Finite Normalized Tight Frames

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Frames are interesting because they provide decompositions in applications where bases could be a liability. Tight frames are valuable to ensure fast convergence of such decompositions. Normalized frames guarantee control of the frame elements. Finite frames avoid the subtle and omnipresent approximation problems associated with the truncation of infinite frames. In this paper the theory of finite normalized tight frames (FNTFs) is developed. The main theorem is the characterization of all FNTFs in terms of the minima of a potential energy function, which was designed to measure the total orthogonality of a Bessel sequence. Examples of FNTFs abound, e.g., in $\mathbb{R}^3$ the vertices of the Platonic solids and of a soccer ball are FNTFs.
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1. Introduction

We shall develop the theory of finite normalized tight frames (FNTFs) for real and complex Euclidean space K^d, $K = \mathbb{R}$, $\mathbb{C}$, respectively; and we shall characterize all FNTFs for K^d in terms of a potential energy function which we designate as the *frame potential*, see Theorem 7.1. FNTFs are developed in Section 2, but, for the sake of this preliminary discussion, an A-FNTF for K^d is a finite sequence $\{x_n : n = 1, \ldots, N\} \subseteq K^d$ for which the Euclidean norm $\|x_n\|$

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is 1 for each x_n, i.e., $\{x_n\}$ is normalized, and for which

$$\exists A > 0 \text{ such that } \forall y \in K^d,\ y = \frac{1}{A}\sum_{n=1}^{N} \langle y, x_n\rangle x_n. \tag{1.1}$$

It is elementary to see that if $\{x_n\}_{n=1}^N$ is an A-FNTF then $A = 1$ if and only if $\{x_n\}_{n=1}^N$ is an orthonormal basis (ONB) for $\mathbb{R}^d$, e.g., Theorem 2.1. (Naturally we choose the terminology "normalized" in the definition of an A-FNTF because of the definition of an ONB, but see Remark 1.1b below.)

Of course, it is reasonable to ask why anyone would want to develop a theory of FNTFs.

First, the theory of frames was initiated by Duffin and Schaeffer [9] in 1952 as part of an on-going development of non-harmonic Fourier series, in which it is an important, fundamental theory. The background for [9] and its relationship to non-uniform sampling theory is the subject of Chapter 1 in [2]. Further, there is a significant role in signal processing for the theory of frames. This was initiated by Daubechies, Grossmann, and Meyer [8] in 1986, and since then there has been a plethora of activity in the area including a landmark treatment in [7] as well as many major theoretical developments and applications in journals such as Applied and Computational Harmonic Analysis and the Journal of Fourier Analysis and Applications.

Second, some of the applications mentioned in the previous paragraph result from the fact that frames give stable signal representations and allow modelling for noisy environments, e.g., [3], [4], [5]. In this setting, tight frames, reflected by the type of decomposition in (1.1), can give fast convergence; and normalization controls the "harmonics" x_n. Also, many of the applications of frames involve modelling and constructing infinite frames for $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ and then invoking a finitization method, such as some sort of truncation, e.g., [13], [17], [18]. For all of the art involved in this approach, there are frequently treacherous problems such as dealing with spurious eigenvalues. Many of these issues are avoided if the initial modelling is done directly in $\mathbb{R}^d$, and so we are led to FNTFs, see also [11], [12].

It turns out that even if there was not a signal processing problem in sight, the theory of FNTFs leads directly to fascinating issues of equidistribution on the unit sphere S^{d-1} of $\mathbb{R}^d$, and to equilibrium problems with respect to various potential energy laws. There are also interesting examples of FNTFs such as the vertices of the Platonic solids and of a soccer ball, see Section 4. Notwithstanding these and other tantalizing examples, e.g., [10], [11], [12], [19], this paper is based upon a fundamental theoretical result which characterizes all FNTFs, see Sections 5-7.

In Section 2 we briefly introduce the notion of a tight frame and note its relationship to the special case of ONBs. Section 3 is devoted to some properties of FNTFs. In particular, we discuss the existence and possible frame constants for FNTFs, including some specialized results for $\mathbb{R}^2$ and $\mathbb{R}^3$.

The basis for our theory in Sections 5-7 was fathoming the notions of equidistribution and equilibrium on S^1 and S^2 in terms of frames. An inspiration in this regard was the article [15] by Saff and Kuijlaars, see also [6], [14], [16]. Our examples and motivation in Section 4 reflect this situation. In particular, the roots of unity on S^1 form FNTFs for $\mathbb{R}^2$, and they are equidistributed; whereas any ONB for $\mathbb{R}^2$ or $\mathbb{R}^3$ is certainly not equidistributed. On the other hand, there is the fundamental problem in electrostatics of finding equivalent point charges on S^2 so that they are in equilibrium with respect to a Coulomb potential law. Balancing these facts from the perspective of frames led to our formulation of frame potential and its development in Sections 5-7.

After defining the notion of frame force in Section 5, we are led to the idea of a *frame force critical set.* The main result of Section 5 is a characterization of frame force critical sets in terms of a partition of mutually orthogonal sequences. Section 6 is devoted to proving quantitative estimates of the corresponding frame potential. Then, in Section 7, we use these results to characterize all FNTFs for K^d in terms of the minimizers of the frame potential. In fact, we demonstrate that both orthonormal sets and FNTFs arise from the same optimization problem, thereby further justifying the perception of FNTFs as the most natural generalization of ONBs.

Remark 1.1. a. At Bommerholz, in September 2000, Edward Saff gave a lecture on the equidistribution of points on S^2 and Hans Feichtinger asked a question about finding the optimal frame constants relation B/A for normalized frames with N elements. The present paper exists because of Saff's lecture and Feichtinger's quesiton. After the fact, Götz Pfander reminded us that we had attended, but apparently not attentively enough, Jelena Kovačević's lecture, essentially on FNTFs, at a University of Maryland Harmonic Analysis Conference in October 1999, see [11] for the article associated with her lecture. In 1998, the older author had even read parts of Goyal, Vetterli, and Thao's article [12] on frames in $\mathbb{R}^d$. In any case, during and immediately after Bommerholz, there was a great deal of activity on FNTFs by Pfander and Zimmermann [19], and we were hooked.

b. The "N" in FNTF stands for "normalized" which means that each frame element x_n has norm $\|x_n\| = 1$. This is compatible with the "N" in ONB. Zimmermann [19] uses "normalized" in this way, as did Kovačević in her aforementioned 1999 lecture. However, in the written version of [11], Kovačević, et al. replace "normalized" by "uniform"; and Balan, Casazza, Heil, and Z. Landau [1] use "normalized" for tight frames in which the frame constant $A = 1$.

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2. Preliminaries

Given a Hilbert space H, a sequence $\{x_n\} \subseteq H$ is a *Bessel sequence* for H if there exists $B > 0$ such that

$$\sum_n |\langle y, x_n \rangle|^2 \leq B\|y\|^2,$$

for all $y \in H$. For any Bessel sequence $\{x_n\}_{n\in I}$, where I is an appropriate countable indexing set, the associated *Bessel map* L is

$$\begin{aligned} L : H &\longrightarrow \ell^2(I), \\ y &\longrightarrow \{\langle y, x_n \rangle\}, \end{aligned}$$

whose corresponding adjoint operator is then

$$\begin{aligned} L^* : \ell^2(I) &\longrightarrow H, \\ \{a_n\} &\longrightarrow \sum_n a_n x_n. \end{aligned}$$

The *frame operator* is the map $S : H \longrightarrow H$ defined as $S = L^*L$. Hence,

$$Sy = \sum_n \langle y, x_n \rangle x_n.$$

The *Grammian operator* is the map $G : \ell^2(I) \longrightarrow \ell^2(I)$ defined as $G = LL^*$.

A Bessel sequence is *normalized* if $\|x_n\| = 1$ for all n. A sequence $\{x_n\} \subseteq H$ is a *frame* for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\|y\|^2 \leq \sum_n |\langle y, x_n \rangle|^2 \leq B\|y\|^2,$$

for all $y \in H$; and a frame $\{x_n\}$ is *A-tight* if there exists a constant $A > 0$ so that

$$\sum_n |\langle y, x_n \rangle|^2 = A\|y\|^2,$$

for all $y \in H$. A *normalized tight frame* (NTF) is a tight frame which is also normalized.

NTFs are often interpreted as generalizations of ONBs. In fact, if $\{x_n\}$ is an ONB for H, then Parseval's equality states $\sum_n |\langle y, x_n \rangle|^2 = \|y\|^2$ for any $y \in H$. Thus, any ONB for H is a NTF with frame constant 1. An equivalent formulation of Parseval's equality states that if $\{x_n\}$ is an ONB for H, then $\sum_n \langle y, x_n \rangle x_n = y$ for all $y \in H$. The following result generalizes this equivalence to NTFs, as well as clarifying the possibilities for tight frame constants.

Theorem 2.1. Let $\{x_n\}$ be a Bessel sequence for H, with frame operator S.

a. $\{x_n\}$ is an A-tight frame $\Leftrightarrow S = AI$, where $I : H \to H$ is the identity map.

b. Let $\{x_n\}$ be an A-NTF. Then $A \geq 1$, and furthermore $A = 1$ if and only if $\{x_n\}$ is an orthonormal basis (ONB) for H.

Remark 2.2. a. Theorem 2.1a is a well-known result which may be rewritten in the useful decomposition formula (1.1).

b. The proof of Theorem 2.1b is elementary. In fact, observe that, for any m,

$$A = A\|x_m\|^2 = \sum_n |\langle x_m, x_n \rangle|^2 = 1 + \sum_{n \neq m} |\langle x_m, x_n \rangle|^2,$$

and so $A \geq 1$; and A is equal to 1 if and only if, for all m and $n \neq m$, $\langle x_m, x_n \rangle = 0$.

NTFs may be thought of as the Bessel sequences that retain the decomposition properties of ONBs while relaxing the need to be a basis. For example, for a positive integer A, A ONBs taken together form an A-NTF. These expansions gain redundancy and stability at the expense of having a unique expansion. Although much work has already been done on NTFs in infinite dimensional Hilbert spaces, the current *intuitive* understanding of NTFs is small compared to that of ONBs.

