb{I} \times \mathbb{T}$, it must have at least one *point of singularity*, i.e., an element of the singular support of h. We expect that if $Z(g)$ has a single singularity, it will give rise to good decay properties of g and $\hat{g}$.

Before proceeding with our construction, we need to introduce a few auxiliary functions.

Definition 2.1 *1) Let $\varphi \in \mathcal{C}^{\infty}(\mathbb{R})$ be a function with the following properties:*

$$\begin{array}{ll} \varphi(x) = -1, & \textit{for } x \in (-\infty, 0], \\ \varphi(x) = 0, & \textit{for } x \in [1, \infty), \\ \varphi(x) \in [-1, 0], & \textit{for all } x \in \mathbb{R}. \end{array}$$

2) Let $\psi(x) = \chi_{[0,\infty)}(x)\, x^a$, where $a > 0$ is a fixed number.

3) Given $0 < \epsilon < \frac{1}{8}$, let $\gamma \in \mathcal{C}^{\infty}(\mathbb{R})$ be a function that satisfies

$$\operatorname{supp}(\gamma) \subseteq [-2\epsilon, 2\epsilon], \quad \gamma(x) = 1 \textit{ for } x \in [-\epsilon, \epsilon],$$

and

$$\gamma(x) \in [0, 1] \textit{ for all } x \in \mathbb{R}.$$

The next lemma is the basis for our whole construction.

Lemma 2.2 *There exists a function $H : \left[-\frac{1}{2}, 1\right] \times \mathbb{I}$, with the following properties:*

1) $H(x, \xi) = 0$ for $x \in \left[-\frac{1}{2}, 0\right]$;

2) $H(x, \xi) = \varphi\left(\frac{\xi}{\psi(x)}\right)$ for $x \in (0, 2\epsilon]$, where $\epsilon > 0$ is chosen for the definition of γ;

3) $H(x, 0) = 0$ for $x \in \left[-\frac{1}{2}, 0\right]$ and $H(x, 0) = -1$ for $x \in (0, 1]$;

4) $H(1 + x, \xi) = H(x, \xi) + (\xi - 1)$ for $x \in \left[-\frac{1}{2}, 0\right]$; and

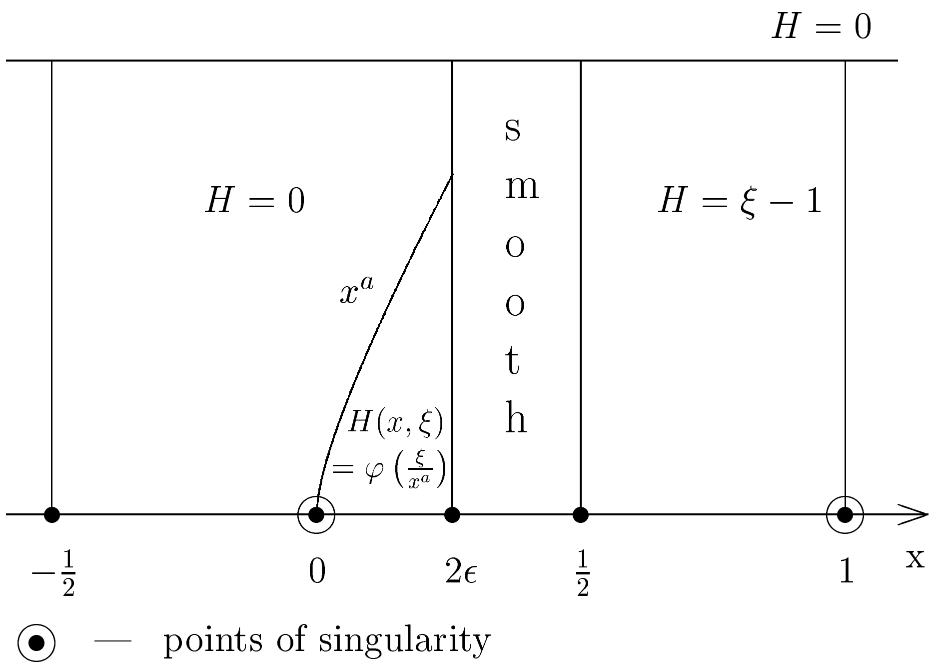
5) the function $e^{2\pi i H(x,\xi)} : \left[-\frac{1}{2}, 1\right] \times \mathbb{T} \to \mathbb{C}$ is of class $\mathcal{C}^{\infty}$ away from the points $(0, 0)$ and $(1, 0)$.

