MULTISCALE RIESZ PRODUCTS AND THEIR SUPPORT PROPERTIES

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Abstract

This work defines and investigates the properties of multiscale Riesz product measures. These are product measures constructed on general locally compact abelian groups in a process similar to that of the original example of Riesz. The multiscale element of the construction is the use of a general homomorphism of the group in place of the dilation factor. Furthermore, this construction allows for the use of generating functions that are piecewise constant, reminiscent of a wavelet approach, as well as trigonometric polynomials, which is a more classical Fourier approach. Results obtained include the characterization of the mutual absolute continuity or singularity of such Riesz products, in the spirit of Zygmund's original dichotomy result. The differences regarding the support properties of measures based on each approach are analyzed and examples are constructed.

Keywords and phrases: Riesz products, thin sets. .

1. Introduction

1.1. Background Riesz products are examples of a construction originating in a celebrated paper by Frédéric Riesz. In his 1918 paper, " Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung " [26], Riesz proved the existence of a continuous function F of bounded variation on $\mathbb{T}$, whose Fourier-Stieltjes coefficients do not vanish at infinity. His example was the pointwise limit of the functions

$$F_N(x) = x + \int_0^x \prod_{n=1}^{N} (1 + \cos(2\pi 4^n t))\, dt.$$

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Notationally, $\mathbb{T}$ is the quotient group $\mathbb{R}/\mathbb{Z}$ of the real numbers $\mathbb{R}$ by the integers $\mathbb{Z}$. Generally our notation is standard, e.g., [28]; in particular, $M(\mathbb{T})$ is the Banach algebra of bounded Radon measures on $\mathbb{T}$, see (1).

Riesz' construction proved to be the source of powerful ideas that can be used to produce concrete examples of measures with a number of desired properties (singularity being the goal of the original construction), and the literature on the subject is extensive. It starts with Antoni Zygmund [30] (1932), [31], who generalized Riesz' example by introducing a bounded sequence of coefficients α_n and a sequence of scales λ_n, and showed that the limit function

$$F(x) = x + \int_0^x \prod_{n=1}^{\infty} (1 + \alpha_n \cos(2\pi\lambda_n t))\, dt$$

is absolutely continuous if and only if $\sum |\alpha_n|^2 < \infty$, i.e., $\{\alpha_n\} \in l^2$, and it is singular otherwise. This is the first of the so-called *dichotomy* results. The introduction of the λ_n here gives flexibility to the construction by allowing a more general concept of scale. In measure theoretic terms, Zygmund's theorem means that the sequence of measures μ_N, defined by

$$d\mu_N = \prod_{n=1}^{N} (1 + \alpha_n \cos(2\pi\lambda_n t))\, dt, \tag{1}$$

converges in the weak-$*$ topology of $M(\mathbb{T})$ to a measure μ_α which is absolutely continuous or singular with respect to Lebesgue measure according to whether the sequence $\{\alpha_n\}$ is square-summable or not. Interestingly, these singular measures have full support in $\mathbb{T}$. Later in the 1930s, Riesz' ideas were used by Schaeffer [29] to obtain measures with Fourier transforms that satisfy prescribed decay rates at infinity.

A seemingly different idea has also been used to answer the same type of questions. Hellinger's example of an infinite product uses the piecewise constant function $\mathrm{sgn}(\sin(2\pi t))$ in place of $\cos(2\pi t)$ and a positive sequence $\{s_n\}$ satisfying $0 < \overline{\lim}\ s_n < 1$ in place of $\{\alpha_n\}$, see Riesz–Sz.-Nagy [25] (page 48) and Hewitt–Stromberg [11] (page 278), cf., [2] (page 128) for a more geometrically intuitive treatment of the original example. The Cantor ternary measure, which is singular and therefore whose support has Lebesgue measure zero, is also constructed by using piecewise constant functions rather than cosines, e.g., see (4).

These examples are special cases of what we call *multiscale Riesz product measures*, characterized by similarity of construction. The next subsection gives an overview of our point of view. A brief outline of the paper is given at the end of Section 1.2.4.

1.2. Perspective In general, the different constructions used to obtain Riesz product measures tend to fall into two main categories: a Fourier approach and a wavelet one.

1.2.1. The Fourier approach views the Riesz product as generated by a dissociate set of characters. The work of Hewitt and Zuckerman [12] defining Riesz products on a general nondiscrete compact abelian group demonstrates this. The following is a short description of their approach.

Let G be a nondiscrete compact abelian group with discrete dual group Γ. Let $\Lambda \subset \Gamma$. Denote by $W(\Lambda)$ the set of all characters $\gamma \in \Gamma$ of the form

$$\gamma = \pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n, \tag{2}$$

where the λ_k are distinct elements of Λ.

A set $\Lambda \subset \Gamma$ is called *dissociate* if each element of $W(\Lambda)$ has a unique representation of the form (2) except for the order of the factors. Given a dissociate set Λ and any complex function α on Λ that is bounded by 1, define

$$P(\Phi, \alpha)(x) = \prod \{1 + Re[\alpha(\lambda)\langle x, \lambda\rangle] : \lambda \in \Phi\} \tag{3}$$

for any finite set $\Phi \subset \Lambda$. In other words, each $P(\Phi, \alpha)$ is a trigonometric polynomial on G. Hewitt and Zuckerman showed that there exists a unique, continuous probability measure $\mu_{\alpha,\Lambda}$ on G such that for all $f \in C(G)$, the space of continuous functions on G, we have

$$\lim_{\Phi} \int f P(\Phi, \alpha) dm = \int f d\mu_{\alpha,\Lambda},$$

where m is normalized Haar measure on G.

Kilmer and Saeki [18] established the dichotomy result for $\mu_{\alpha,\Lambda}$ by showing that it is absolutely continuous with respect to Haar measure if and only if $\alpha \in l^2(\Gamma)$, and their work follows the work of Brown and Moran [8].

This is also the point of view taken by Host, Méla, and Parreau [13], when they defined generalized Riesz products using sets of dissociate blocks Λ_n. A sequence $\Lambda = \{\Lambda_n\}$ of finite subsets of Γ is a set of *dissociate blocks* if each $\gamma \in \Gamma$ has at most one representation, up to order of factors, of the form

$$\gamma = \pm\lambda_1 \pm \lambda_2 \ldots \pm \lambda_k,$$

where each λ_k belongs to Λ_k. Then, for a given sequence H_n of trigonometric polynomials on G such that each H_n is composed of frequencies from Λ_n and each $\|H_n\|_\infty \leq 1$, they showed that there exists a measure $\mu \in M(G)$ that is the weak-$*$ limit of $\prod(1 + H_n)$. Independently of [13], Choksi and Nadkarni [9] devised an equivalent formulation for $\mathbb{T}$, and Dooley and Eigen [10] proved the dichotomy result in this case.

Most dichotomy results regarding Riesz products of the type (1) have been greatly influenced by Peyrière's work. In his 1973 paper [24], Peyrière proved the following theorem. Given two complex sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which give rise to the generalized Riesz product measures μ_α and μ_β as defined by (1), the condition $\sum |\alpha_n - \beta_n|^2 = \infty$ implies that μ_α and μ_β are mutually singular, whereas if $\sum |\alpha_n - \beta_n|^2 < \infty$ and $\sup |\beta_n| < \frac{1}{2}$, then μ_α is absolutely continuous with respect to μ_β . Peyrière's work shares a deep connection with Kakutani's 1948 result [15], where he established a similar dichotomy criterion for the equivalence of infinite product measures.