In this paper we seek to further the understanding of NTFs, particularly in the most natural finite-dimensional Hilbert spaces, $H = K^d$ where $K = \mathbb{R}$ or $\mathbb{C}$. Though it is still possible to have an infinite Bessel sequence in a finite dimensional space, the next elementary result shows that a normalized Bessel sequence in a finite dimensional space must be a sequence of finite length.

Theorem 2.3. Let H be a Hilbert space containing the Bessel sequence $\{x_n\}$.

a. If H is finite-dimensional, then $\{\|x_n\|\}$ is square-summable.

b. If H is finite-dimensional and $\{x_n\}$ is a normalized Bessel sequence, then $\{x_n\}$ is a finite sequence.

c. If $\{x_n\}$ is a finite sequence which is a frame for H, then H is finite-dimensional.

Proof. a. Let $\{x_n\}$ be a Bessel sequence with constant B, and let $\{e_j\}_{j=1}^d$ be an orthonormal basis for H. Then

$$\sum_n \|x_n\|^2 = \sum_n \sum_{j=1}^d |\langle e_j, x_n\rangle|^2 = \sum_{j=1}^d \sum_n |\langle e_j, x_n\rangle|^2 \leq \sum_{j=1}^d B\|e_j\|^2 = Bd,$$

and so $\{\|x_n\|\}$ is square-summable.

b. Assume H is finite dimensional and that $\|x_n\| = 1$ for all n. By part a, the constant sequence $\{\|x_n\|\}$ must be square-summable, and therefore of finite length.

c. Let H_1 be the orthogonal complement of $span\{x_n\}$ in H. If $y \in H_1$ with $y \neq 0$, then $A\|y\|^2 \leq \sum_n |\langle y, x_n\rangle|^2 = 0$, a contradiction. Thus $H = span\{x_n\}$ and so $\dim(H) = \dim(span\{x_n\}) < \infty$. □

Being interested in normalized tight frames in finite dimensions, we shall therefore focus upon finite normalized tight frames (FNTFs). For a given finite Bessel sequence in a finite-dimensional space H, the associated Bessel map as well as the frame and Grammian operators become linear operators to and from finite-dimensional spaces. In particular, with $H = K^d$ each of these operators will have a matrix representation with respect to the standard basis $\{e_j\}_{j=1}^d$. Writing the Bessel sequence elements $\{x_n\}_{n=1}^N$ as column vectors, the adjoint of the Bessel map has the representation $L^* = [x_1|\ldots|x_N]$. The matrix representation of the frame operator is $S = L^*L$, and the matrix representation of the Grammian operator is $G = LL^*$ whose $(m,n)th$ entry is

$$G(m,n) = \langle Ge_n, e_m\rangle = \langle LL^*e_n, e_m\rangle = \langle L^*e_n, L^*e_m\rangle = \langle x_n, x_m\rangle.$$

Thus, for a normalized Bessel sequence, the associated Grammian matrix has ones along the diagonal. As a side note, recall that if an orthonormal basis for K^d is arranged as the columns of a matrix U, then $U^*U = I = UU^*$. An FNTF is therefore a generalization of an ONB in the sense that L^*L must be a constant multiple of the identity, and that LL^* must have ones along the diagonal.

3. Finite Normalized Tight Frames

Finite frames which are both tight *and* normalized possess a significant structure which has only recently begun to be investigated. One observation that has been made is that there is only one choice for the frame constant of an FNTF of N elements for a d-dimensional Hilbert space. We note that this result was observed in [12], and proven independently in both [11] and [19] for the case $H = K^d$. Our result is somewhat more general, a fact we shall need later on in Section 7. Also, our proof is unlike the others in that we do not rely upon matrix theory.

Theorem 3.1. If $\{x_n\}_{n=1}^N$ is an A-FNTF for a d-dimensional Hilbert space H, then $A = N/d$.

Proof. Let $\{e_j\}_{j=1}^d$ be an orthonormal basis for H. Then

$$Ad = \sum_{j=1}^d A\|e_j\|^2 = \sum_{j=1}^d\sum_{n=1}^N |\langle e_j, x_n\rangle|^2 = \sum_{n=1}^N\sum_{j=1}^d |\langle e_j, x_n\rangle|^2 = \sum_{n=1}^N \|x_n\|^2 = N.$$

□

This result is natural in the sense that the constant N/d reflects the redundancy of the system.

Another question of interest is if FNTFs of a given N elements exist for a Hilbert space H. This was answered for $H = K^d$ by the following result, which was arrived at independently in [11] and [19].

Theorem 3.2. Given any d and $N \geq d$, then there exists an FNTF for K^d of N elements.

The proofs in [11] and [19] are the same, providing an explicit construction of the desired FNTF. Though the construction depends upon whether $K = \mathbb{R}$ or $K = \mathbb{C}$, both are forms of overcomplete discrete Fourier series. The insight of the authors in question was to realize what the frames *should* be, since verifying that a specific Bessel sequence is indeed a FNTF is not difficult in this situation. We wish to note that Zimmermann also provided us with a proof of Theorem 3.2 which was different than the one he used in [19], employing an inductive approach and Gram-Schmidt orthogonalization. Furthermore, in Section 7 we provide an independent non-constructive proof of this same result. In fact, we justify a reasoning as to why FNTFs *must* exist as mimimizers of a certain continuous function over a compact set.

Of course, beyond the existence question of FNTFs in general, there is also the issue of characterizing all FNTFs for a given Hilbert space H. For practical applications of frames in the physical world, one is drawn to FNTFs for $\mathbb{R}^2$ and $\mathbb{R}^3$. For $\mathbb{R}^2$, one is naturally lead to thinking in terms of polar coordinates and the complex plane. The following result was arrived at independently by Goyal, Kovačević, and Kelner in [11], as well as by one of us in [10].

Theorem 3.3. Normalized tight frames for $\mathbb{R}^2$ of N elements correspond to sequences $\{z_n\}_{n=1}^N \subseteq \mathbb{C}$, with $|z_n| = 1$ for all n and for which

$$\sum_{n=1}^{N} z_n^2 = 0.$$

The proof follows from writing down the frame elements in terms of polar coordinates, computing the associated 2×2 frame operator S, and writing the equation $S = (N/2)I$ in terms of a single equation in complex variables. A similar technique may be used to find equations that characterize FNTFs for $\mathbb{R}^3$. Here, one identifies points on the sphere S^2 with their projections inside the unit disc of the complex plane.

Theorem 3.4. Normalized tight frames for $\mathbb{R}^3$ of N elements correspond to sequences $\{z_n\}_{n=1}^N \subseteq \mathbb{C}$ with $|z_n| \leq 1$ for all n that satisfy

$$\sum_{n=1}^{N} |z_n|^2 = \frac{2}{3}N, \ \sum_{n=1}^{N} z_n^2 = 0, \ \sum_{n=1}^{N} z_n \sqrt{1 - |z_n|^2} = 0.$$

As hinted at above, the proof of Theorem 3.4, found in [10], involves writing the frame elements in terms of cylindrical (or spherical) coordinates and rewriting $S = (N/3)I$.

Despite the fact that each of these results about FNTFs is useful and interesting in its own right, they nevertheless fail to provide an intuitive characterization of frames. However, recently several fascinating results about FNTFs have been discovered. In [12] the authors demonstrate that if one randomly chooses unit vectors according to a uniform distribution on a sphere, the resulting Bessel sequence is asymptotically tight. Furthermore, in [11] FNTFs are shown to be the optimal decompositions to use under a particular model of data loss. We shall demonstrate yet another way in which FNTFs are naturally motivated, and present a characterization in those terms. Our inspiration in that regard are certain "famous" sets in S^1 and S^2 that also turn out to be examples of FNTFs for $\mathbb{R}^2$ and $\mathbb{R}^3$.

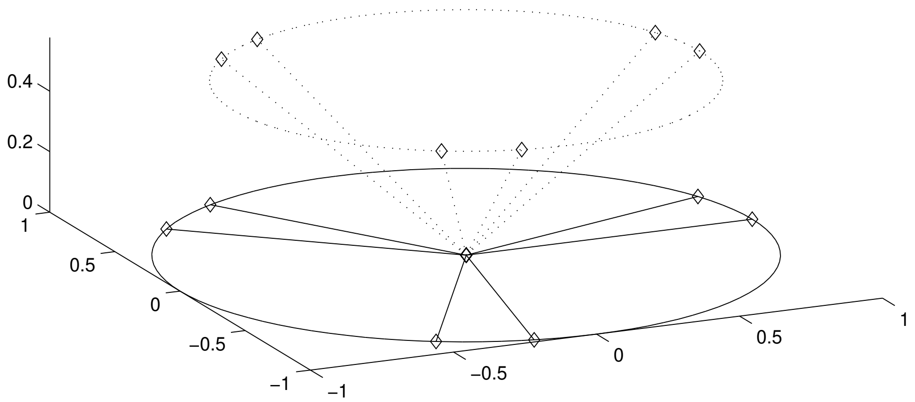
4. Examples and Motivation

4.1. Examples

In order to formulate a general characterization of FNTFs, we first examine some simple examples in low dimensional real spaces. As the examples of FNTFs in $\mathbb{R}^1$ are trivial, we focus on the intuitive parallels between certain examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 4.1. Theorem 3.3 provides a straightforward computational means of finding some FNTFs for $\mathbb{R}^2$. Specifically, for $N > 2$, the N vectors in $\mathbb{R}^2$ identified with Nth roots of unity are an FNTF. To see this, note that when one squares the set of Nth roots of unity, one obtains either the same set, as is the case when N is odd, or, when N is even, one obtains two copies of the $N/2$ roots of unity. In either case, the sum of the squares of the Nth roots of unity is 0. On the other

Figure 1. “Lifting” a frame to a higher dimension



hand, we note that for $N = 2$, the Nth roots of unity do not form an FNTF, and are not even a basis for $\mathbb{R}^2$. The role of the square here is important, as the most trivial example of an FNTF for $\mathbb{R}^2$ is the ONB corresponding to $\{1, i\}$.