Proof. It is easy to see that if $(x,\xi) \in \left[-\frac{1}{2}, 1\right] \times \mathbb{I}$, then

$$\begin{aligned} H(x,\xi) &= \gamma\left(\frac{x}{2}\right) \chi_{(0,1]}(x) \varphi\left(\frac{\xi}{\psi(x)}\right) \\ &+ \left(1 - \gamma\left(\frac{x}{2}\right)\right) \chi_{(0,1]}(x)(\xi - 1), \end{aligned}$$

satisfies properties 1 through 5. ■

Note that the function H has two singularities in $\left[-\frac{1}{2}, 1\right] \times \mathbb{I}$. However, it is easy to find a unit square in the domain of H with a single point of singularity - the minimal singular support we have mentioned earlier.



We are now in position to define the functions h and g of (2.1). The function g will be the generating function for a Gabor system orthonormal basis for $L^2(\mathbb{R})$ with the decay and regularity conditions asserted in Theorems 1.2 and 3.11.

Definition 2.3 *1) Let $a > 0$ be fixed as in Definition 2.1, define H as in Lemma 2.2, and set*

$$h(x,\xi) = e^{2\pi i H(x,\xi)},$$

for $(x,\xi) \in \left[-\frac{1}{2}, 1\right] \times \mathbb{T}$. By property 4) of Lemma 2.2, the function h can be extended to $\mathbb{R} \times \mathbb{T}$ by means of the formula

$$h(k+x,\xi) = h(x,\xi)\, e^{2\pi i k(\xi-1)} = h(x,\xi)\, e^{2\pi i k\xi}.$$

2) Now, define the function $g = g_a$ on $\mathbb{R}$ by means of the inverse Zak transform of h. Thus, we set

$$g(k+x) = Z^{-1}(h)(k+x),$$

where $k \in \mathbb{Z}$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.
3) Let us also divide the function g into singular and smooth parts, g_1 and g_2, respectively, i.e., $g = g_1 + g_2$, where

$$g_1(k+x) = \gamma(x) g(k+x)$$

and

$$g_2(k+x) = (1-\gamma(x)) g(k+x).$$

(This assertion will be evident in Propositions 3.1 and 3.2.)

Note that in the above definition the function H is defined on $\left[-\frac{1}{2}, 1\right] \times \mathbb{I}$, while the function h is defined on $\left[-\frac{1}{2}, 1\right] \times \mathbb{T}$. Moreover, we do not extend the function H to a larger domain, but only h, thanks to the fact that $h(x,0) = h(x,1)$.

For future use, an easy calculation shows that

$$g(k+x) = (\mathcal{F}_2^{-1} h)(x,k) = (\mathcal{F}_2^{-1} h)(k+x,0),$$

where $\mathcal{F}_2$ denotes the Fourier transform in the second variable.

Notice that the function h, defined as an exponential function with imaginary exponent, is equal to 1 in absolute value. Thus, from Criterion 1.3, it follows that the function g, which is the inverse Zak transform of h, generates an orthonormal basis for $L^2(\mathbb{R})$. Moreover, it is not difficult to see that the function h extended periodically to $(0,1) \times \mathbb{R}$ is infinitely differentiable. Therefore, due to property (1.5), the function h extended to $\mathbb{R} \times \mathbb{R}$ is of class $\mathcal{C}^\infty$ away from the set $\mathbb{Z} \times \{0\}$, which, in view of our previous discussion, is the smallest singular support that one expects for the Zak transform of a generator of a Gabor orthonormal basis.

In the next section, we prove the main result, Theorem 3.11, of this paper, which provides us with the regularity estimates of the function $g = g_a$, $a > 0$.

3 Estimates and the Main Theorem

First, we reduce the situation to dealing with a neighborhood of the set $\mathbb{Z}$. This is accomplished by a decomposition of g into two parts $g = g_1 + g_2$, as in Definition 2.3. We show in the next proposition that $g_2 \in \mathcal{S}(\mathbb{R})$, and so any integral estimates of g may be reduced to estimates of g_1, which is supported on the set $\bigcup_{k \in \mathbb{Z}} [k - 2\epsilon, k + 2\epsilon]$.