1.2.2. The wavelet based approach to Riesz products is fundamentaly a multi-scale one, and it uses functions that need not be characters. It turns out that a natural setting for this type of product can be found in probability theory, the characters being replaced by random variables of zero expected value. A certain orthogonality or " near orthogonality ", see Ritter [27], is still required, and this translates into independence for the random variables. In this vein, a number of results have been obtained about Riesz products constructed by using Rademacher functions. The examples of Riesz–Sz.-Nagy and Hewitt–Stromberg have already been mentioned, and we also point out the following work of Karanikas [16]. Given any set E of positive measure in a locally compact metrizable (and not necessarily abelian) group, Karanikas constructed a Rademacher system for $L^2(E)$, and obtained singular measures based on this system, cf., Bisbas [4]. In a related approach, given any integer base $r > 1$, Koumandos [19] constructed an r-adic Rademacher system for $L^2([0,1))$ and showed that Riesz product measures based on that system are bounded measures on $[0,1)$. In the same paper, Koumandos established a criterion that determines whether the product measure is going to be absolutely continuous or singular depending on the sequence of coefficients used. His proofs are based on martingale theory, and follow Kakutani's work.

1.2.3. The Fourier and wavelet approaches where first combined in [1] by introducing the concept of a pyramidal Riesz product

$$P_N(t) = \prod_{n=1}^{N} \left(1 + Re[\alpha_n H_{\epsilon_n}(\lambda_n t)]\right), \ \epsilon_n = 0, 1, \tag{4}$$

where

(i) H_0, H_1 are trigonometric polynomials on $\mathbb{T}$ with frequencies within $[-L, L]$,

(ii) $\{\lambda_n\}$ is L-lacunary, i.e., $\frac{\lambda_{n+1}}{\lambda_n} \geq 2L + 1$ for all n, and

(iii) $\{\alpha_n\}_{n\in\mathbb{N}}$ is bounded by a number that ensures that all of the product terms are non-negative.

The term "pyramidal" is used since P_N depends on the choice of ϵ_n at each level n, and since various sequences $\{\epsilon_n\}$ correspond to paths down a binary tree. This is an infinite analogue of wavelet packet algorithms. In this paper we shall not investigate the properties of measures stemming from the full ϵ_n flexibility of their pyramidal structure, but essentially only for a constant sequence $\epsilon_1 = \epsilon_2 = \dots$. However, for an arbitrary sequence $\{\epsilon_n\}$, the authors in [1] proved that there exists a continuous measure $\mu_\alpha \in M(\mathbb{T})$ which is a weak-$*$ limit point of the sequence $\{P_N\}$, and that μ_α is absolutely continuous if and only if $\sum |\alpha_n|^2 < \infty$.

It turns out that more general functions than trigonometric polynomials can give rise to generalized Riesz products, and dichotomy results have been established in exchange for a stronger lacunarity condition. We present a brief description.

Consider a function $H : \mathbb{T} \to \mathbb{R}$, such that

$$|H(x)| \leq 1, \; H(x + \frac{1}{2}) = -H(x),$$

and there exists L for which

$$|H(x) - H(y)| \leq L\,|x - y| \; \text{ for } x, y \in \left(0, \frac{1}{2}\right).$$

Modica and Mortola [22] showed that the sequence of products

$$P_N(x) = \prod_{n=1}^{N} \left(1 + \alpha_n H(2^{h_n} x)\right), \tag{5}$$

where $0 < \alpha_n \leq 1$, converges weak-$*$ to a continuous measure μ whenever $\{h_n\}$ is a strictly increasing sequence of positive integers satisfying

$$\liminf_n n^{-1}(h_{n+1} - h_n) > 1. \tag{6}$$

(6) is a rather strong lacunarity condition. R. T. Bott [5] extended their results by establishing dichotomy criteria for determining whether the products in (5) yield absolutely continuous or singular measures. His results are based on Brown's work [7] and again make use of the square summability criterion for the sequence of coefficients α_n, cf., Meyer [21], Katznelson [17] for related work.

1.2.4. The common thread among all the approaches described in the previous subsections is what we call a multiscale Riesz product. The notion is formulated in the following definition.

Definition 1.1. Let G be a locally compact abelian group with Haar measure m and dual group Γ. $M(G)$ is the Banach algebra of bounded Radon (complex regular Borel) measures on G. Given

(i) a homomorphism $T : G \to G$,

(ii) a real-valued function H on G, and

(iii) a real-valued sequence of coefficients $\alpha_n, \ n \in \mathbb{N}$.

A *multiscale Riesz product measure* is the weak-$*$ limit in $M(G)$ of the sequence $\{\mu_N\} \subset M(G)$, where

$$d\mu_N = \prod_{n=1}^{N} \left(1 + \alpha_n H(T^{n-1}x)\right) dm,$$

when such a limit exists. We shall call T the *scaling function* and H the *generating function* of the multiscale Riesz product.

Section 2 is devoted to generalizing the results obtained in [1], using trigonometric polynomials as generating functions. In Section 3 we focus on multiscale Riesz products using piecewise constant generating functions , which allows us to obtain measures of thin support, as explained in Section 4.

1.3. Notation We have already established some of our notation. In this subsection we repeat some of it in order to define our more specialized notions in terms of a more familiar setting. For a standard reference in harmonic analysis on locally compact abelian groups see Rudin [28].

Let G be a nondiscrete locally compact abelian group, and let Γ be its dual group. We shall use additive notation for both groups. We denote the action of the character γ on the element $x \in G$ by $\langle x, \gamma \rangle$.

$M(G)$ is the Banach algebra under convolution of bounded Radon (complex regular Borel) measures on G. The *Fourier transform* of $\mu \in M(G)$ is defined by

$$\forall \, \gamma \in \Gamma, \qquad \widehat{\mu}(\gamma) = \int_G \langle x, \gamma \rangle d\mu(x).$$

$C(G)$ is the space of all bounded continuous complex-valued functions on G. A trigonometric polynomial is a function of the form

$$H(x) = \sum_{m=1}^{M} h[m] \langle x, \gamma_m \rangle,$$

where the coefficients $h[m]$ are complex numbers and $\gamma_m \in \Gamma$. The set spec $H = \{\gamma_m \in \Gamma : m = 1, \ldots, M\}$ is called the *spectrum* of H. H has *zero mean* if $0 \notin$ spec H. Notice that trigonometric polynomials are elements of $C(G)$.

If a trigonometric polynomial H is real-valued, we can write H as a sum

$$H(x) = \sum_{m=1}^{M} h[m] \frac{1}{2} \left(\langle x, \gamma_m \rangle + \langle x, -\gamma_m \rangle \right),$$

where the coefficients $h[m]$ are now real numbers. Thus, the spectrum of a real-valued trigonometric polynomial H is symmetric, that is, spec $H = -$spec H.