Example 4.2. In a similar fashion, Theorem 3.4 may be used to find some FNTFs for $\mathbb{R}^3$. In fact, any FNTF for $\mathbb{R}^2$ whose elements also sum to zero may be converted into a FNTF for $\mathbb{R}^3$. To see this, observe that if $\{z_n\}_{n=1}^N \subseteq S^1$ satisfies $\sum z_n = \sum z_n^2 = 0$, then $\{\sqrt{2/3} z_n\}$ satisfies the conditions of Theorem 3.4. In this fashion, for $N > 2$, the Nth roots of unity may be “lifted” to the latitude $z = 1/\sqrt{3}$ of the sphere S^2 to form an FNTF for $\mathbb{R}^3$, as seen in Figure 1.

Example 4.3. Other examples of FNTFs for $\mathbb{R}^3$ may be found by looking at sets which generalize the geometric regularity of the roots of unity to three dimensions. Since the Nth roots of unity may be viewed as the vertices of a regular N-sided polygon, in three dimensions we are led to the vertices of the regular polyhedra. Indeed, the collection of vertices of any of the Platonic solids, pictured in Figure 2, can be shown to form an FNTF for $\mathbb{R}^3$, cf., [10]. The most trivial means of verifying this is to take the coordinates of the vertices, form the associated Bessel matrices L, and verify that $L^*L = cI$. The same method may be used to demonstrate that other sets of high symmetry, such as the vertices of the truncated icosahedron, i.e., the soccer ball pictures in Figure 3, also form FNTFs for $\mathbb{R}^3$.

Figure 2. The five Platonic solids

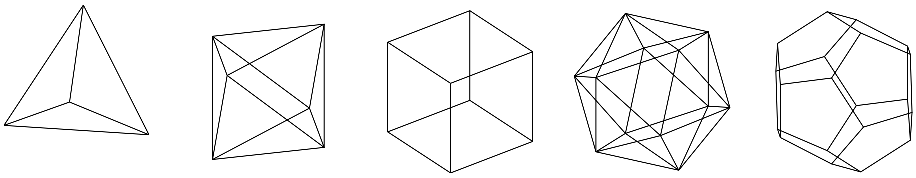
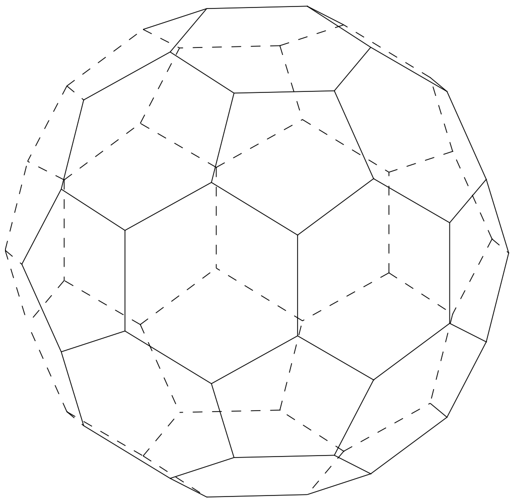


Figure 3. The "soccer ball", a truncated icosahedron



Though such examples of FNTFs in three dimensions are interesting, they are at the same time unsatisfying. While roots of unity will generate FNTFs of an arbitrary order for $\mathbb{R}^2$, the fact that there are only five Platonic solids leads us to believe that any general characterization of FNTFs will not be phrased in terms of geometric regularity. So rather than pursuing a strictly geometric analysis of FNTFs any further, we instead realized that sets such as the roots of unity and the vertices of the Platonic solids also play an important role in the study of equidistribution, ultimately leading us to the study of equilibrium.

4.2. Motivation

Problems of *spherical equidistribution* involve finding the optimal way in which to place N points on the sphere S^{d-1} in $\mathbb{R}^d$ so that the points are as far away from each other as possible. In two dimensions, the intuitively equidistributed set of N points on S^1 is a rotation of the set of Nth roots of unity. However, in three dimensions, the question of how to define what it means for a set to be equidistributed is not resolved. In fact, there are many different methods of measuring equidistribution on S^2; [15] is an excellent survey of such methods. Though these different formalisms seem to lead asymptotically to the same optimal configurations when the number of points is large, for a small number of points these different notions of distribution lead to different optimally distributed sets.

Example 4.4. Some of the means employed to measure the uniformity of a distribution arose from classical problems in physics. In these cases, one tends to think of an equidstributed set in terms of sets which are in equilibrium under the presence of a certain force. For example, imagine one has N electrons upon a conductive spherical shell. In the presence of no external forces, one electron will push upon another according to the Coulomb force law. Specifically, the force of a point $b \in S^2$ upon a distinct point $a \in S^2$ is $F(a,b) = \frac{a-b}{\|a-b\|^3}$, up to a physical constant factor.

Thinking of this situation dynamically, the electrons will try to rearrange themselves so that the resulting distribution will contain as little potential energy as possible. The total potential energy contained within a set is therefore an intuitive and natural means of measuring how close one is to being equidistributed. Of course, there may be "local" minimizers of the potential energy, namely arrangements of electrons which are stable, and yet do not have as small of a potential energy as possible. However, there intuitively must exist some global minimizer, namely an arrangement which distributes the electrons as evenly as possible under the Coulomb force.

To find such arrangements, one must first find an explicit formula for the potential by taking the negative antigradient of the vector field in the case of a conservative force. For Coulomb's law, the potential between two distinct points is $\frac{1}{\|a-b\|}$, again up to a physical constant factor. The total potential inside a set is then defined to be the sum of all of the pairwise potentials, namely $\sum_{m=1}^{N} \sum_{n \neq m} \frac{1}{\|x_m - x_n\|}$. Having the potential in hand, one seeks to minimize the potential to find the equidistributed arrangements. Although viewing the potential as a function of several real variables allows the use of techniques from calculus, finding the optimal arrangements is much easier said than done. In fact, the optimal distributions of points under the Coulomb force are known only for a relatively small number of points, a search which is complicated by the fact the the number of local minima grows geometrically with the number of points [15].

Example 4.5. If one assumes that the points on the sphere act against each other in a manner different from Coulomb's, one naturally expects to find different notions of equidistribution. This has lead to the study of minimizers of potentials having the form $\sum_{m=1}^{N}\sum_{n\neq m}\|x_m - x_n\|^\alpha$. Beyond this, there is Tammes' problem, which is intuitively obtained by taking α to $-\infty$, e.g., see [15]. In this case one seeks to minimize the maximum of pairwise distances of the points. There has also been work on minimizing $\sum_{m=1}^{N}\sum_{n\neq m}\log\frac{1}{\|x_m-x_n\|}$, which is equivalent to maximizing the product $\prod_{m=1}^{N}\prod_{n\neq m}\|x_m - x_n\|$.

What is remarkable about Platonic solids, signaling a connection between FNTFs and equidistribution, is that the vertices of the Platonic solids not only form FNTFs for $\mathbb{R}^3$, but also are natural candidates for equidistributed sets under a very broad class of forces, as defined explicitly below. Although it is true that, for a given force, the verticies of the Platonic solids are not always the optimal configurations themselves, they naturally will satisfy Lagrange's equations in the constrained optimization problems one encounters when attempting to minimize the potential.

Of course, the simplest cases of FNTFs, namely ONBs, are not intuitively well-distributed, and so we do not expect to characterize FNTFs purely in terms of equidistribution. Rather, we seek to adapt a notion from equidistribution, namely equilibrium under a force, and characterize FNTFs as sets which are in equilibrium under a certain force law. Before presenting the appropriate force for frames in the following section, we first discuss the formalism of investigating sets in equilibrium under a force in general.

4.3. A Formalism for Equilibrium

4.3.1. Central Forces

A *force* between any two distinct points on the sphere is a function $F : S^{d-1}\times S^{d-1}\setminus D\to\mathbb{R}^d$, where D is the diagonal set, $D = \{(x,x) : x\in S^{d-1}\}$. For our purposes, it is sufficient to consider only those forces that are directed along the axis connecting two points. Specifically, a *central* force F corresponds to a continuous function $f : (0,2]\to\mathbb{R}$ for which $F(a,b) = f(\|a-b\|)(a-b)$ for all $a, b \in S^{d-1}$, $a \neq b$. The quantity $F(a,b)$ will be interpreted as the force of the point b upon the point a.

4.3.2. Potential Energy

Given a central force F, the *potential* corresponding to F is a function $P : S^{d-1} \times S^{d-1} \setminus D \to \mathbb{R}$, defined by $P(a,b) = p(\|a-b\|)$, where $p : (0,2] \to \mathbb{R}$ satisfies $p'(x) = -xf(x)$. Note that the potential is a symmetric function, and is defined only up to an additive constant. Holding b constant, one may verify that $\nabla_a P(a,b) = -F(a,b)$, and so every central force is a conservative vector field.

In the presence of a central force, the total potential contained within a certain arrangement of distinct points is the sum of all the pairwise potentials. Specifically, the *total potential* is a function

$$TP : \underbrace{S^{d-1} \times \dots \times S^{d-1}}_{N \text{ times}} \setminus D_N \longrightarrow \mathbb{R}$$

$$\{x_n\}_{n=1}^N \longrightarrow \sum_{m=1}^N \sum_{n \neq m}^N P(x_m, x_n),$$

where D_N is the generalized diagonal subset of $(S^{d-1})^N$ that consists of all N-tuples in which at least two coordinates are the same.

4.3.3. Equilibrium

Intuitively, a set of points which is in equilibrium under a given force is an arrangement that remains constant over time, and if disturbed slightly will eventually return to the original configuration. As any physical system will seek to achieve the lowest postential energy possible, a set which is in equilibrium must have an associated potential energy which is less than all the neighboring arrangements. Formally, a set of points on the sphere is in *equilibrium* under a given central force if it is a local minimizer of the associated total potential function in the topology of $(S^{d-1})^N$.

In the following sections we shall characterize FNTFs in terms of the sets which are in equilibrium under a particular central force.

4.3.4. Critical Sets

The concept of a critical set is important in motivating the proper force to choose. In general, when looking for the minimizers of a differentiable function one is naturally interested in critical points. As the total potential is a differentiable function of several constrained variables, sets which are in equilibrium must satisfy a certain set of Lagrange equations.