Proposition 3.1 *The function g_2 is in the Schwartz class $\mathcal{S}(\mathbb{R})$.*

Proof. If $y \in \operatorname{supp}(g_2)$, then y is of the form $y = k + x,\ k \in \mathbb{Z},\ x \in [\epsilon, 1 - \epsilon]$. We compute

$$\begin{aligned} \partial_x^l g_2(y) &= \partial_x^l g_2(k+x) = \partial_x^l (1 - \gamma(x))(\mathcal{F}_2^{-1} h)(x,k) \\ &= \sum_{m=0}^{l} \binom{l}{m} \partial_x^{l-m} (1 - \gamma(x)) \partial_x^m (\mathcal{F}_2^{-1} h)(x,k) \end{aligned}$$

and

$$\partial_x^m (\mathcal{F}_2^{-1} h)(x,k) = \partial_x^m \int_{\mathbb{T}} e^{2\pi i k \xi} h(x,\xi)\, d\xi$$

$$
\begin{aligned}
&= \int_{\mathbb{T}} e^{2\pi i k\xi} \partial_x^m h(x,\xi)\, d\xi \\
&= \frac{1}{(-2\pi i k)^n} \int_{\mathbb{T}} e^{2\pi i k\xi} \partial_\xi^n \partial_x^m h(x,\xi)\, d\xi.
\end{aligned}
$$

The function h is smooth on $[\epsilon, 1-\epsilon] \times \mathbb{T}$. Therefore, $\left|\partial_x^l g_2(y)\right| \leq C_{l,n} \frac{1}{1+|k|^n}$, and so $g_2 \in \mathcal{S}(\mathbb{R})$.

■

We also have the following useful formula for g_1.

Proposition 3.2 *The function g_1 can be written in the form*

$$
g_1(k+x) = \gamma(x) \left(\delta(k) + \chi_{[0,\frac{1}{2})}(x) x^a F(kx^a) \right),
$$

where $F \in \mathcal{S}(\mathbb{R})$, $k \in \mathbb{Z}$, and $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.
(Here, δ denotes the Kronecker delta function: $\delta(k) = 0$ for $k \neq 0$ and $\delta(0) = 1$.)

Proof. For $x \in \left[-\frac{1}{2}, 0\right]$,

$$
\begin{aligned}
g_1(k+x) &= \gamma(x)(\mathcal{F}_2^{-1} h)(x,k) = \gamma(x) \int_{\mathbb{T}} h(x,\xi) e^{2\pi i k\xi}\, d\xi \\
&= \gamma(x) \int_{\mathbb{T}} 1 \cdot e^{2\pi i k\xi}\, d\xi = \gamma(x)\delta(k).
\end{aligned}
$$

We introduce the function $\Phi(\xi) = e^{2\pi i \varphi(\xi)} - 1$. Then, $\operatorname{supp}(\Phi) \subseteq [0,1]$, $\Phi \in \mathcal{C}^\infty(\mathbb{R})$, and, for $x \in \left[0, \frac{1}{2}\right)$,

$$
\begin{aligned}
g_1(k+x) &= \gamma(x) \int_{\mathbb{T}} e^{2\pi i \varphi\left(\frac{\xi}{x^a}\right)} \cdot e^{2\pi i k\xi}\, d\xi \\
&= \gamma(x) \left(\delta(k) + \int_{\mathbb{T}} \left(e^{2\pi i \varphi\left(\frac{\xi}{x^a}\right)} - 1 \right) e^{2\pi i k\xi}\, d\xi \right) \\
&= \gamma(x) \left(\delta(k) + \int_{\mathbb{T}} \Phi\left(\frac{\xi}{x^a}\right) e^{2\pi i k\xi}\, d\xi \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma(x)\left(\delta(k) + x^a \int_{\mathbb{R}} \Phi(\xi) e^{2\pi i (kx^a)\xi}\, d\xi\right) \\
&= \gamma(x)\left(\delta(k) + x^a \hat{\Phi}(-kx^a)\right) \\
&= \gamma(x)\left(\delta(k) + x^a F(kx^a)\right),
\end{aligned}
$$

where $F(x) = \hat{\Phi}(-x)$ is a Schwartz function.

∎

We further reduce the question of finding good estimates for the function g by subtracting $\gamma \in \mathcal{S}(\mathbb{R})$ from g_1.

Definition 3.3 *Let*

$$
g_0(k+x) = \gamma(x)\chi_{[0,\frac{1}{2})}(x) x^a F(kx^a), \quad \text{where } k \in \mathbb{Z}, x \in \left[-\frac{1}{2}, \frac{1}{2}\right).
$$

Clearly, $g_1(x) - g_0(x) = \gamma(x)$. Thus, to discuss the decay and regularity of g, it is enough to deal with g_0.

We first estimate g "on the x-side".