For a given homomorphism $T : G \to G$, T^* denotes the dual homomorhism $T^* : \Gamma \to \Gamma$ defined by

$$\forall\, x \in G, \quad \forall\, \gamma \in \Gamma, \qquad \langle T(x), \gamma\rangle = \langle x, T^*(\gamma)\rangle.$$

EXAMPLE 1.2. The following are examples of homomorphisms T and their duals T^*. If I_G denotes the identity mapping on G and $n \in \mathbb{Z}$, then $T = nI_G$ is a homomorphism with dual homomorphism $T^* = nI_\Gamma$. If $G = \mathbb{R}^d$, and T is multiplication by any matrix $M \in GL(d, \mathbb{R})$, then T^* is multiplication by the adjoint matrix M^*.

For a fixed integer $R \in \mathbb{N}$ with $R > 1$, we let $B = \{0, 1, \ldots, R-1\}^d$ and $r = |B| = R^d$.

In order to represent points in $[0,1)^d$ we use the following conventions. We let $W_n = \{b_1 \ldots b_n \ : \ b_k \in B \text{ for } k = 1, \ldots, n\}$ where each b_k is an ordered d-tuple $(b_k(1), b_k(2), \ldots, b_k(d))$ with $0 \le b_k(d) \le R-1$
.

For $w \in W_n$ we let

$$\begin{aligned}
(.w)_R &= \frac{b_1}{R} + \ldots + \frac{b_n}{R^n} \\
&= \left(\frac{b_1(1)}{R} + \ldots + \frac{b_n(1)}{R^n}, \ldots, \frac{b_1(d)}{R} + \ldots + \frac{b_n(d)}{R^n}\right) \\
&= ((.w(1))_R, \ldots, (.w(d))_R)
\end{aligned}$$

so that each $(.w)_R$ corresponds to a rational point in $[0,1)^d$.

If $w \in W_N$, we let $\mathbf{1}_w$ denote the characteristic function of the set

$$I(w) = [(.w(1))_R, (.w(1))_R + \frac{1}{R^N}) \times \ldots \times [(.w(d))_R, (.w(d))_R + \frac{1}{R^N}) + \mathbb{Z}^d$$

in $\mathbb{R}^d$. Notice that there are r^N different words of length N (the elements of W_N) and to each such word w there corresponds a $\mathbf{1}_w$ which is just the characteristic function of one of the r^N cubes we obtain from dividing $[0,1)^d$ into equal cubes, periodized over $\mathbb{R}^d$. For example, when $d = 1$ and $R = 3$, W_N consists of all the words with N letters that can be formed using the digits $\{0, 1, 2\}$ as letters. Thus, in the case $N = 1$, $\mathbf{1}_0$ is the characteristic function of $I(0) = [0, \frac{1}{3}) + \mathbb{Z}$; in the case $N = 2$, $\mathbf{1}_{10}$ the characteristic function of $I(10) = [\frac{1}{3}, \frac{1}{3} + \frac{1}{9}) + \mathbb{Z}$; and in the case $N = 3$, $\mathbf{1}_{222}$ is the characteristic function of $I(222) = [1 - \frac{1}{27}, 1) + \mathbb{Z}$.

Definition 1.3. A function $H : \mathbb{T}^d \to \mathbb{R}$ is of *type* R if there exists an integer $R > 1$ such that $H = \sum_{b \in B} c_b \mathbf{1}_b$, with $c_b \in \mathbb{C}$ and $b = b_1 \ldots b_n = w \in W_n$, $n \in \mathbb{N}$. In particular, $\mathbf{1}_b$ is the characteristic function of $I(w)$.

Clearly, if a function is of type R, it is also of type Rm for all $m \in \mathbb{N}$ with $m \geq 1$.

2. Trigonometric polynomials

2.1. Definitions and notation In this section we shall define multiscale Riesz product measures on a nondiscrete locally compact abelian group G, using trigonometric polynomials as generating functions. We begin by introducing an appropriate notion of convergence.

Definition 2.1. A sequence $\{\mu_N\} \subset M(G)$ converges to a measure μ *weakly* if

$$\forall f \in C(G) \quad \lim_N \int_G f d\mu_N = \int_G f d\mu.$$

Notice that if G is compact, this is the same as the usual analytic notion of weak-$*$ convergence because $M(G)$ is the dual space of $C(G)$. On the other hand, this is not the usual functional analytic notion of weak convergence.

The following definition introduces the concept of a dissociate pair, which plays the role of lacunarity in this general setting. It is very close to Hewitt and Zuckerman's definition of a dissociate set of characters.

Definition 2.2. Let H be a real-valued trigonometric polynomial, of zero mean value, and let T be a homomorphism of the group G. We define

$$\begin{aligned} S(H,T) &= \{0\} \cup \bigcup_{N=1}^{\infty} \sum_{n=1}^{N} (T^*)^{n-1} \text{spec } H \\ &= \{0\} \cup \text{spec } H \cup (\text{spec } H + T^*(\text{spec } H)) \cup \dots . \end{aligned}$$

(H,T) is called a *dissociate pair* if elements of $S(H,T)$ have unique expansions as sums of the form

$$\gamma = \sum_{n=1}^{N} (T^*)^{n-1}(\gamma_{m_n}), \quad \gamma_{m_n} \in \{0\} \cup \text{spec } H$$

up to rearrangement of factors.

The motivation behind this definition is the following example of a dissociate pair as given in [1].

Let $G = \mathbb{T}$. Pick $\gamma \in \Gamma = \mathbb{Z}$, and let $H(x) = \frac{1}{2}(\langle x, \gamma \rangle + \langle x, -\gamma \rangle) = \cos(2\pi\gamma x)$ for all $x \in \mathbb{T}$. Let T be multiplication by any positive integer greater than or equal to $2|\gamma| + 1$. It is easy to see that the pair (H,T) is dissociate according to our definition. The case $\gamma = 1$ and $T(x) = 4x$ was the original example of Riesz, and $S(H,T)$ is a dissociate set according to the definition of Hewitt and Zuckerman [12].

Having defined dissociateness, we are ready to introduce our notion of a multiscale Riesz product. We begin with the following definition.

Definition 2.3. Let (H,T) be a dissociate pair and let $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ be a real-valued sequence. We define the *Riesz polynomials associated with the triple* (H,T,α) to be

$$\forall x \in G, \quad P_N(H,T,\alpha)(x) = \prod_{n=1}^{N} \left(1 + \alpha_n H(T^{n-1}x)\right).$$

2.2. Results We derive some properties of Riesz polynomials.

LEMMA 2.4. *Let (H,T) be a dissociate pair. Then*

$$\textit{spec } P_N = \{0\} \cup \sum_{n=1}^{N} (T^*)^{n-1} \textit{spec } H.$$

PROOF.

$$H(T^{n-1}x) = \sum_{m=1}^{M} h[m]\langle T^{n-1}x, \gamma_m\rangle = \sum_{m=1}^{M} h[m]\langle x, (T^*)^{n-1}\gamma_m\rangle,$$

and so

$$\text{spec } H \circ T^{n-1} = (T^*)^{n-1}\text{spec } H.$$

Thus,

$$\text{spec}(1 + \alpha_n H) \circ T^{n-1} = \{0\} \cup (T^*)^{n-1}\text{spec } H.$$