Theorem 4.6. For a given d, N and central force F, if $\{x_n\}_{n=1}^N$ is a local minimzer for the associated total potential function, then

$$\forall\ m = 1, ..., N,\ \exists\ c_m \in \mathbb{R} \text{ so that } c_m x_m = \sum_{n \neq m} F(x_m, x_n). \tag{4.1}$$

Proof. For any $l = 1, ..., N$, let TP_l be the function obtain from TP by holding all but the lth coordinate constant in the domain of TP. Specifically,

$$\begin{aligned} TP_l : S^{d-1} &\longrightarrow \mathbb{R} \\ x &\longrightarrow TP(\{x_1, ..., x, ..., x_N\}) = 2\sum_{n \neq l} P(x, x_n) + \sum_{m \neq n, m, n \neq l} P(x_m, x_n). \end{aligned}$$

As $\{x_n\}_{n=1}^N$ is a local minimizer for TP, then $x = x_l$ is a local minimizer for TP_l. Since x is constrained to move on the sphere, then x is subject to the constraint $\|x\|^2 = 1$. The local minimizer $x = x_l$ must therefore satisfy the Lagrange equations

$$\nabla TP_l(x)|_{x=x_l} = c_l \nabla TP_l \|x\|^2|_{x=x_l}$$

for some $c_l \in \mathbb{R}$. Because

$$\begin{aligned} \nabla TP_l(x) &= \nabla TP_l(x)(2\sum_{n \neq l} P(x, x_n) + \sum_{m \neq n, m, n \neq l} P(x_m, x_n)) \\ &= 2\sum_{n \neq l} \nabla P(x, x_n) = -2\sum_{n \neq l} \nabla F(x, x_n), \end{aligned}$$

the Lagrange equations are therefore

$$-2\sum_{n \neq l} \nabla F(x_l, x_n) = c_l 2 x_l,$$

and the result is proved. □

Any set of distinct normalized vectors in $\mathbb{R}^d$ is said to be a *critical set* for a central force F if (4.1) is satisfied. In seeking to characterize FNTFs in terms of extreme points for the potential corresponding to a particular force, one would necessarily have to have that any FNTF is a critical set under that force. Recalling that for an FNTF $\{x_n\}$ we have $S = cI$, we can assert that

$$\forall m,\ cx_m = Sx_m = \sum_n \langle x_m, x_n \rangle x_n. \tag{4.2}$$

It then becomes relatively easy to choose a central force F so that (4.1) becomes (4.2).

5. The Frame Force

We define the *frame force* between any two points on the unit sphere in $\mathbb{R}^d$ as

$$FF : S^{d-1} \times S^{d-1} \longrightarrow \mathbb{R}^d,\ FF(a,b) = \langle a, b \rangle (a-b).$$

As the theory described in the last section involves only central forces, we must find a function f so that $FF(a,b) = f(\|a-b\|)(a-b)$. To do so, we observe that if a and b are two real vectors of norm 1, then

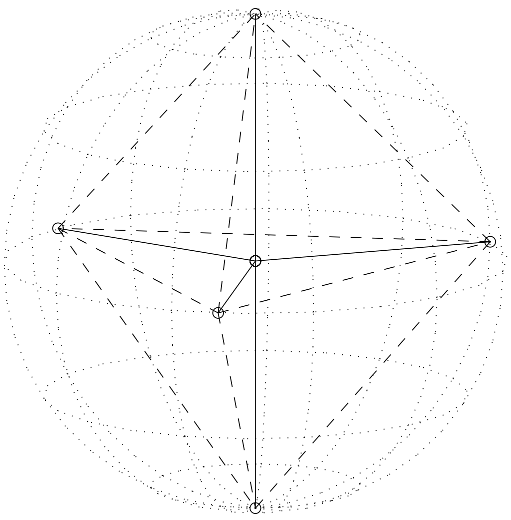
$$\|a-b\|^2 = \|a\|^2 - 2\langle a, b \rangle + \|b\|^2 = 2(1 - \langle a, b \rangle).$$

Thus, the frame force is a central force with $f(x) = 1 - \frac{x^2}{2}$. We note that, unlike many traditional forces found in nature, the frame force between two coincident points is well-defined, being 0. Therefore, we shall regard a collections of points on the sphere under the frame force as a *sequence* of not necessarily distinct vectors.

Our goal is to characterize FNTFs in terms of the sequences of points on the sphere which are in equilibrium under this force. As alluded to in the previous section, the sets which are in equilibrium are the local minima of the associated potential, and therefore these sets must satisfy the corresponding Lagrange equations. Specifically, a sequence $\{x_n\}_{n=1}^N$ in equilibrium under the frame force must be critical under the frame force. Thus, for any m, there must exist $c_m \in \mathbb{R}$ such that

$$\begin{aligned} c_m x_m &= \sum_{n \neq m} F(x_m, x_n) = \sum_{n \neq m} \langle x_m, x_n \rangle (x_m - x_n) \\ &= \sum_{n=1}^N \langle x_m, x_n \rangle x_m - \sum_{n=1}^N \langle x_m, x_n \rangle x_n = \left(\sum_{n=1}^N \langle x_m, x_n \rangle \right) x_m - Sx_m, \end{aligned}$$

Figure 4. The five Platonic solids



where S is the associated frame operator. Equivalently, we write $Sx_m = \lambda_m x_m$ for some $\lambda_m \in \mathbb{R}$. We therefore say a sequence $\{x_n\}_{n=1}^N$ is *FF-critical* if each x_n is an eigenvector for the corresponding frame operator S.

Since we seek to classify FNTFs for both real *and* complex Euclidean spaces, observe that this definition of an FF-critical sequence is well-defined for the complex case.

Turning again to the problem at hand, we observe that any FNTF is an FF-critical sequence. To see this, note that if $\{x_n\}$ is an FNTF then $S = cI$, and so every vector is an eigenvector of S, including each of the x_n. However, it is not true that every FF-critical sequence is a FNTF, as demonstrated by the sequence $\{x_n\}_{n=1}^5$ in S^2 given by

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_5 = \begin{bmatrix} 0 \\ 0 \\ \text{-1} \end{bmatrix}.$$

This is the classical trigonal-bipyrimidal arrangement found in nature as the structure of the PF_5 molecule, formed by taking the cube roots of unity on the equator, along with the north and south poles, as seen in Figure 4. An elementary computation shows that

$$S = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and so this sequence is not an FNTF for $\mathbb{R}^3$. However, x_1, x_2, and x_3 are eigenvectors of S with eigenvalue $3/2$, while x_4 and x_5 are eigenvectors with eigenvalue 2, and so this sequence is FF-critical.

Consequently, the notion of an FF-critical sequence does not characterize FNTFs. However, every FNTF is an FF-critical sequence. Furthermore, when $H = \mathbb{R}^d$, Theorem 4.6 shows that every local minimizer of the associated potential is an FF-critical sequence. Therefore, in the next section we shall explore the connection between FNTFs and the potential corresponding to the frame force.

For now, since FF-critical sequences themselves play an important role in the following sections, we state and prove a statement about their structure. We demonstrate that, although FF-critical sequences are not always FNTFs, the two concepts are intimately related.

Theorem 5.1. A finite sequence of unit vectors $\{x_n\}$ is FF-critical if and only if the sequence may be partitioned into a collection of mutually orthogonal sequences, each of which is a normalized tight frame for its span.

Furthermore, the partition may be chosen explicitly to be $\{E_\lambda\}$, where $E_\lambda = \{x_n : Sx_n = \lambda x_n\}$. Also, the frame constant of E_λ is λ, and the spans of the $\{E_\lambda\}$ are precisely the non-zero eigenspaces of S.

Proof. We first show that a finite sequence of unit vectors FF-critical if and only if the sequence may be partitioned into a collection of mutually orthogonal sequences, each of which is a normalized tight frame for its span.

($\Rightarrow$) Assuming $\{x_n\}$ is an FF-critical sequence, define $E_\lambda = \{x_n : Sx_n = \lambda x_n\}$. Since each x_n is in one of the E_λ by assumption, then $\{E_\lambda\}$ is a partition of $\{x_n\}$. To show that the members of the partition are mutually orthogononal, take any x_{n_1}, x_{n_2} in E_{λ_1}, E_{λ_2}, respectively. Since $S = L^*L$ is self-adjoint, then from classical linear algebra we know that $\lambda_1 \neq \lambda_2$ implies $\langle x_{n_1}, x_{n_2} \rangle = 0$.

To show that E_{λ_1} is an FNTF for $\mathit{span}\, E_{\lambda_1}$ with frame constant λ_1, take any $y \in \mathit{span}\, E_{\lambda_1}$. Then $Sy = \lambda_1 y$, and so $\langle y, x_n \rangle = 0$ for any $x_n \neq E_{\lambda_1}$. Thus, letting S_{λ_1} denote the frame operator for the Bessel sequence E_{λ_1}, we have

$$S_{\lambda_1} y = \sum_{x_n \in E_{\lambda_1}} \langle y, x_n \rangle x_n = \sum_n \langle y, x_n \rangle x_n = Sy = \lambda_1 y.$$

The result then follows from Theorem 2.1a.

($\Leftarrow$) Assume that $\{x_n\}$ is a normalized sequence which is partitioned into a collection of mutually orthogonal sequences, each of which is a normalized tight frame for its span. Denote this partition by $\{E_\gamma\}_{\gamma \in I}$, where I is an appropriate indexing set. Take any $x_m \in E_\gamma$, where E_γ is an FNTF for its span with frame constant λ and frame operator S_γ. Then since $x_m \in \mathit{span}\, E_\gamma$, and since $\langle x_m, x_n \rangle = 0$ for $x_n \notin E_\lambda$ by assumption, we have

$$\lambda x_m = S_\gamma x_m = \sum_{x_n \in E_\gamma} \langle x_m, x_n \rangle x_n = \sum_n \langle x_m, x_n \rangle x_n = S x_m.$$

Thus, each x_m is an eigenvector for S, and therefore $\{x_n\}$ is an FF-critical sequence.

Having proven the equivalence, we note that the first half of the proof also shows that the partition may be taken to be $\{E_\lambda\}$, where $E_\lambda = \{x_n : Sx_n = \lambda x_n\}$ is an FNTF for its span with frame constant λ. Letting $F_\lambda = \{x : Sx = \lambda x\}$ denote the λ-eigenspace of S, what remains to be shown is that $\mathit{span}\, E_\lambda = F_\lambda$ for all $\lambda \neq 0$.