Theorem 3.4 *Let $a > 0$ be fixed as in Definition 2.1. If $A = 1 + \frac{1}{a}$ and $\epsilon > 0$, then*

$$
\int_{\mathbb{R}} |g(x)|^2 \frac{1 + |x|^A}{\log^{1+\epsilon}(2 + |x|)}\, dx < \infty.
$$

Proof. As we have mentioned before, it is enough to consider the above integral, with g replaced by g_0. Here, and in the sequel, C denotes a constant, which depends only on a, and not on either x or ξ.

$$
\int_{\mathbb{R}} |g_0(x)|^2 \frac{1 + |x|^A}{\log^{1+\epsilon}(2 + |x|)}\, dx
$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{2}} |\gamma(x) x^a F(kx^a)|^2 \frac{1+|k+x|^A}{\log^{1+\epsilon}(2+|k+x|)}\, dx \\
&\le C + C \sum_{|k|\ge 2} \int_0^{\frac{1}{2}} |x^a F(kx^a)|^2 \frac{|k|^A}{\log^{1+\epsilon}(|k|)}\, dx \\
&= C + C \sum_{|k|\ge 2} \int_0^{k\left(\frac{1}{2}\right)^a} |\frac{y}{k} F(y)|^2 \frac{|k|^A}{\log^{1+\epsilon}(|k|)} |k|^{-\frac{1}{a}} \frac{1}{a} |y|^{\frac{1}{a}-1}\, dy \\
&\le C + C \sum_{|k|\ge 2} \frac{|k|^{-2+A-\frac{1}{a}}}{\log^{1+\epsilon}(|k|)} \frac{1}{a} \int_{\mathbb{R}} |y|^{1+\frac{1}{a}} |F(y)|^2\, dy \\
&= C + C \sum_{|k|\ge 2} \frac{|k|^{-1}}{\log^{1+\epsilon}(|k|)} < \infty,
\end{aligned}
$$

where we see that the integral

$$\int_{\mathbb{R}} |y|^{1+\frac{1}{a}} |F(y)|^2\, dy$$

is convergent for any fixed $a > 0$ because of the rapid decay of the function F.

■

In a similar way we have the following fact.

Proposition 3.5 *The function g_0 belongs to $L^1(\mathbb{R})$.*

Proof. We estimate

$$
\begin{aligned}
\int_{\mathbb{R}} |g_0(x)|\, dx &= \sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{2}} |\gamma(x) x^a F(kx^a)|\, dx \\
&\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |x^a F(kx^a)|\, dx \le C \sum_{k\in\mathbb{Z}} |k|^{-1-\frac{1}{a}} < \infty.
\end{aligned}
$$

■

Our next goal, Theorem 3.10, is to estimate g “on the ξ-side”. To this end we begin with the following calculation for $l \in \mathbb{Z}$ and $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right)$:

$$
\begin{aligned}
\hat{g}_0(l+\xi) &= \int_{\mathbb{R}} g_0(x) e^{-2\pi i x(l+\xi)}\, dx \\
&= \sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{2}} \gamma(x) x^a F(kx^a) e^{-2\pi i(k+x)(l+\xi)}\, dx \qquad (3.1)\\
&= \int_0^{\frac{1}{2}} \gamma(x) \left(\sum_{k\in\mathbb{Z}} x^a F(kx^a) e^{-2\pi i k\xi} \right) e^{-2\pi i x(l+\xi)}\, dx,
\end{aligned}
$$

where the change of integration and summation is legitimate by Proposition 3.5.

Definition 3.6 *Let us denote the sum in the last expression of (3.1) by $G_\xi(x)$, i.e.,*

$$
G_\xi(x) = \sum_{k\in\mathbb{Z}} x^a F(kx^a) e^{-2\pi i k\xi},
$$

where $x \in [0,1)$ and $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.

It is easy to check that the function G satisfies conditions

$$
G_\xi^{(s)}(x) = x^{a-s} \cdot \sum_{0\le n\le s} C_{s,n,a} \left(\sum_{k\in\mathbb{Z}} F^{(n)}(kx^a)(kx^a)^n e^{-2\pi i k\xi} \right), \qquad (3.2)
$$

for $x \neq 0$, and

$$
G_\xi \in C^\infty((0,1)) \quad \text{for all } \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right).
$$

Because the function F is defined as a function which satisfies the equation $x^a F(kx^a) = \int_{\mathbb{T}} \Phi(\xi/x^a) e^{2\pi i k\xi} d\xi$, 0 is a point of singularity of the function G_ξ. To prove our main result, we need an estimate for $|G_\xi^{(s)}(x)|$. In the process we shall make use of the following definition.