Now, for all trigonometric polynomials f and g we have spec $f \cdot g \subseteq$ spec $f +$ spec g, but dissociateness ensures that spec $H_n \cdot H_m =$ spec $H_n +$ spec H_m. Since $P_N = \prod_{n=1}^{N}(1 + \alpha_n H) \circ T^{n-1}$, we have spec $P_N = \{0\} \cup \sum_{n=1}^{N}(T^*)^{n-1}$spec H. □

LEMMA 2.5. *Let (H,T) be a dissociate pair. Then*

$$\forall\ N \in \mathbb{N} \textit{ and } \forall\ x \in G, \quad P_N(x) = \sum_{\gamma \in \textit{spec } P_N} p_N(\gamma)\langle x, \gamma\rangle,$$

where, given $\gamma = \sum_{n=1}^{N}(T^*)^{n-1}(\gamma_{m_n}) \in$ spec P_N,

$$p_N(\gamma) = \prod_{n=1}^{N} h[m_n]\alpha_n.$$

PROOF. The claim is true for $N = 1$. In fact,

$$P_1(x) = 1 + \alpha_1 H(x) = 1 + \alpha_1 \sum_{m=1}^{M} h[m]\langle x, \gamma_m\rangle = \sum_{\gamma \in \{0\} \cup \text{spec } H} p_1(\gamma)\langle x, \gamma\rangle,$$

where $p_1(0) = 1$ and $p_1(\gamma_m) = \alpha_1 h[m]$; and since spec $P_1 =$ spec $H \cup \{0\}$ by Lemma 2.4, the claim is obtained for $N = 1$.

Assuming that the claim is true for $N-1$, we need to show that it holds for N. We have

$$\begin{aligned}
P_N(x) &= \prod_{n=1}^{N} \big(1 + \alpha_n H(T^{n-1}x)\big) \\
&= \big(1 + \alpha_N H(T^{N-1}x)\big) \prod_{n=1}^{N-1} \big(1 + \alpha_n H(T^{n-1}x)\big) \\
&= \big(1 + \alpha_N H(T^{N-1}x)\big)\, P_{N-1}(x) \\
&= \big(1 + \alpha_N H(T^{N-1}x)\big) \sum_{\gamma \in \text{spec } P_{N-1}} p_{N-1}(\gamma)\langle x, \gamma\rangle \\
&= \sum_{\gamma \in \text{spec } P_{N-1}} p_{N-1}(\gamma)\langle x, \gamma\rangle \\
&\quad + \alpha_N H(T^{N-1}x) \sum_{\gamma \in \text{spec } P_{N-1}} p_{N-1}(\gamma)\langle x, \gamma\rangle.
\end{aligned}$$

Now,

$$\begin{aligned}
&\alpha_N \sum_{m=1}^{M} h[m]\langle T^{N-1}x, \gamma_m\rangle \sum_{\gamma \in \text{spec } P_{N-1}} p_{N-1}(\gamma)\langle x, \gamma\rangle \\
&= \alpha_N \sum_{m=1}^{M} h[m]\langle x, (T^*)^{N-1}\gamma_m\rangle \sum_{\gamma \in \text{spec } P_{N-1}} p_{N-1}(\gamma)\langle x, \gamma\rangle \\
&= \sum_{m=1}^{M} \sum_{\gamma \in \text{spec } P_{N-1}} \alpha_N h[m] p_{N-1}(\gamma)\langle x, (T^*)^{N-1}\gamma_m\rangle\langle x, \gamma\rangle \\
&= \sum_{m=1}^{M} \sum_{\gamma \in \text{spec } P_{N-1}} \alpha_N h[m] p_{N-1}(\gamma)\langle x, (T^*)^{N-1}(\gamma_m) + \gamma\rangle \\
&= \sum_{m=1}^{M} \sum_{\gamma \in \text{spec } P_{N-1}} p_N(\gamma)\langle x, (T^*)^{N-1}(\gamma_m) + \gamma\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_N(x) &= \sum_{\gamma \in \text{spec } P_{N-1}} p_N(\gamma)\langle x, \gamma\rangle \\
&\quad + \sum_{m=1}^{M} \sum_{\gamma \in \text{spec } P_{N-1}} p_N(\gamma)\langle x, (T^*)^{N-1}(\gamma_m) + \gamma\rangle \\
&= \sum_{\gamma \in \text{spec } P_N} p_N(\gamma)\langle x, \gamma\rangle + 0,
\end{aligned}$$

since (H,T) is a dissociate pair; and so

$$\text{spec } P_{N-1} \cap \big((T^*)^{N-1}(\gamma_m) + \text{spec } P_{N-1}\big) = \emptyset$$

for all $\gamma_m \in$ spec H. Thus, the two different sums are over two disjoint sets of indices whose union is spec P_N. $\square$

LEMMA 2.6. *Let (H,T) be a dissociate pair. There exists a unique $p : \Gamma \to \mathbb{C}$ such that if $\gamma \notin S(H,T)$, then*

$$p(\gamma) = 0;$$

and

$$\forall\ N \in \mathbb{N} \text{ and } \forall\ x \in G, \quad P_N(x) = \sum_{\gamma \in spec\ P_N} p(\gamma)\langle x, \gamma\rangle.$$

Moreover, given $\gamma = \sum_{n=1}^{N} (T^*)^{n-1}(\gamma_{m_n}) \in S(H,T)$, we have

$$p(\gamma) = \prod_{n=1}^{N} h[m_n]\alpha_n.$$

PROOF. For $\gamma \in S(H,T)$ let $p(\gamma) = \lim_N p_N(\gamma)$, which exists by Lemma 2.5. $\square$

The idea is that the Riesz product associated with the Riesz polynomials should be the "inverse Fourier transform" of this function p. When G is compact this is indeed the case: p is the Fourier transform of a probability measure $\mu \in M(G)$, and the Riesz polynomials P_N converge to μ weakly. The key to this argument is that Haar measure on a compact group belongs to $M(G)$. Since Haar measure on a locally compact abelian group is not always a bounded measure, we compensate by making use of an auxiliary bounded measure σ, e.g., see Kilmer and Saeki [18].

Definition 2.7. Let (H,T) be a dissociate pair, $\{\alpha_n : n \in \mathbb{N}\}$ a real-valued sequence bounded by $\frac{1}{\max|H|}$, and $\sigma \in M(G)$ a probability measure. We define the sequence $\mu_N = \mu_N(H,T,\alpha,\sigma)$ of *partial Riesz product measures associated with* (H,T,α,σ) by

$$\forall\ f \in C(G), \quad \int_G f d\mu_N = \int_G f P_N(H,T,\alpha) d\sigma.$$

We use a rather important and essentially well-known result, stated as Proposition A in Kilmer and Saeki [18], also see M. Loève [20] (Chapter IV).

PROPOSITION 2.8. *Let $\{\mu_N\}$ be a norm-bounded sequence in $M(G)$, and let ϕ be a bounded Haar measurable function on Γ such that*

$$\widehat{\mu}_N \to \phi \text{ weak-}* \text{ in } L^\infty(\Gamma).$$

Then there exists a measure $\mu \in M(G)$ such that

1. $\widehat{\mu} = \phi$ *locally Haar a.e. on* Γ, *and*

2. $\{\mu_N\}$ *converges to* μ *in the weak-$*$ topology of* $M(G)$.