Clearly, for any λ we have $\mathit{span}\, E_\lambda \subseteq F_\lambda$. Letting $H_\lambda = E_\lambda^\perp \cap F_\lambda$, we shall show that if $\lambda \neq 0$, then $H_\lambda = \{0\}$. Given $x \in H_\lambda$, then $x \in E_\lambda^\perp$ implies $\langle x, x_n \rangle = 0$ for all $x_n \in E_\lambda$. Also, $x \in F_\lambda$, and so x is orthogonal to every distinct eigenspace of S, implying $\langle x, x_n \rangle = 0$ for all $x_n \notin E_\lambda$. Thus,

$$Sx = \sum_n \langle x, x_n \rangle x_n = \sum_{x_n \in E_\lambda} \langle x, x_n \rangle x_n + \sum_{x_n \notin E_\lambda} \langle x, x_n \rangle x_n = 0,$$

and so either $x \in E_0$ or $x = 0$. $\square$

This result is best understood in terms of the example given at the beginning of this section, namely that of the FF-critical sequence of 5 elements for $\mathbb{R}^3$. Although not being an FNTF for $\mathbb{R}^3$, this sequence is nevertheless a union of an FNTF with frame constant 3/2 for the plane, and an FNTF with frame constant 2 for the z-axis, which is perpendicular to the plane.

Having explored a necessary aspect of the structure of sequences which are in equilibrium under the frame force, we now turn directly to minimizing the associated potential.

6. The Frame Potential

Recall that for any central force $F(a,b) = f(\|a-b\|)(a-b)$, the associated potential between two points is $P(a,b) = p(\|a-b\|)$, where $p'(x) = -xf(x)$. In the case of the frame force we have $f(x) = 1 - \frac{x^2}{2}$, and so we take $p(x) = -\frac{x^2}{2}(1-\frac{x^2}{4})$. Again, using the fact that $\|a-b\|^2 = 2(1-\langle a,b\rangle)$, we compute

$$P(a,b) = p(\|a-b\|) = -\frac{\|a-b\|^2}{2}\left(1-\frac{\|a-b\|^2}{4}\right) = \frac{1}{2}(\langle a,b\rangle^2 - 1).$$

Consequently, the total potential contained within a sequence of points $\{x_n\}_{n=1}^N$ under the frame force is given by

$$TP(\{x_n\}_{n=1}^N) = \sum_{m=1}^{N}\sum_{n\neq m} P(x_m,x_n) = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}\langle x_m,x_n\rangle^2 - \frac{N^2}{2}.$$

As we shall only be interested in the minimizers of this potential, we may forego the constants. Furthermore, our desire to work in both real and complex Euclidean space leads us to define the *frame potential* of a sequence $\{x_n\}_{n=1}^N$ in K^d to be

$$FP(\{x_n\}_{n=1}^N) = \sum_{m=1}^{N}\sum_{n=1}^{N}|\langle x_m,x_n\rangle|^2. \tag{6.1}$$

Although the frame potential is well-defined on any sequence of vectors in K^d, we shall only consider sequences of unit vectors, due to our focus upon normalized frames. Letting $S(K^d) = \{x \in K^d : \|x\| = 1\}$ denote the unit sphere in K^d, the frame potential may be viewed as a function

$$FP : S(K^d)^N = \underbrace{S(K^d)\times\ldots\times S(K^d)}_{N \text{ times}} \longrightarrow [0,\infty).$$

We formally define the sequences which are in *equilibrium under the frame force* to be the local minimizers of the frame potential (6.1).

The principal result of this paper is the characterization of FNTFs for K^d as the sets which are in equilibrium under the frame force. Note that, as with our definition of an FF-critical sequence, our definition of frame potential extends a real physical concept to possibly complex spaces. However, in doing so we emphasize the fact that classical results for real spaces must now be proven again in the complex setting. In particular, in the next section we shall adapt one such result relating the local minimizers of the potential and critical sets, e.g., Theorem 7.3.

Before addressing the minimization problem directly, we devote the rest of this section to finding explicit upper and lower bounds on the frame potential. The following result is similar to, and partly inspired by, Zimmermann's insightful use of the trace operator in his proof of Theorem 3.1 in [19].

Theorem 6.1. Let $\{x_n\}$ be a finite sequence in K^d with associated frame operator S. Then $FP(\{x_n\}) = \mathrm{Tr}(S^2)$.

Proof. We first observe that the frame potential of a sequence (6.1) is actually a type of L^2 norm squared of the values $\{\langle x_m, x_n\rangle\}_{m,n}$. As the (n,m)th entry of the Gramian matrix $G = LL^*$ is $G(n,m) = \langle x_m, x_n\rangle$, then the frame potential is in fact an L^2 norm squared on the entries of G.

We therefore recall that for any matrix A, the Frobenius norm of A is defined by $\|A\|_F^2 = \sum_m \sum_n |A(m,n)|^2$. A simple computation shows that $\|A\|_F^2 = \mathrm{Tr}(A^*A)$, and therefore we have

$$FP(\{x_n\}_{n=1}^N) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_m, x_n\rangle|^2 = \sum_{m=1}^N \sum_{n=1}^N |G(m,n)|^2 = \|G\|_F^2 = \mathrm{Tr}(G^*G).$$

Using the classical fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, even for non-square matrices A and B so long as their products are square, we have

$$FP(\{x_n\}_{n=1}^N) = \mathrm{Tr}(G^*G) = \mathrm{Tr}(LL^*LL^*) = \mathrm{Tr}(L^*LL^*L) = \mathrm{Tr}(S^2).$$

□

One immediate consequence of Theorem 6.1 is that we may explicitly compute the frame potential for any FNTF. To see this, note that if $\{x_n\}$ is an FNTF of N elements for K^d, then Theorem 3.1 states that $S = (N/d)I$. Therefore $FP(\{x_n\}) = \mathrm{Tr}((N^2/d^2)I) = N^2/d$. Thus, we see that all FNTFs of N elements for K^d have the same frame potential. In addition to providing upper and lower bounds on the frame potential in terms of N and d, the next result also shows that the *only* Bessel sequences with frame potential $\frac{N^2}{d}$ are FNTFs, and that this is the minimum possible potential energy.

Theorem 6.2. Given any d and N, let $\{x_n\}_{n=1}^N \subseteq S(K^d)$. Then

$$N \max\left(1, \frac{N}{d}\right) \leq FP(\{x_n\}_{n=1}^N) \leq N^2.$$

In particular, a lower bound on the frame potential is N if $N \leq d$ and N^2/d if $N \geq d$. Furthermore, the lower bounds of N and $\frac{N^2}{d}$ are achieved if and only if $\{x_n\}$ is an orthonormal set or an FNTF for K^d, respectively.

Proof. The upper bound is trivial, since for any normalized sequence $\{x_n\}_{n=1}^N$, the Cauchy-Schwarz inequality implies

$$FP(\{x_n\}_{n=1}^N) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_m, x_n\rangle|^2 \leq \sum_{m=1}^N \sum_{n=1}^N \|x_m\|^2 \|x_n\|^2 = \sum_{m=1}^N \sum_{n=1}^N 1 = N^2.$$

To obtain the first lower bound, note that for any normalized sequence $\{x_n\}_{n=1}^N$,

$$FP(\{x_n\}_{n=1}^N) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_m, x_n\rangle|^2 = N + \sum_{m=1}^N \sum_{n \neq m} |\langle x_m, x_n\rangle|^2 \geq N,$$

with equality if and only if

$$\sum_{m=1}^N \sum_{n \neq m} |\langle x_m, x_n\rangle|^2 = 0.$$

Thus, the frame potential of a sequence may equal the lower bound N if and only if $\langle x_m, x_n\rangle = 0$, for all m, and all $n \neq m$, that is, if and only if $\{x_n\}_{n=1}^N$ is orthonormal.

To obtain the second lower bound, let S be the frame operator corresponding to $\{x_n\}$, and let $\{\lambda_k\}_{k=1}^d$ denote the eigenvalues of S, counting multiplicities. Since the eigenvalues of S^2 are $\{\lambda_k^2\}_{k=1}^d$, we have by Theorem 3.1 that

$$FP(\{x_n\}) = \mathrm{Tr}(S^2) = \sum_{k=1}^d \lambda_k^2. \tag{6.2}$$

To obtain a lower bound for the frame potential we therefore seek to minimize (6.2) over a sufficient set of possible parameters $\{\lambda_k\}$. As $S = L^*L$ is a positive (and therefore self-adjoint) semi-definite operator, we know the values $\{\lambda_k\}$ may be taken to be real and nonnegative. However, because the condition, that $\{\lambda_k\}$ is the sequence of eigenvalues of the frame operator of a normalized Bessel sequence, is cumbersome to work with, we instead seek a broader constraint.

Since $\{x_n\}_{n=1}^N$ is a normalized sequence, the diagonal elements of the corresponding Grammian matrix are of the form $\langle x_n, x_n\rangle = \|x_n\|^2 = 1$. Thus G is an $N \times N$ matrix with ones along the diagonal, and hence

$$\sum_{k=1}^{d} \lambda_k = \mathrm{Tr}(S) = \mathrm{Tr}(L^*L) = \mathrm{Tr}(LL^*) = \mathrm{Tr}(G) = N.$$

Therefore, to obtain a lower bound for the frame potential, we minimize the function $\sum_{k=1}^d \lambda_k^2$ subject to the constraint that $\{\lambda_k\}_{k=1}^d$ is a sequence of nonnegative values which sum to N.

This is the classical problem of finding a point on a hyperplane in $\mathbb{R}^d$ which is closest to the origin, and it may be solved using the corresponding Lagrange equations. Of course, the unique minimizer occurs when all of the λ_k are equal, and therefore all equal to $\frac{N}{d}$. The minimum value is therefore

$$\sum_{k=1}^{d} \lambda_k^2 = \sum_{k=1}^{d} \left(\frac{N}{d}\right)^2 = d\left(\frac{N}{d}\right)^2 = \frac{N^2}{d}.$$

Note that the only way that this minimum is achieved is to have each λ_k equal to $\frac{N}{d}$. Consequently, if $FP(\{x_n\}) = N^2/d$, then the eigenvalues of the corresponding frame operator S are all equal to N/d, and so $S = (N/d)I$, i.e., $\{x_n\}$ is a FNTF for K^d.