Definition 3.7 *For a function $I : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, we define*

$$\tilde{I}(x) = \sup_{|y| \geq x} |I(y)|.$$

Note that $\tilde{I}(x) = \tilde{I}(0)$ for all $x \leq 0$. The following simple estimate will be useful:

$$\begin{aligned} \sum_{m \in \mathbb{Z}} |I(my)| &\leq |I(0)| + 2 \sum_{m \in \mathbb{N}^+} \int_{m-1}^{m} \tilde{I}(z|y|)\, dz \\ &= |I(0)| + \frac{2}{|y|} \int_{\mathbb{R}^+} \tilde{I}(z)\, dz. \end{aligned} \tag{3.3}$$

The right hand side or both sides can equal to infinity.

Proposition 3.8 *For all $x \in (0, 1)$, $\xi \in [-\frac{1}{2}, \frac{1}{2})$, and $s \in \mathbb{N}$, the function $G_\xi^{(s)}$ can be estimated by*

$$|G_\xi^{(s)}(x)| \leq C x^{-s} \min \left(\left(\frac{x^a}{|\xi|} \right)^{\frac{1}{2}}, 1 \right).$$

Recall that C depends on a, but not on x or ξ. In this case C also depends on s.

Proof. For brevity we write

$$T(x) = F^{(n)}(x) x^n.$$

Then the equality (3.2) says that $G_\xi^{(s)}(x) \cdot x^{s-a}$ is a finite sum of expressions of the form $C \sum_{k \in \mathbb{Z}} T(kx^a) e^{-2\pi i k \xi}$. The rest of the proof is devoted to estimating such an expression. Clearly, $T \in \mathcal{S}(\mathbb{R})$.
From the inequality (3.3) we can derive that, for any $x \in (0, 1)$,

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}} T(kx^a) e^{-2\pi i k \xi} \right| &\leq \sum_{k \in \mathbb{Z}} |T(kx^a)| \leq |T(0)| + \frac{2}{x^a} \int_{\mathbb{R}^+} \tilde{T}(z)\, dz \\ &\leq C(1 + x^{-a}) \leq C x^{-a}; \end{aligned}$$

and this in turn proves that

$$\left|G_\xi^{(s)}(x)\right| \leq Cx^{-s}\cdot 1, \quad \text{if } x \in (0,1),$$

and that

$$\left|G_\xi^{(s)}(x)\right| \leq Cx^{-s}\left(\frac{x^a}{|\xi|}\right)^{\frac{1}{2}}, \quad \text{if } x \in (0,1) \text{ and } x^a \geq |\xi|.$$

It remains to prove that

$$\left|\sum_{k\in\mathbb{Z}} T(kx^a)e^{-2\pi ik\xi}\right| \leq C(|\xi|x^a)^{-\frac{1}{2}}, \quad \text{if } 0 < x^a \leq |\xi|.$$

To this end, we divide the whole sum into sums of K components, with a constant K to be chosen. We have the estimate

$$\begin{aligned}
\left|\sum_{k\in\mathbb{Z}} T(kx^a)e^{-2\pi ik\xi}\right| &\leq \sum_{l\in\mathbb{Z}}\left|\sum_{k=lK}^{(l+1)K-1} T(kx^a)e^{-2\pi ik\xi}\right| \\
&= \sum_{l\in\mathbb{Z}}\left|T(lKx^a)\frac{e^{-2\pi ilK\xi}-e^{-2\pi i(l+1)K\xi}}{1-e^{-2\pi i\xi}}\right. \\
&+ \left.\sum_{k=0}^{K-1} kx^aT'((lK+\theta_k k)x^a)e^{-2\pi i(lK+k)\xi}\right| \\
&\leq \sum_{l\in\mathbb{Z}}\left(|T(lKx^a)|\frac{C}{|\xi|}+K^2x^a\widetilde{T'}\left((|l|-1)Kx^a\right)\right) \\
&= \frac{C}{|\xi|}\sum_{l\in\mathbb{Z}}|T(lKx^a)| \qquad (3.4) \\
&+ K^2x^a\left(2\widetilde{T'}(0)+\sum_{l\in\mathbb{Z}}\widetilde{T'}\left(|l|Kx^a\right)\right) \\
&\leq \frac{C}{|\xi|}\left(1+\frac{1}{Kx^a}\right)+CK^2x^a\left(1+\frac{1}{Kx^a}\right),
\end{aligned}$$

where we have used the inequality (3.3). Next, we set $K = \left[(|\xi| x^a)^{-\frac{1}{2}}\right]$ ($[x]$ denotes the greatest integer less or equal to x), so that $Kx^a \leq 1$ and $1 + \frac{1}{Kx^a} \leq \frac{2}{Kx^a}$. Consequently, (3.4) allows us to make the estimate,

$$\begin{aligned} \left|\sum_{k\in\mathbb{Z}} T(kx^a)e^{-2\pi ik\xi}\right| &\leq \frac{C}{|\xi|}\frac{1}{Kx^a} + CK^2x^a\frac{1}{Kx^a} \\ &= C\left(\frac{1}{K|\xi|x^a} + K\right) \leq C(|\xi|x^a)^{-\frac{1}{2}}, \end{aligned}$$

and this is the desired result.