If, in addition,

$$\lim_N \|\mu_N\| = \|\mu\|,$$

then $\lim_N \mu_N = \mu$ *weakly.*

We require that the sequence $\alpha = \{\alpha_n\}$ be bounded to ensure positivity of our measures, as the following lemma shows.

LEMMA 2.9. *Let* (H, T) *be a dissociate pair. If* $\|\alpha\|_\infty \le \frac{1}{\max|H|}$, *then all* P_N *are positive, and* $\|p\|_\infty \le 1$.

PROOF. By hypothesis,

$$\|\alpha\|_\infty \max|H| \le 1.$$

Hence,

$$\forall\ n \in \mathbb{N} \text{ and } \forall\ x \in G, \quad |\alpha_n| \cdot \left|H(T^{n-1}x)\right| \le 1,$$

and therefore

$$\forall\ n \in \mathbb{N} \text{ and } \forall\ x \in G, \quad \left|\alpha_n H(T^{n-1}x)\right| + \alpha_n H(T^{n-1}x) \le 1 + \alpha_n H(T^{n-1}x).$$

Since $|a| + a \ge 0$ for all $a \in \mathbb{R}$, we must have

$$\forall\ n \in \mathbb{N} \text{ and } \forall\ x \in G, \quad 1 + \alpha_n H(T^{n-1}x) \ge 0.$$

This means that every term in the product $\prod_{n=1}^N \left(1 + \alpha_n H(T^{n-1}x)\right)$ is greater than or equal to 0, and so the product itself is also positive.

Since $\max|H| \ge \max\{|h[m]| : m = 1, \ldots, M\}$, the hypothesis

$$\|\alpha\|_\infty \le \frac{1}{\max|H|}$$

implies

$$\|\alpha\|_\infty \le \frac{1}{\max\{|h[m]| : m = 1, \ldots, M\}}.$$

Now, if $p(\gamma) \ne 0, \gamma$ must be of the form $\sum_{n=1}^N (T^*)^{n-1}\gamma_{m_n}$, and

$$\begin{aligned}
|p(\gamma)| = \left|\prod_{n=1}^N h[m_n]\alpha_n\right| &= \prod_{n=1}^N |h[m_n]\alpha_n| \\
&\le \prod_{n=1}^N \max\{|h[m]| : m = 1, \ldots, M\}\,|\alpha_n| \\
&\le \prod_{n=1}^N \max\{|h[m]| : m = 1, \ldots, M\}\,\|\alpha\|_\infty \\
&\le \prod_{n=1}^N 1 = 1.
\end{aligned}$$

$\square$

LEMMA 2.10. *Let* (H,T) *be a dissociate pair and assume* $\|\alpha\|_\infty \le \frac{1}{\max|H|}$. *Suppose that there exists* C *such that for all* $\lambda \in \Gamma$ *we have*

$$\sum_{\gamma \in S(H,T)} |\widehat{\sigma}(\lambda - \gamma)| \le C.$$

Then

$$\|\mu_N\| \le C \textit{ for all } N \in \mathbb{N}.$$

PROOF. For all $N \in \mathbb{N}$ we have

$$\|\mu_N\| = \|P_N \sigma\| = \int_G |P_N|\, d\,|\sigma| = \int_G P_N d\sigma,$$

since $P_N \ge 0$. Therefore,

$$\begin{aligned}
\|\mu_N\| = \int_G P_N d\sigma = \sum_{\gamma \in \text{spec } P_N} p_N(\gamma)\widehat{\sigma}(0-\gamma) &= \sum_{\gamma \in \text{spec } P_N} p(\gamma)\widehat{\sigma}(0-\gamma) \\
&\le \sum_{\gamma \in \text{spec } P_N} |p(\gamma)|\,|\widehat{\sigma}(0-\gamma)| \\
&\le \sum_{\gamma \in \text{spec } P_N} |\widehat{\sigma}(0-\gamma)| \\
&\le \sum_{\gamma \in S(H,T)} |\widehat{\sigma}(0-\gamma)| \le C.
\end{aligned}$$

$\square$

LEMMA 2.11. *Let* (H,T) *be a dissociate pair and assume* $\|\alpha\|_\infty \le \frac{1}{\max|H|}$. *Suppose that there exists* C *such that for all* $\lambda \in \Gamma$ *we have*

$$\sum_{\gamma \in S(H,T)} |\widehat{\sigma}(\lambda - \gamma)| \le C.$$

Then there exists a unique bounded Haar measurable function ϕ *on* Γ *such that*

$$\widehat{\mu}_N \to \phi \textit{ pointwise on } \Gamma.$$

PROOF. Let $\lambda \in \Gamma$ be given. $\|\alpha\|_\infty \le \frac{1}{\max|H|}$ implies $\|p\|_\infty \le 1$, so we have

$$\left|\sum_{\gamma \in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda-\gamma)\right| \le \sum_{\gamma \in S(H,T)} |p(\gamma)\widehat{\sigma}(\lambda-\gamma)| \le \sum_{\gamma \in S(H,T)} |\widehat{\sigma}(\lambda-\gamma)| \le C.$$

Thus, the series $\sum_{\gamma\in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda-\gamma)$ converges absolutely. Let

$$\phi(\lambda) = \sum_{\gamma\in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda-\gamma),$$

so that $|\phi(\lambda)| \le C$. Also,

$$\widehat{\mu}_N(\lambda) = \sum_{\gamma\in \text{spec } P_N} p(\gamma)\widehat{\sigma}(\lambda-\gamma) \to \sum_{\gamma\in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda-\gamma),$$

since

$$\sum_{\gamma\in S(H,T)} |\widehat{\sigma}(\lambda-\gamma)| \le C.$$

Finally, $\phi : \Gamma \to \mathbb{C}$ is Haar measurable since each $\widehat{\mu}_N$ is measurable. $\square$

PROPOSITION 2.12. *Let (H,T) be a dissociate pair and assume $\|\alpha\|_\infty \le \frac{1}{\max|H|}$. Suppose that there exists C such that for all $\lambda \in \Gamma$ we have*

$$\sum_{\gamma\in S(H,T)} |\widehat{\sigma}(\lambda-\gamma)| \le C.$$

Then there exists a unique measure $\mu_\sigma \in M(G)$ such that

1. *$\widehat{\mu}_\sigma(\lambda) = \sum_{\gamma\in\Gamma} p(\gamma)\widehat{\sigma}(\lambda-\gamma)$ for almost every (Haar) $\lambda \in K, K$ a compact subset of Γ, and*

2. *$\{\mu_N\}$ converges to μ_σ in the weak-$*$ topology of $M(G)$.*

PROOF. The result follows from Proposition 2.8 and the Lemmata 2.9 – 2.11. $\square$

It follows from Proposition 2.8 that if the function ϕ that appears in the proof of Lemma 2.11 is continuous, then $\lim \mu_N = \mu$ weakly in Proposition 2.12. In fact, if ϕ is continuous then $\phi(\lambda) = \widehat{\mu}(\lambda)$ for all $\lambda \in \Gamma$. Further, μ is a positive measure since all the μ_N are positive measures. By the Lebesgue Dominated Convergence Theorem we see that $\|\mu\| = \widehat{\mu}(0) = \lim \widehat{\mu}(0) = \lim \|\mu_N\|$, and Proposition 2.8 applies.