Writing all three bounds together in a single inequality yields

$$N \max\left(1, \frac{N}{d}\right) = \max\left(N, \frac{N^2}{d}\right) \le FP(\{x_n\}_{n=1}^N) \le N^2,$$

with $FP(\{x_n\}) = N$ if and only if $\{x_n\}$ is orthonormal, and $FP(\{x_n\}) = \frac{N^2}{d}$ if and only if $\{x_n\}$ is a FNTF for K^d. $\square$

We observe that a similar approach was independently, and at about the same time, used to prove the main result in [11]. There the authors optimized a quantity equivalent to $\sum_k 1/\lambda_k$ subject to $\sum_k \lambda_k = N$.

Theorem 6.2 hints at the fact that FNTFs, provided they exist, should be the global minimizers of the frame potential. We now turn to proving that this statement is indeed true, as well as addressing the issue of local minimizers.

7. Classification of FNTFs

The main results of this paper are summarized in the following theorem.

Theorem 7.1. For a given d and N, consider the frame potential

$$\begin{aligned} FP : S(K^d)^N &\longrightarrow [0, \infty) \\ \{x_n\}_{n=1}^N &\longrightarrow \sum_{m=1}^N \sum_{n=1}^N |\langle x_m, x_n\rangle|^2. \end{aligned}$$

Then

a. Every local minimizer of the frame potential is also a global minimizer.

b. If $N \leq d$, the minimum value of the frame potential is N, and the minimizers are precisely the orthonormal sequences in K^d.

c. If $N \geq d$, the minimum value of the frame potential is N^2/d, and the minimizers are precisely the FNTFs for K^d.

Theorem 7.1 is important for several reasons. First, the result reveals some of the physical characteristics of the frame force, as described in Remark 7.2a. Also, even if one were to completely disregard the mechanical motivation, the frame potential as defined by (6.1) is a natural measurement on the *total orthogonality* of a sequence of normalized vectors in a Hilbert space. Under this interpretation, minimizing the frame potential is equivalent to finding a normalized sequence whose elements are as orthogonal to each other as possible. When $N \leq d$ the obvious means of doing this is to find an orthonormal set. However, when $N \geq d$, Theorem 7.1 says that the sequences which are as close to orthogonal as possible are precisely the FNTFs of N elements for K^d. Theorem 7.1 also strengthens the perception of FNTFs as extensions of ONBs. Though this idea is of course supported by (1.1), we now have a result which has both orthonormal sets and FNTFs arising from the same minimization problem, differentiated only by whether N is less than, equal to, or greather than d.

The proof of Theorem 7.1 is the subject of the remainder of this paper. We found several different ways of proving most of these results, as seen in Remark 7.2.b, and present the proof which requires the least reliance upon other ideas.

Remark 7.2. a. Recall that sequences which are in equilibrium under a given force are defined as the local minimizers of the associated potential. Theorem 7.1 therefore shows that the sequences which are in equilibrium under the frame force are either orthonormal sets or FNTFs. In fact, we have that if $N \leq d$, namely if the number of points on the sphere is less than the dimension of the surrounding space, then the sequences in equilibrium are orthonormal sets. In this case, each point actually exhibits no force upon the other points on the sphere. However, if the number of points on the sphere exceeds the dimension of the space it becomes impossible for the points to be completely perpendicular to each other. To achieve equilibrium in this case, one intuitively spreads the frame force energy equally among all the points. The remarkable fact is that in doing so one obtains an FNTF, namely a sequence which satisfies the identity (1.1).

Theorem 7.1 also illustrates an important difference between the frame force and the classical forces of interest, such as Coulomb's force, when studying equidistribution on the sphere. In the classical cases, one is often confronted with many local minima of the associated potential, making it difficult to compute the optimal energy numerically. In fact, in some cases the number of local minima has been shown to grow exponentially with N [15]. However, as the local minimizers of the frame potential are necessarily global minimizers, we have that if $N \geq d$ then a local minimzer of the frame potential must be an FNTF. This could be practically useful, as a steepest descent method could be used to compute numerically some FNTFs.

b. We developed two independent approaches to proving much of Theorem 7.1. The first approach is to regard Theorem 7.1 as a minor consequence of Theorem 6.2. Indeed, by invoking the existence of ON sets and FNTFs from Theorem 3.2, we see that the the lower bound results from Theorem 6.2 are exact. That is, the lower bounds of Theorem 6.2 are indeed minimal values, proving Theorem 7.1b,c.

The rest of this section is devoted to a different approach to proving Theorem 7.1, one which still depends on, but is much more involved than, the results of Section 6. In this section we not only prove all three statements of Theorem 7.1; but we do so without relying upon the known existence of FNTFs from Theorem 3.2. As a matter of fact, we shall obtain existence as a consequence of material presented here, thereby providing a non-constructive proof of Theorem 3.2.

Calculus is the first step in understanding the minima of the frame potential. We must take a derivative, and understand what the critical points are in this context. Of course, since the frame potential is a function of several constrained variables, finding the critical points will involve Lagrange's equations. In Section 4, we mentioned the fact that given a central force F, any extreme point of the corresponding potential must be a critical set, i.e., it must satisfy (4.1). In

Sections 5 and 6, the notion of a set which is critical under the frame force, and the corresponding potential energy, were extended to possibly complex spaces. The following result verifies that the property which inspired the original definitions still holds in the more general setting.

Theorem 7.3. The local minimizers of the frame potential are FF-critical sequences.

Proof. Let $\{x_n\}_{n=1}^N \subseteq S(K^d)$ be a local minimizer for the frame potential. For any $l = 1, ..., N$, we shall demonstrate that x_l is an eigenvector of S. Define the lth frame potential, denoted by FP_l, as the function obtained by taking the frame potential at $\{x_n\}_{n=1}^N$, and letting the lth term vary over the sphere. Formally,

$$\begin{aligned} FP_l : S(K^d) &\longrightarrow [0,\infty) \\ x &\longrightarrow 2\sum_{n\neq l} |\langle x, x_n\rangle|^2 + 1 + \sum_{m\neq l}\sum_{n\neq l} |\langle x_m, x_n\rangle|^2. \end{aligned}$$

Since $\{x_n\}_{n=1}^N$ is a local minimizer for the frame potential in the topology of $S(K^d)^N$, then $x = x_l$ is a local minimizer for FP_l in the topology of $S(K^d)$. By writing each variable of the form $x \in K^d$ as $x = a + ib$ with $a, b \in \mathbb{R}^d$ gives

$$|\langle x, x_n\rangle^2| = |\langle a + ib, a_n + ib_n\rangle|^2 = (\langle a, a_n\rangle + \langle b, b_n\rangle)^2 + (\langle b, a_n\rangle - \langle a, b_n\rangle)^2.$$

As $x \in S(K^d)$, then $\|a\|^2 + \|b\|^2 = \|x\|^2 = 1$. Letting $S = \{(a, b) \in \mathbb{R}^d \times \mathbb{R}^d : \|a\|^2 + \|b\|^2 = 1\}$, rewrite the lth frame potential function, FP_l, as a function of the real and imaginary components of x, namely $\widetilde{FP}_l : S \longrightarrow [0,\infty)$,

$$(a, b) \longrightarrow 2\sum_{n\neq l}(\langle a, a_n\rangle + \langle b, b_n\rangle)^2 + (\langle b, a_n\rangle - \langle a, b_n\rangle)^2 + 1 + \sum_{m\neq l}\sum_{n\neq l} |\langle x_m, x_n\rangle|^2.$$

In this form, $(a, b) = (a_l, b_l)$ is a local minimizer for $\widetilde{FP}_l$ in terms of the topology of S, and so the corresponding Lagrange equations must be satisfied. Namely there must exist some scalar $c \in \mathbb{R}$ such that

$$\nabla_a \widetilde{FP}_l(a, b)\Big|_{(a,b)=(a_l,b_l)} = c\nabla_a(\|a\|^2 + \|b\|^2)\Big|_{(a,b)=(a_l,b_l)}, \tag{7.1}$$

$$\nabla_b \widetilde{FP}_l(a, b)\Big|_{(a,b)=(a_l,b_l)} = c\nabla_b(\|a\|^2 + \|b\|^2)\Big|_{(a,b)=(a_l,b_l)}, \tag{7.2}$$

$$\|a_l\|^2 + \|b_l\|^2 = 1.$$

Now

$$\nabla_a \widetilde{FP}_l(a,b) = \nabla_a\Big(2\sum_{n\neq l}(\langle a,a_n\rangle + \langle b,b_n\rangle)^2 + (\langle b,a_n\rangle - \langle a,b_n\rangle)^2$$
$$= 4\sum_{n\neq l}(\langle a,a_n\rangle + \langle b,b_n\rangle)a_n - (\langle b,a_n\rangle - \langle a,b_n\rangle)b_n,$$

and so

$$\nabla_a \widetilde{FP}_l(a,b)\Big|_{(a,b)=(a_l,b_l)} = 4\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)a_n - (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)b_n.$$

In a similar fashion, we have

$$\nabla_b \widetilde{FP}_l(a,b)\Big|_{(a,b)=(a_l,b_l)} = 4\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)b_n + (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)a_n.$$

Next we observe that

$$\nabla_a(\|a\|^2 + \|b\|^2)\Big|_{(a,b)=(a_l,b_l)} = 2a_l, \quad \nabla_b(\|a\|^2 + \|b\|^2)\Big|_{(a,b)=(a_l,b_l)} = 2b_l,$$

and so the first two Lagrange equations (7.1) and (7.2) become

$$2\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)a_n - (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)b_n = ca_l,$$
$$2\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)b_n + (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)a_n = cb_l.$$

Writing these two equations as a single equation in complex variables gives

$$\begin{aligned} cx_l &= c(a_l + ib_l) = ca_l + ib_l \\ &= 2\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)a_n - (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)b_n \\ &\quad +2i\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)b_n + (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)a_n. \end{aligned}$$

Finally, letting S denote the frame operator corresponding to $\{x_n\}_{n=1}^N$,

$$\begin{aligned} Sx_l &= \sum_{n=1}^N \langle x_l,x_n\rangle x_n = x_l + \sum_{n\neq l}\langle a_l + ib_l, a_n + ib_n\rangle(a_n + ib_n) \\ &= x_l + \sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)a_n - (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)b_n \\ &\quad +i\sum_{n\neq l}(\langle a_l,a_n\rangle + \langle b_l,b_n\rangle)b_n + (\langle b_l,a_n\rangle - \langle a_l,b_n\rangle)a_n \\ &= x_l + \frac{1}{2}cx_l = (1 + \frac{1}{2}c)x_l. \end{aligned}$$

Thus x_l is an eigenvector of the frame operator S with eigenvalue $1 + \frac{1}{2}c$. Since l is arbitrary, we see that $\{x_n\}_{n=1}^N$ is an FF-critical sequence. $\square$

Combined with Theorem 5.1, Theorem 7.3 tells us that any local minimizer of the frame potential may be partitioned into a collection of mutually orthogonal sequences, each of which is a FNTF for its span. The next result builds upon this characterization, and is the fundamental result required to prove Theorem 7.1

Theorem 7.4. A local minimizer for the frame potential is either an orthonormal set or a FNTF for K^d.