■

It follows from Proposition 3.5 that $\hat{g}_0 \in L^\infty(\mathbb{R})$, and, thus, it is only relevant to study the behavior of $\hat{g}_0$ at infinity. Therefore, we shall restrict our consideration to the set

$$\{l + \xi : l \in \mathbb{Z}, |l| \geq 2, \xi \in (0,1)\}.$$

Then, equation (3.1) says that

$$\begin{aligned} \hat{g}_0(l+\xi) &= \int_0^{\frac{1}{2}} \gamma(x)G_\xi(x)e^{-2\pi ix(l+\xi)}\,dx \\ &= \frac{1}{(2\pi i(l+\xi))^s}\int_0^{\frac{1}{2}} (\gamma G_\xi)^{(s)}(x)e^{-2\pi ix(l+\xi)}\,dx. \end{aligned} \tag{3.5}$$

We can integrate by parts as long as $(\gamma G_\xi)^{(r)} \in L^1(\mathbb{R})$, for each $0 \leq r \leq s$, and $(\gamma G_\xi)^{(r)}(x) \to 0$ as $x \to 0$, for each $0 \leq r \leq s-1$. By Proposition 3.8, we can integrate as long as the exponents satisfy the inequality

$$\frac{1}{2}a - s > -1 \Leftrightarrow s < \frac{1}{2}a + 1.$$

For any $m \in \mathbb{N}$ and $x > 0$

$$\left|(\gamma G_\xi)^{(m)}(x)\right| = \left|\sum_{n=0}^{m} \binom{m}{n} \gamma^{(m-n)}(x) G_\xi^{(n)}(x)\right| \leq C x^{-m} \cdot \left(\frac{x^a}{|\xi|}\right)^{\frac{1}{2}}.$$

Let us choose the maximal s, i.e., choose the integer $s \in \left[\frac{1}{2}a, \frac{1}{2}a + 1\right)$. Then, performing integration by parts twice, we obtain:

$$\begin{aligned}
& \left|\int_0^{\frac{1}{2}} (\gamma G_\xi)^{(s)}(x) e^{-2\pi i x(l+\xi)}\, dx\right| \\
= & \left|\int_0^{|l|^{-1}} (\gamma G_\xi)^{(s)}(x) e^{-2\pi i x(l+\xi)}\, dx\right. \\
- & (\gamma G_\xi)^{(s)}(x) \frac{e^{-2\pi i x(l+\xi)}}{2\pi i(l+\xi)}\Bigg|_{|l|^{-1}}^{\frac{1}{2}} - (\gamma G_\xi)^{(s+1)}(x) \frac{e^{-2\pi i x(l+\xi)}}{(2\pi i(l+\xi))^2}\Bigg|_{|l|^{-1}}^{\frac{1}{2}} \\
+ & \left.\int_{|l|^{-1}}^{\frac{1}{2}} (\gamma G_\xi)^{(s+2)}(x) \frac{e^{-2\pi i x(l+\xi)}}{(2\pi i(l+\xi))^2}\, dx\right| \\
\leq & \int_0^{|l|^{-1}} \left|(\gamma G_\xi)^{(s)}(x)\right|\, dx + \frac{C\,|(\gamma G_\xi)^{(s)}(|l|^{-1})|}{|l|} \\
+ & \frac{C\,|(\gamma G_\xi)^{(s+1)}(|l|^{-1})|}{|l|^2} + \frac{C}{|l|^2} \int_{|l|^{-1}}^{\frac{1}{2}} |(\gamma G_\xi)^{(s+2)}(x)|\, dx \\
\leq & C \int_0^{|l|^{-1}} \frac{x^{\frac{a}{2}-s}}{|\xi|^{\frac{1}{2}}}\, dx + \frac{C\,|l|^{-(\frac{a}{2}-s)}}{|l||\xi|^{\frac{1}{2}}} \\
+ & \frac{C\,|l|^{-(\frac{a}{2}-s-1)}}{|l|^2|\xi|^{\frac{1}{2}}} + \frac{C}{|l|^2} \int_{|l|^{-1}}^{\frac{1}{2}} \frac{|x|^{\frac{a}{2}-s-2}}{|\xi|^{\frac{1}{2}}}\, dx \\
\leq & C\, \frac{|l|^{s-\frac{a}{2}-1}}{|\xi|^{\frac{1}{2}}}.
\end{aligned}$$