In the case that G is compact, Γ must be discrete, so ϕ is continuous, and we can state the following theorem.

THEOREM 2.13. *Let G be a compact abelian group. Let (H,T) be a dissociate pair and assume $\|\alpha\|_\infty \le \frac{1}{\max|H|}$. There exists a unique probability measure $\mu \in M(G)$ such that $\lim \mu_N = \mu$ weakly.*

PROOF. Take σ to be Haar measure on G. $\square$

Clearly, μ is *the* multiscale Riesz product measure associated with the triple (H, T, α). Note that $\widehat{\mu} = p$ on $S(H, T)$.

Similarly, if we can assume that $S(H, T)$ is discrete, we have the following theorem.

THEOREM 2.14. *Let $G = \mathbb{R}^d$, and let (H, T) be a dissociate pair. Assume that $S(H, T)$ is discrete, and that $\|\alpha\|_\infty \leq \frac{1}{\max|H|}$. There exists a unique measure μ on G such that $\lim P_N = \mu$ weakly in $M(K_G)$ for any compact set $K_G \subset G$.*

PROOF. Let K_G be a compact subset of G. We need to show that there exists a unique measure μ_{K_G} such that

$$\lim_N P_N \mathbf{1}_{K_G} = \mu_{K_G} \text{ weakly in } M(K_G).$$

Since the Fourier transform of $\mathbf{1}_{K_G}$ does not necessarily have the right properties, we shall smooth it. Let W be a bounded, open set containing K_G. There exists $f \in L^1(\Gamma)$ such that $\widehat{f} = 1$ on K_G and $\widehat{f} = 0$ outside W, see, for example, Rudin [28] (Theorem 2.6.2). Let $d\sigma = f dm$ and consider $\mu_N = \mu_N(H, T, \alpha, \sigma)$. It is enough to show that there exists a unique measure μ_f such that

$$\lim_N \mu_N = \mu_f \text{ weakly in } M(K_G).$$

In view of Proposition 2.12 we first need to show that there exists C such that for all $\lambda \in \Gamma$ we have

$$\sum_{\gamma \in S(H,T)} |\widehat{\sigma}(\lambda - \gamma)| \leq C.$$

This holds since the support of $\widehat{\sigma}$ is bounded ($\widehat{\sigma} = \widehat{f}$ and $\operatorname{supp}\widehat{f}$ is a subset of the closure of W) and $S(H, T)$ is discrete. Next, we shall show that $\phi(\lambda)$ is continuous on Γ by using the Weierstrass M-test on its series expansion

$$\sum_{\gamma \in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda - \gamma).$$

Indeed, for any $\lambda \in \Gamma$ we have

$$\left| \sum_{\gamma \in S(H,T)} p(\gamma)\widehat{\sigma}(\lambda - \gamma) \right| \leq \|p\|_\infty \sum_{\gamma \in S(H,T)} |\widehat{\sigma}(\lambda - \gamma)| \leq C,$$

so that the M-test applies. □

As a final remark, we note that our construction only uses functions with finite spectra. This poses a question with a number theoretic flair, namely to identify which subsets of Γ, other than finite ones, can be spectra for generating functions H in a multiscale Riesz product. This in turn relies (in $\mathbb{R}$) on identifying which sets $S \subset \mathbb{Z}$ such that $0 \notin S$ and $S = -S$ can be associated with dissociate pairs. For example, take H and T for which $\operatorname{spec} H \subset \mathbb{Z}$ and $T(x) = \pi x$.

3. Generating functions of type R

In this section we consider multiscale Riesz product measures for which the generating function H is of type R on the d-dimensional torus $\mathbb{T}^d$. Functions of type R are piecewise constant, and so do not belong to $C(\mathbb{T}^d)$, as opposed to trigonometric polynomials. They are, however, related to the theory of periodic wavelet transforms, see [3], and hence to characters of another group.

We begin by fixing an integer R greater than 1. As a scaling function we shall use the homomorphism $T : \mathbb{T}^d \to \mathbb{T}^d, x \mapsto Rx$, i.e., multiplication by R. For the rest of this section, the generating function H will be a function of type R, see Definition 1.3.

Definition 3.1. Let H be a function of type R and let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a non-negative, bounded sequence. We define the *Riesz polynomials* associated with the triple (H, R, α) by

$$P_N(H, R, \alpha)(t) = \prod_{n=1}^{N} \left(1 + \alpha_n H(R^{n-1}t)\right).$$

We shall write P_N instead of $P_N(H, R, \alpha)$ whenever the choice of H and α can be clearly inferred from the context. We shall also write $H_n(x) = H(R^{n-1}x)$.

Definition 3.2. A triple (H, R, α) is called *admissible* if H is a function of type R with $\int_{\mathbb{T}^d} H(x)dx = 0$ and the sequence α satisfies $0 \le \alpha_n \le \frac{1}{|\min H|}$ for all $n \in \mathbb{N}$.

Our main results are Theorems 3.7, 3.9 and 3.10. We shall use Propositions 3.3 and 3.6 as the main tools for the proofs of the theorems.

PROPOSITION 3.3. *Let (H, R, α) be an admissible triple and take $M \in \mathbb{N}$. For all $N > M$ and for every finite sequence $\{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}$ for which $M \le n_1 < n_2 < \cdots < n_k \le N$, we have*

$$\forall\ w \in W_N, \quad \int_{\mathbb{T}^d} \mathbf{1}_w \cdot H_{n_1} H_{n_2} \ldots H_{n_k} = 0 \tag{7}$$

and

$$\int_{\mathbb{T}^d} H_M \cdot H_{n_1} H_{n_2} \dots H_{n_k} = 0. \tag{8}$$

PROOF. To prove (7), we proceed by induction on N.

Let $N = M+1$. For all $w \in W_M$ we have

$$\int_{\mathbb{T}^d} \mathbf{1}_w \cdot H_{n_1} H_{n_2} \dots H_{n_k} = \int_{\mathbb{T}^d} \mathbf{1}_w H_{M+1} = \int_{I(w)} H(R^M x) dx = \int_{\mathbb{T}^d} H(x) dx.$$

Assuming (7) holds for $N-1$, we compute

$$\begin{aligned}
\int_{\mathbb{T}^d} \mathbf{1}_w \cdot H_{n_1} H_{n_2} \dots H_{n_k} &= \int_{I(w)} H_{n_1} H_{n_2} \dots H_{n_k} \\
&= \sum_{b \in B} \int_{I(wb)} H_{n_1} H_{n_2} \dots H_{n_k} \\
&= \dots \\
&= \sum_{v \in W_{n_1+1-M}} \int_{I(wv)} H(R^{n_1} x) H(R^{n_2} x) \dots H(R^{n_k} x) dx \\
&= \sum_{v \in W_{n_1+1-M}} \int_{I_{wv}} H((.wv)_R) H(R^{n_2} x) \dots H(R^{n_k} x) dx,
\end{aligned}$$

since H_{n_1} is constant on each $I(wv)$. Also, $\int_{I(wv)} H(R^{n_2} x) \dots H(R^{n_k} x) dx = 0$ by the induction hypothesis, and so

$$\begin{aligned}
&\int_{\mathbb{T}^d} \mathbf{1}_w \cdot H_{n_1} H_{n_2} \dots H_{n_k} \\
&= \sum_{v \in W_{n_1+1-M}} H((.wv)_R) \int_{I(wv)} H(R^{n_2} x) \dots H(R^{n_k} x) dx \\
&= \sum_{v \in W_{n_1+1-M}} 0 = 0.
\end{aligned}$$

Hence, (7) is also valid for N. Consequently, (7) is valid for all $N \in \mathbb{N}$.