Proof. Let $\{x_n\}_{n=1}^N$ be a local minimum for the frame potential, and let λ_1 denote the largest eigenvalue of the corresponding frame operator S.

The previous result tells us that $\{x_n\}$ is an FF-critical sequence, and we know by Theorem 5.1 that every non-zero eigenvalue of S arises as the tight frame constant of an FNTF for a subspace of K^d. By Theorem 2.1b, the tight frame constant must be at least 1 in value. Thus, we have that every non-zero eigenvalue of S is at least 1 in value. In particular, $\lambda_1 \geq 1$. We shall show that if $\lambda_1 = 1$ then $\{x_n\}$ is an orthonormal set, and if $\lambda_1 > 1$, then $\{x_n\}$ is an FNTF for K^d.

If $\lambda_1 = 1$, then every non-zero eigenvalue of S must equal 1. By Theorem 5.1, the eigenvectors E_1 form a FNTF for their span with frame constant 1. By Theorem 2.1.b, the eigenvectors E_1 must therefore be an orthonormal basis for their span. However, since 1 is the only non-zero eigenvalue of S, then Theorem 5.1 also tells us that E_1 is the only nonempty partition element of $\{x_n\}$. Thus, $E_1 = \{x_n\}$, and so altogether we have that $\{x_n\}$ is an orthonormal basis for its span. In other words, we have that $\{x_n\}$ is an orthonormal set for K^d.

What remains to be shown is that if $\lambda_1 > 1$, then $\{x_n\}$ is an FNTF for K^d. To do this, we assume to the contrary that $\{x_n\}$ is not an FNTF, and we shall obtain a contradiction, namely that $\{x_n\}$ is not a local minimizer of the frame potential. To do this we shall explicitly provide a sequence $\{\tilde{x}_n\}$ which may be taken to be as close to $\{x_n\}$ as desired, and so that $FP(\{\tilde{x}_n\}) < FP(\{x_n\})$.

To do so, note that if $\{x_n\}$ is not a normalized tight frame, then $S \neq AI$ for any A, and so there exists an eigenvalue λ_2 of S which is distinct from λ_1. By assumption, λ_1 is the largest eigenvalue of S, and so $\lambda_2 < \lambda_1$. Let I denote the set of indices of $\{x_n\}$ that correspond to λ_1, namely $I = \{m : Sx_m = \lambda_1 x_m\}$, and let u be any unit eigenvector for the eigenvalue λ_2. Now, because $\{x_n\}$ is an FF-critical sequence, then each x_n is an eigenvector of S. Since S is self-adjoint, we see that the eigenvectors corresponding to distinct eigenvalues of S are orthogonal. Therefore,

$$\forall m \in I, n \notin I, \qquad \langle x_m, x_n \rangle = \langle x_m, u \rangle = 0. \tag{7.3}$$

Again referring to Theorem 5.1, note that $\{x_n\}_{n\in I} = E_{\lambda_1}$ is a normalized tight frame for its span with frame constant λ_1. By Theorem 3.1, $\lambda_1 = N_1/d_1$ where N_1 is the number of elements in E_1, and d_1 is the dimension of the span of E_1. Since $\lambda_1 > 1$ by assumption, then E_1 is linearly dependent set. Therefore, there exist scalars $\{\beta_n\}_{n\in I}$ not all zero such that $\sum_{n\in I} \overline{\beta_m} x_n = 0$, the reason for the complex conjugation becoming apparent later. Since there are only a finite number of the β_n, we assume without loss of generality that $|\beta_n|^2 \leq \frac{1}{2}$ for all $n = 1, ..., N$.

To form the sequence $\{\widetilde{x}_n\}$, we shall perturb the sequence $\{x_n\}$ in a specific, and non-intuitive manner. Namely, we shall "push" the points $\{x_m\}_{m\in I}$ that correspond to the large eigenvalue λ_1 into the direction u corresponding to the smaller eigenvalue λ_2. However, the "pushing" shall not be uniform, as the rate of moving into the direction of u depends upon the linear dependence constants $\{\beta_n\}_{n\in I}$. We also must be careful that our new points remain on the sphere.

Specifically, given any $\epsilon \in [0, 1]$, we define the sequence $\{\widetilde{x}_n\}$ as

$$\widetilde{x}_m = \begin{cases} \sqrt{1 - \epsilon^2 |\beta_m|^2}\, x_m + \epsilon \beta_m u, & m \in I \\ x_m, & m \notin I. \end{cases}$$

Note that since $|\beta_m| \leq \frac{1}{2}$ and since $\epsilon \leq 1$, then $\{\widetilde{x}_n\}$ is a well-defined arrangement on $S(K^d)$. Also note that, as $\epsilon \to 0$, we have $\{\widetilde{x}_n\} \to \{x_n\}$ in the topology of $S(K^d)^N$. Beyond that, the reason for defining $\{\widetilde{x}_n\}$ in this manner will not be apparent until the end of the proof. We compute the value of the frame potential at $\{\widetilde{x}_m\}$, and verify that for all sufficiently small ϵ, the new value is strictly less than the frame potential at $\{x_n\}$.

The frame potential of $\{\widetilde{x}_n\}_{n=1}^N$ is computed in several pieces. In particular, note that

$$\begin{aligned} FP(\{\widetilde{x}_n\}) &= \sum_{m=1}^{N} \sum_{n=1}^{N} |\langle \widetilde{x}_m, \widetilde{x}_n \rangle|^2 \\ &= \sum_{m\notin I} \sum_{n\notin I} |\langle \widetilde{x}_m, \widetilde{x}_n \rangle|^2 + 2 \sum_{m\in I} \sum_{n\notin I} |\langle \widetilde{x}_m, \widetilde{x}_n \rangle|^2 + \sum_{m\in I} \sum_{n\in I} |\langle \widetilde{x}_m, \widetilde{x}_n \rangle|^2. \end{aligned}$$

An expression for $|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2$ must therefore be found for all possibilities of m and n.

If both indices m and n are not in I, then $\langle\widetilde{x}_m,\widetilde{x}_n\rangle=\langle x_m,x_n\rangle$.

Next, if $m\in I$ and $n\notin I$, then (7.3) implies

$$|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2=|\langle\sqrt{1-\epsilon^2|\beta_m|^2}x_m+\epsilon\beta_m u,x_n\rangle|^2=\epsilon^2|\beta_m|^2|\langle u,x_n\rangle|^2.$$

Summing over all such m and n,

$$\sum_{m\in I}\sum_{n\notin I}|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2=\sum_{m\in I}\sum_{n\notin I}\epsilon^2|\beta_m|^2|\langle u,x_n\rangle|^2=\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big)\Big(\sum_{n\notin I}|\langle u,x_n\rangle|^2\Big).$$

Again using (7.3) we have

$$\begin{aligned}\sum_{n\notin I}|\langle u,x_n\rangle|^2&=\sum_{n\notin I}|\langle u,x_n\rangle|^2+\sum_{n\in I}|\langle u,x_n\rangle|^2=\sum_{n=1}^{N}|\langle u,x_n\rangle|^2\\&=\sum_{n=1}^{N}\langle u,x_n\rangle\langle x_n,u\rangle=\langle\sum_{n=1}^{N}\langle u,x_n\rangle x_n,u\rangle=\langle Su,u\rangle=\lambda_2\langle u,u\rangle=\lambda_2.\end{aligned}$$

At this point we have

$$\begin{aligned}FP(\{\widetilde{x}_n\})=&\sum_{m\notin I}\sum_{n\notin I}|\langle x_m,x_n\rangle|^2+2\lambda_2\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big)\\&+\sum_{m\in I}\sum_{n\in I}|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2.\end{aligned}\tag{7.4}$$

Now, if both m and n are in I, then $\langle x_m,u\rangle=\langle u,x_n\rangle=0$, and so

$$\begin{aligned}\langle\widetilde{x}_m,\widetilde{x}_n\rangle&=\langle\sqrt{1-\epsilon^2|\beta_m|^2}\,x_m+\epsilon\beta_m u,\sqrt{1-\epsilon^2|\beta_n|^2}\,x_n+\epsilon\beta_n u\rangle\\&=\sqrt{1-\epsilon^2|\beta_m|^2}\,\sqrt{1-\epsilon^2|\beta_n|^2}\,\langle x_m,x_n\rangle+\epsilon^2\beta_m\overline{\beta_n}.\end{aligned}$$

Since $|z+w|^2=|z|^2+2\Re(z\overline{w})+|w|^2$ for any complex numbers z and w, we compute

$$\begin{aligned}|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2=&(1-\epsilon^2|\beta_m|^2)(1-\epsilon^2|\beta_n|^2)|\langle x_m,x_n\rangle|^2\\&+2\Re(\sqrt{1-\epsilon^2|\beta_m|^2}\,\sqrt{1-\epsilon^2|\beta_n|^2}\,\langle x_m,x_n\rangle\epsilon^2\overline{\beta_m}\beta_n)+\epsilon^4|\beta_m|^2|\beta_n|^2\\=&|\langle x_m,x_n\rangle|^2-\epsilon^2(|\beta_m|^2+|\beta_n|^2)|\langle x_m,x_n\rangle|^2\\&+\epsilon^4|\beta_m|^2|\beta_n|^2(|\langle x_m,x_n\rangle|^2+1)\\&+2\Re(\sqrt{1-\epsilon^2|\beta_m|^2}\,\sqrt{1-\epsilon^2|\beta_n|^2}\,\langle x_m,x_n\rangle\epsilon^2\overline{\beta_m}\beta_n).\end{aligned}$$