The above estimate, combined with (3.5), gives

$$|\hat{g}_0(l+\xi)| \leq C\, \frac{|l|^{-\frac{a}{2}-1}}{|\xi|^{\frac{1}{2}}}.$$

Notice, that if the parameter $a/2$ was not an integer, then we need one less integration by parts. In this case the expression

$$\frac{1}{|l|}\int_{|l|^{-1}}^{\frac{1}{2}} |(\gamma G_\xi)^{(s+1)}(x)|\, dx\,,$$

can be estimated by $C\,\frac{|l|^{s-\frac{a}{2}-1}}{|\xi|^{\frac{1}{2}}}$. But, if $a/2$ was an integer, we would get an additional factor $\log(|l|)$. We also know, by Proposition 3.5, that $\hat{g}_0$ is a bounded function, and thus we have proved:

Proposition 3.9 *The function $\hat{g}_0$ satisfies the estimate*

$$|\hat{g}_0(l+\xi)| \leq C\ \min\left(\frac{|l|^{-\frac{a}{2}-1}}{|\xi|^{\frac{1}{2}}}, 1\right).$$

As a consequence, we derive the following result.

Theorem 3.10 *Let $a > 0$ be fixed as in Definition 2.1. If $B = 1 + a$ and $\epsilon > 0$, then*

$$\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 \frac{1+|\xi|^B}{\log^{2+\epsilon}(2+|\xi|)}\, d\xi < \infty.$$

Proof. Without loss of generality, and as we did in the proof of Theorem 3.4, we can replace g by g_0.

$$\begin{aligned}
&\int_{\widehat{\mathbb{R}}} |\hat{g}_0(\xi)|^2 \frac{1+|\xi|^B}{\log^{2+\epsilon}(2+|\xi|)}\, d\xi \\
\leq\ & C + C\sum_{|l|\geq 2}\int_0^1 |\hat{g}_0(l+\xi)|^2 \frac{|l|^B}{\log^{2+\epsilon}(|l|)}\, d\xi \\
\leq\ & C + C\sum_{|l|\geq 2}\frac{|l|^B}{\log^{2+\epsilon}(|l|)}\left(\int_0^{|l|^{-a-2}} 1\, d\xi + \int_{|l|^{-a-2}}^1 \left(\frac{|l|^{-\frac{a}{2}-1}}{|\xi|^{\frac{1}{2}}}\right)^2 d\xi\right) \\
\leq\ & C + C\sum_{l\geq 2}\frac{l^B}{\log^{2+\epsilon}(l)}\left(l^{-a-2} + l^{-a-2}\left|\log(l^{-a-2})\right|\right)
\end{aligned}$$

$$\leq \quad C + C \sum_{l \geq 2} \frac{1}{l \log^{1+\epsilon}(l)} < \infty.$$

■

We shall conclude this section with the following generalization of Theorems 3.4 and 3.10.

Theorem 3.11 *If $\frac{1}{A} + \frac{1}{B} = 1$ and $\epsilon > 0$, then there exists a function $g \in L^2(\mathbb{R})$ with the property that $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is a Gabor orthonormal basis for $L^2(\mathbb{R})$, and such that*

$$\int_{\mathbb{R}} |g(x)|^2 \frac{1 + |x|^A}{\log^{1+\epsilon}(2 + |x|)}\, dx < \infty,$$

and

$$\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 \frac{1 + |\xi|^B}{\log^{2+\epsilon}(2 + |\xi|)}\, d\xi < \infty.$$

Proof. We can take $g = g_a$, with $a = 1/(A-1) = B - 1$.

■

In particular, the assertion in the Abstract is a consequence of Theorem 3.11. Further, for $A = B = 2$ we have Theorem 1.2, and thus we have constructed a function that is almost "optimal" from the point of view of Balian–Low theorem, Theorem 1.1; that is, both g and $\hat{g}$ are square integrable with respect to all weights $|x|^\alpha$, $\alpha < 2$. Thus, one cannot make the Balian–Low theorem any more restrictive by changing the exponents.

We would also like to stress that the choice of the function $\log^{d+\epsilon}(2 + |x|)$, $d = 1, 2$, is only an example of a function that makes the appropriate series converge in Theorem 3.11. There are many other functions $\alpha(k)$, that have a slower growth at infinity, and yet they still yield convergence of the series $\sum 1/(k\alpha(k))$. For example, one may take $\alpha(k) = \log(k)\log^{1+\epsilon}(\log(k))$.