A similar argument is employed to prove (8). □

Proposition 3.3 is used to prove the following lemmata.

LEMMA 3.4. *Let (H, R, α) be an admissible triple and take $M \in \mathbb{N}$.*

(a) For all $N > M$ and for all $w \in W_M$,

$$\int_{\mathbb{T}^d} \mathbf{1}_w(x) \prod_{n=M+1}^{N} (1 + \alpha_n H_n(x))\, dx = \frac{1}{r^M}. \tag{9}$$

(b) For all $N > M$,

$$\int_{\mathbb{T}^d} H_M(x) \prod_{n=1, n\neq M}^{N} (1 + \alpha_n H_n(x))\, dx = 0. \tag{10}$$

(c) For all $N > M$,

$$\int_{\mathbb{T}^d} H_M^2(x) \prod_{n=1, n\neq M}^{N} (1 + \alpha_n H_n(x))\, dx = \int_{\mathbb{T}^d} H_M^2(x) dx. \tag{11}$$

PROOF.

$$\begin{aligned}
&\int_{\mathbb{T}^d} \mathbf{1}_w(x) \prod_{n=M+1}^{N} (1 + \alpha_n H_n(x))\, dx \\
&= \int_{\mathbb{T}^d} \mathbf{1}_w(x) 1 dx + \int_{\mathbb{T}^d} \mathbf{1}_w(x) \sum_{n=M+1}^{N} \alpha_n H_n(x) dx \\
&\qquad + \int_{\mathbb{T}^d} \mathbf{1}_w(x) \sum_{M+1 \leq n_1 < n_2 \leq N} \alpha_{n_1} H_{n_1}(x) \alpha_{n_2} H_{n_2}(x) dx \\
&\qquad + \ldots + \int_{\mathbb{T}^d} \mathbf{1}_w(x) \prod_{n=M+1}^{N} (\alpha_n H_n(x))\, dx \\
&= \int_{\mathbb{T}^d} \mathbf{1}_w(x) dx + \sum_{n=M+1}^{N} \int_{\mathbb{T}^d} \mathbf{1}_w(x) \alpha_n H_n(x) dx \\
&\qquad + \sum_{M+1 \leq n_1 < n_2 \leq N} \int_{\mathbb{T}^d} \mathbf{1}_w(x) \alpha_{n_1} H_{n_1}(x) \alpha_{n_2} H_{n_2}(x) dx \\
&\qquad + \ldots + \int_{\mathbb{T}^d} \mathbf{1}_w(x) \prod_{n=M+1}^{N} (\alpha_n H_n(x))\, dx \\
&= \int_{\mathbb{T}^d} \mathbf{1}_w + 0 + \ldots + 0 = \frac{1}{r^M},
\end{aligned}$$

where the penultimate equality is justified by Proposition 3.3.

A similar argument proves the other two identities. □

LEMMA 3.5. *Let (H, R, α) be an admissible triple and take $M \in \mathbb{N}$. Then*

$$\forall\ w \in W_M, \quad \lim_N \int_{I(w)} P_N = \prod_{n=1}^{M} \frac{1}{r} \left(1 + \alpha_n H(R^{n-1}(.w)_R)\right)$$

PROOF. Fix $M \in \mathbb{N}$ and let $N > M, w \in W_M$. By construction of H, the product $\prod_{n=1}^{M} \left(1+\alpha_n H_n\big((.w)_R\big)\right)$ is constant on each $I(w)$. Thus we have

$$\begin{aligned}
\int_{I(w)} P_N &= \int_{I(w)} \prod_{n=1}^{N} (1+\alpha_n H_n(x))\, dx \\
&= \prod_{n=1}^{M} \left(1+\alpha_n H_n\big((.w)_R\big)\right) \int_{I(w)} \prod_{n=m+1}^{n} (1+\alpha_n H_n(x))\, dx \\
&= \frac{1}{r^M} \cdot \prod_{n=1}^{M} \left(1+\alpha_n H_n\big((.w)_R\big)\right) \\
&= \prod_{n=1}^{M} \frac{1}{r} \left(1+\alpha_n H_n\big((.w)_R\big)\right);
\end{aligned}$$

and so

$$\lim_N \int_{I(w)} P_N = \prod_{n=1}^{M} \frac{1}{r} \left(1+\alpha_n H_n\big((.w)_R\big)\right).$$

□

PROPOSITION 3.6. *Let $R \in \mathbb{N}$ with $R > 1$, and assume the following:*

(a) $\psi_1 : \mathbb{T}^d \to \mathbb{R}$ is periodic with respect to the lattice $\frac{1}{R^M}\mathbb{Z}^d$ for some $M \in \mathbb{N}$, and

(b) $\psi_2 : \mathbb{T}^d \to \mathbb{R}$ has the property that there exists $L \geq 0$ such that

$$\forall\ k \in \mathbb{Z}^d \text{ and } \forall\ x,\ y \in I(w),\ w \in W_M,\ |\psi_2(x) - \psi_2(y)| \leq L\,|x-y|\,.$$

Then

$$\left| \int_{\mathbb{T}^d} \psi_1\psi_2 - (\int_{\mathbb{T}^d} \psi_1)(\int_{\mathbb{T}^d} \psi_2) \right| \leq \frac{2L\sqrt{d}}{R^M} \int_{\mathbb{T}^d} |\psi_1|\,. \tag{12}$$

PROOF.