As the nonlinearity of the last term would make further simplification impossible, we shall use Taylor's Theorem to extract the dominant terms of

$\sqrt{1-\epsilon^2|\beta_m|^2}\,\sqrt{1-\epsilon^2|\beta_n|^2}$ in terms of ϵ. To facilitate this, we add and subtract the constant component, namely 1, to obtain

$$\begin{aligned}|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2 = {} & |\langle x_m,x_n\rangle|^2-\epsilon^2(|\beta_m|^2+|\beta_n|^2)|\langle x_m,x_n\rangle|^2+2\epsilon^2\Re(\langle\overline{\beta_m},\overline{\beta_n}\rangle)\\ & +\epsilon^4|\beta_m|^2|\beta_n|^2(|\langle x_m,x_n\rangle|^2+1) \qquad (7.5)\\ & +2\epsilon^2(\sqrt{(1-\epsilon^2|\beta_m|^2)(1-\epsilon^2|\beta_n|^2)}-1)\Re(\overline{\beta_m}\beta_n\langle x_m,x_n\rangle). \qquad (7.6)\end{aligned}$$

Seeking to bound (7.5) in terms of ϵ^4, we observe that

$$\left|\epsilon^4|\beta_m|^2|\beta_n|^2(|\langle x_m,x_n\rangle|^2+1)\right|\le\epsilon^4(\|x_m\|^2\|x_n\|^2+1)\le 2\epsilon^4$$

by the Cauchy-Schwarz inequality. To bound (7.6) in terms of ϵ^4, we bound

$$f(\epsilon)=\sqrt{(1-\epsilon^2|\beta_m|^2)(1-\epsilon^2|\beta_n|^2)}-1$$

in terms of ϵ^2 using Taylor's Theorem about $\epsilon=0$. Specifically, one may show that $f(0)=f'(0)=0$. We therefore know that $f(\epsilon)=(f''(\eta)/2)\epsilon^2$ for some $0<\eta<\epsilon$. Using the assumption that $|\beta_m|^2\le 1/2$ for all $m\in I$, a computation shows that $|f''(\eta)|\le 38$ over $[0,\epsilon]$. Therefore $|f(\epsilon)|\le 19\epsilon^2$, and so

$$\begin{aligned}&|2\epsilon^2(\sqrt{(1-\epsilon^2|\beta_m|^2)(1-\epsilon^2|\beta_n|^2)}-1)\Re(\overline{\beta_m}\beta_n\langle x_m,x_n\rangle)|\\ &=2\epsilon^2|f(\epsilon)||\Re(\overline{\beta_m}\beta_n\langle x_m,x_n\rangle)|\le(2\epsilon^2)(19\epsilon^2)|\overline{\beta_m}\beta_n\langle x_m,x_n\rangle|\\ &\le 38\epsilon^4(1/\sqrt{2})^2\|x_m\|\|x_n\|\le 19\epsilon^4.\end{aligned}$$

By the triangle inequality, the total effect of (7.5) and (7.6) in the expansion of $|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2$ is of magnitude less than or equal to $21\epsilon^4$.

Summing over all indicies $m,n\in I$ we obtain

$$\begin{aligned}\sum_{m\in I}\sum_{n\in I}|\langle\widetilde{x}_m,\widetilde{x}_n\rangle|^2 = {} & \sum_{m\in I}\sum_{n\in I}|\langle x_m,x_n\rangle|^2-\epsilon^2\sum_{m\in I}\sum_{n\in I}(|\beta_m|^2+|\beta_n|^2)|\langle x_m,x_n\rangle|^2\\ & +2\epsilon^2\sum_{m\in I}\sum_{n\in I}\Re(\langle\overline{\beta_m}x_m,\overline{\beta_n}x_n\rangle)+R(\epsilon)\epsilon^4,\end{aligned}$$

where $R(\epsilon)$ is bounded in magnitude. By exploiting symmetry and by using (7.3), we have

$$\begin{aligned}&\sum_{m\in I}\sum_{n\in I}(|\beta_m|^2+|\beta_n|^2)|\langle x_m,x_n\rangle|^2=2\sum_{m\in I}\sum_{n\in I}|\beta_m|^2|\langle x_m,x_n\rangle|^2\\ &\quad=2\sum_{m\in I}|\beta_m|^2\sum_{n=1}^{N}|\langle x_m,x_n\rangle|^2=2\sum_{m\in I}|\beta_m|^2\langle\sum_{n=1}^{N}\langle x_m,x_n\rangle x_n,x_m\rangle\end{aligned}$$

$$= 2\sum_{m\in I}|\beta_m|^2\langle Sx_m, x_m\rangle = 2\sum_{m\in I}|\beta_m|^2\lambda_1\langle x_m, x_m\rangle = 2\lambda_1\Big(\sum_{m\in I}|\beta_m|^2\Big).$$

Furhermore, recalling that $\{\beta_m\}_{m\in I}$ was chosen so that $\sum_{m\in I}\overline{\beta_m}x_m = 0$, we have

$$\sum_{m\in I}\sum_{n\in I}\Re(\langle\overline{\beta_m}x_m, \overline{\beta_n}x_n\rangle) = \Re(\langle\sum_{m\in I}\overline{\beta_m}x_m, \sum_{n\in I}\overline{\beta_n}x_n\rangle) = \|\sum_{m\in I}\overline{\beta_m}x_m\|^2 = 0.$$

This fact was the major motivation in defining $\{\tilde{x}_n\}$ in the manner we did.

Putting all of this into (7.4) yields

$$\begin{aligned} FP(\{\tilde{x}_n\}) &= \sum_{m\notin I}\sum_{n\notin I}|\langle x_m, x_n\rangle|^2 + 2\lambda_2\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big) \\ &\quad + \sum_{m\in I}\sum_{n\in I}|\langle\tilde{x}_m, \tilde{x}_n\rangle|^2 \\ &= \sum_{m\notin I}\sum_{n\notin I}|\langle x_m, x_n\rangle|^2 + 2\lambda_2\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big) \\ &\quad + \sum_{m\in I}\sum_{n\in I}|\langle x_m, x_n\rangle|^2 - 2\lambda_1\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big) + R(\epsilon)\epsilon^4. \end{aligned}$$

Using (7.3) again we see that

$$\begin{aligned} FP(\{x_n\}) &= \sum_{m=1}^{M}\sum_{n=1}^{N}|\langle x_m, x_n\rangle|^2 \\ &= \sum_{m\notin I}\sum_{n\notin I}|\langle x_m, x_n\rangle|^2 + 2\sum_{m\in I}\sum_{n\notin I}|\langle x_m, x_n\rangle|^2 + \sum_{m\in I}\sum_{n\in I}|\langle x_m, x_n\rangle|^2 \\ &= \sum_{m\notin I}\sum_{n\notin I}|\langle x_m, x_n\rangle|^2 + \sum_{m\in I}\sum_{n\in I}|\langle x_m, x_n\rangle|^2, \end{aligned}$$

and so we have

$$FP(\{\tilde{x}_n\}) = FP(\{x_n\}) + 2(\lambda_2 - \lambda_1)\epsilon^2\Big(\sum_{m\in I}|\beta_m|^2\Big) + R(\epsilon)\epsilon^4$$

Since $\lambda_2 < \lambda_1$, and since not all of the β_m are 0, we can conclude that the coefficient of ϵ^2 is negative. Since $R(\epsilon)$ is bounded, by taking ϵ sufficiently small the square term will dominate, and so $FP(\{\tilde{x}_n\}_{n=1}^N) < FP(\{x_n\}_{n=1}^N)$, a contradiction to the assumption that $\{x_n\}$ is a local minimizer of the frame potential. Thus $\{x_n\}$ must be a FNTF for K^d. □

The proof of Theorem 7.1 follows from this result.

Proof of Theorem 7.1. For a given d and N, the frame potential is a continuous function on the compact set $S(K^d)^N$, and so has a global minimizer $\{x_n\}$. As this is also a local minimizer, the previous result states that $\{x_n\}$ must either be an orthonormal set or an FNTF for K^d.

If $N \leq d$, we claim that $\{x_n\}$ must be an orthonormal set in K^d. In fact, if $N < d$, then $\{x_n\}$ clearly cannot be an FNTF for K^d, or even a basis, and so it must be an orthonormal set. If $N = d$, then $\{x_n\}$ is either an orthonormal set or an FNTF with frame constant $N/d = 1$. By Theorem 2.1, $\{x_n\}$ is an orthonormal basis for K^d and is therefore an orthonormal set. Thus, $\{x_n\}$ must be an orthonormal set, whose frame potential achieves the lower bound set forth in Theorem 6.2. Thus, the lower bound of N is indeed a minimal value for the frame potential. The fact that the minimizers are precisely the orthonormal sets in K^d is simply a consequence of Theorem 6.2.

On the other hand, if $N \geq d$, we claim that $\{x_n\}$ must be an FNTF for K^d. If $N > d$, then $\{x_n\}$ cannot be an orthonormal set, or even linearly independent, and so $\{x_n\}$ is therefore an FNTF for K^d. If $N = d$, then either $\{x_n\}$ is an FNTF for K^d, or $\{x_n\}$ is an orthonormal set of d vectors in K^d. In the latter case, $\{x_n\}$ is an orthonormal basis for K^d, and so is an FNTF. Thus, in either case, $\{x_n\}$ is an FNTF whose frame potential achieves the lower bound of N^2/d from Theorem 6.2. As above, the fact that the minimizers are precisely the FNTFs of K^d is a consequence of Theorem 6.2.

The fact that every local minimzer is actually a global minimizer then follows from the fact that the previous result, combined with Theorem 6.2, demonstrates that the local minimizers achieve the global lower bounds for the frame potential. $\square$

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