Note that Theorem 3.11, besides providing us with an example of an almost optimal generating function for a Gabor orthonormal basis, from the point of view of Balian–Low theorem, may also be seen as a motivation for yet another version of the uncertainty principle. In fact, given a Gabor orthonormal basis generating function g, it is natural to ask if it is possible that both $\|g\|_{L^2(|x|^A dx)}$ and $\|\hat{g}\|_{L^2(|\xi|^B d\xi)}$ are finite, for some A and B for which $1/A + 1/B = 1$.

4 Remarks and Open Questions

In this section we would like to consider some questions related to our construction of an "optimal" example of a generating function for a Gabor orthonormal basis. The first two remarks are derived from the method of the proof of the main result.

• The estimates for the function g are much simpler to prove than the estimates for its Fourier transform $\hat{g}$. Having in mind that we start with a function which is the Zak transform of g, is it possible to find a more direct proof of Theorem 3.10? It is especially insteresting in view of the following inversion formulas for the Zak transform, see, e.g., [14]:

$$f(x) = \int_0^1 Z(f)(x,\xi)\,d\xi,$$

$$\hat{f}(\xi) = \int_0^1 Z(f)(x,\xi)\,e^{-2\pi i x\cdot\xi}\,dx.$$

These two equations show that given $Z(f)$ it is equally difficult to retrieve both f and $\hat{f}$.

A weaker version of Theorem 3.10, can be obtained in a much easier way by means of the equation,

$$\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 \left(1+|\xi|^{2n}\right) d\xi = \|g\|_{L^2}^2 + \|\partial^n g\|_{L^2}^2, \qquad \text{for } n \in \mathbb{N}.$$

Note that $\|\partial^n g\|_{L^2} < \infty$ if and only if $\|\partial^n g_0\|_{L^2} < \infty$, and, for $k \in \mathbb{Z}$ and $x \in (0,1)$,

$$\begin{aligned} &\partial^n g_0(k+x) = \partial^n \left(\gamma(x)\chi_{[0,\frac{1}{2})}(x)x^a F(kx^a)\right) \\ &= \sum_{n_1+n_2+n_3=n} C_{n_1,n_2,n_3}\chi_{[0,\frac{1}{2})}(x)\partial^{n_1}\gamma(x)x^{a-n_2-n_3}F^{(n_3)}(kx^a)(kx^a)^{n_3}. \end{aligned}$$

One easily checks, with computations as in the proofs of Theorems 3.4, 3.6, and 3.10, that if $2n < a+1$, then each component of the above sum is square integrable. Thus, we obtain

$$\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 \left(1+|\xi|^{2n}\right) d\xi < \infty \qquad \text{for } 2n < a+1, n \in \mathbb{N}.$$

This doesn't help us with establishing whether the standard Balian–Low result is sharp, but it is approximately good for estimates with large values of a.

• There is a certain asymmetry in the statements of Theorems 3.4 and 3.10. Namely, in the latter, we require an additional power of the logarithm in the denominator. Is this due to our method of proof, and, if so, could it be avoided?

Among other questions we would like to state a few possible directions for generalizations of our results.

• Since our motivation came from the study of different versions of the uncertainty principle, e.g., [18], it would seem interesting to consider expressions of the form

$$\int_{\mathbb{R}}\int_{\widehat{\mathbb{R}}} |g(x)|^2|\hat{g}(\xi)|^2 w(x,\xi)\, dx d\xi,$$

where $w(x, \xi)$ is a nonseparable weight function, for example with mixed variables, such as $w(x, \xi) = \frac{|x|^A |\xi|^B}{\log^{2+\epsilon}(2+|x|+|\xi|)}$.

• Our results are stated in one dimension, for the case of Gabor systems with the particular choice of oversampling constants $a = b = 1$. Is it possible to generalize these results, as well as the Balian–Low theorem, to higher dimensions, and more general systems?

• There is a generalization of the Balian–Low result to the so called *Amalgam BLT*, cf., [6], [10]. How would one rephrase the question of optimality of this result, and could our construction be useful in this context?

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John J. Benedetto
Department of Mathematics
University of Maryland
College Park, MD 20742
jjb@math.umd.edu

Wojciech Czaja
Mathematical Institute
University of Wrocław
50-384 Wrocław, pl. Grunwaldzki 2/4, Poland
czaja@math.uni.wroc.pl
and
Department of Mathematics
University of Maryland
College Park, MD 20742
wojtek@math.umd.edu

Przemysław Gadziński
Mathematical Institute
University of Wrocław
50-384 Wrocław, pl. Grunwaldzki 2/4, Poland
pgadz@math.uni.wroc.pl

Alexander M. Powell
Department of Mathematics
University of Maryland
College Park, MD 20742
apowell@math.umd.edu