$$\begin{aligned}
&\int_{\mathbb{T}^d} \psi_1\psi_2 - (\int_{\mathbb{T}^d} \psi_1)(\int_{\mathbb{T}^d} \psi_2) \\
&= \sum_{w\in W_M} \left[\int_{I(w)} \psi_1\psi_2 - \int_{\mathbb{T}^d} \psi_1 \int_{I(w)} \psi_2 \right] \\
&= \sum_{w\in W_M} \int_{I(w)} \left[\psi_1(x)\psi_2(x) - (\int_{\mathbb{T}^d} \psi_1(y)dy)\psi_2(x) \right] dx \\
&= \sum_{w\in W_M} \int_{I(w)} \psi_2(x) \left[\psi_1(x) - (\int_{\mathbb{T}^d} \psi_1(y)dy) \right] dx,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{w\in W_M}\int_{I(w)}\psi_2((.w)_R)\left[\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right]dx\\
&=\sum_{w\in W_M}\psi_2((.w)_R)\int_{I(w)}\left[\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right]dx\\
&=\sum_{w\in W_M}\psi_2((.w)_R)\int_{I(w)}\psi_1(x)dx-\sum_{w\in W_M}\psi_2((.w)_R)\int_{I(w)}(\int_{\mathbb{T}^d}\psi_1(y)dy)dx\\
&=\sum_{w\in W_M}\psi_2((.w)_R)\frac{1}{r^M}\int_{\mathbb{T}^d}\psi_1(x)dx-\sum_{w\in W_M}\psi_2((.w)_R)\frac{1}{r^M}\int_{\mathbb{T}^d}\psi_1(y)dy=0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d}\psi_1\psi_2-(\int_{\mathbb{T}^d}\psi_1)(\int_{\mathbb{T}^d}\psi_2)\right|\\
&=\left|\sum_{w\in W_M}\int_{I(w)}[\psi_2(x)-\psi_2((.w)_R)\left[\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right]dx\right|\\
&\le\sum_{w\in W_M}\int_{I(w)}\left|[\psi_2(x)-\psi_2((.w)_R)\left[\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right]\right|dx\\
&\le\sum_{w\in W_M}\int_{I(w)}L\left|x-(.w)_R\right|\left|\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right|dx\\
&\le L\frac{\sqrt{d}}{R^M}\sum_{w\in W_M}\int_{I(w)}\left|\psi_1(x)-(\int_{\mathbb{T}^d}\psi_1(y)dy)\right|dx\\
&\le L\frac{\sqrt{d}}{R^M}\left(\sum_{w\in W_M}\int_{I(w)}\left|\psi_1(x)\right|dx+\left|\int_{\mathbb{T}^d}\psi_1(y)dy\right|\sum_{w\in W_M}\int_{I(w)}dx\right)\\
&\le L\frac{\sqrt{d}}{R^M}2\int_{\mathbb{T}^d}|\psi_1|
\end{aligned}$$

□

THEOREM 3.7. *Let (H,R,α) be an admissible triple and let $\{P_N\}$ be the sequence of associated partial Riesz polynomials. If $\overline{\lim}\ \alpha_n<\frac{R-1}{\max|H|}$, then $\{P_N\}_{N\in\mathbb{N}}$ converges weak-$*$ to a probability measure $\mu\in M(\mathbb{T}^d)$, which is continuous with respect to Lebesgue measure, and where Lebesgue measure is obtained from the admissible triple $(H,R,\alpha)=(H,R,\{0\})$.*

PROOF. (a) The existence of μ is proven in three steps.

(i) First, notice that admissibility of the triple (H,T,α) ensures that all the P_N are positive, and so $\|P_N\|_1=\int P_N=1$ by Lemma 3.4. Thus, the sequence $\{\|P_N\|_1\}$ converges, and if μ exists, then $\|\mu\|=1$.

(ii) Second, we shall show that for every G in a dense subset X of $C(\mathbb{T}^d)$, the sequence $\left\{\int_{\mathbb{T}^d} GP_N\right\}$ is a Cauchy sequence in $\mathbb{C}$.

We shall choose the set of Lipschitz continuous functions on $\mathbb{T}^d$ to be the dense subset of $C(\mathbb{T}^d)$. Let $G : \mathbb{T}^d \to \mathbb{R}$ be a Lipschitz continuous function, with Lipschitz constant L_G. We compute

$$\begin{aligned}
&\int_{\mathbb{T}^d} G(x)P_N(x)dx - \int_{\mathbb{T}^d} G(x)P_M(x)dx \\
&= \int_{\mathbb{T}^d} G(x)P_M(x) \prod_{n=M+1}^{N} \left(1+\alpha_n H(R^{n-1}x)\right) dx - \int_{\mathbb{T}^d} G(x)P_M(x)dx \\
&= \int_{\mathbb{T}^d} G(x)P_M(x) \prod_{n=M+1}^{N} \left(1+\alpha_n H(R^{n-1}x)\right) dx \\
&\quad - \int_{\mathbb{T}^d} \prod_{n=M+1}^{N} \left(1+\alpha_n H(R^{n-1}x)\right) dx \int_{\mathbb{T}^d} G(x)P_M(x)dx,
\end{aligned}$$

since

$$\int_{\mathbb{T}^d} \prod_{n=M+1}^{N} \left(1+\alpha_n H(R^{n-1}x)\right) dx = 1$$

by Lemma 3.4.

Now, Proposition 3.6 applies, with $L \leq L_G \|P_M\|_\infty \leq L_G \prod_{n=1}^{M}(1 + \alpha_n \max|H|)$, and hence

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} G(x)P_N(x)dx - \int_{\mathbb{T}^d} G(x)P_M(x)dx\right| \\
&\leq 2L_G \prod_{n=1}^{M}(1+\alpha_n \max|H|)\frac{\sqrt{d}}{R^M} \int_{\mathbb{T}^d} \prod_{n=M+1}^{N} \left(1+\alpha_n H(R^{n-1}x)\right) dx \\
&= 2L_G\sqrt{d}\frac{1}{R^M} \prod_{n=1}^{M}(1+\alpha_n \max|H|) \longrightarrow_M 0,
\end{aligned}$$

since $\overline{\lim}\, (1 + \alpha_n \max|H|) < R$.

(iii) Finally, we shall show that for any $F \in C(\mathbb{T}^d)$, the sequence $\left\{\int_{\mathbb{T}^d} FP_N\right\}$ is also a Cauchy sequence in $\mathbb{C}$.

Given any $\epsilon > 0$, choose G to be a Lipschitz continuous function such that $\|F-G\|_\infty < \frac{\epsilon}{2}$. Then

$$\begin{aligned}
\left|\int_{\mathbb{T}^d} FP_N - \int_{\mathbb{T}^d} FP_M\right| &= \left|\int_{\mathbb{T}^d} F(P_N - P_M)\right| \\
&= \left|\int_{\mathbb{T}^d} (F-G)(P_N-P_M) + \int_{\mathbb{T}^d} G(P_N-P_M)\right| \\
&\leq \left|\int_{\mathbb{T}^d} (F-G)(P_N-P_M)\right| + \left|\int_{\mathbb{T}^d} G(P_N-P_M)\right| \\
&\leq \|F-G\|_\infty \|P_N - P_M\|_1 + \left|\int_{\mathbb{T}^d} G(P_N-P_M)\right| \\
&\leq \frac{\epsilon}{2}(\|P_N\|_1 + \|P_M\|_1) + \left|\int_{\mathbb{T}^d} G(P_N-P_M)\right| \\
&= \epsilon + \left|\int_{\mathbb{T}^d} GP_N - \int_{\mathbb{T}^d} GP_M\right|.
\end{aligned}$$

Since the sequence $\left\{\int_{\mathbb{T}^d} GP_N\right\}$ is Cauchy in $\mathbb{C}$, the sequence $\left\{\int_{\mathbb{T}^d} FP_N\right\}$ is also Cauchy in $\mathbb{C}$. Hence, there exists a measure $\mu \in M(\mathbb{T}^d)$ such that the sequence $\{P_N\}$ converges weak-$*$ to μ.

(b) Continuity for μ is established in the following way.

Since $\mu(\{x\}) \leq \mu(I(w))$ for all $I(w)$ containing x, it is enough to verify that $\mu(I(w))$ converges to zero for any sequence $\{I(w_M)\}_{M\in\mathbb{N}}$ containing x, where each w belongs to W_M. In fact, for any $w \in W_M$, we have

$$\mu(I(w)) = \int_{I(w)} d\mu \leq \lim_N \int_{I(w)} P_N = \prod_{n=1}^{M} \frac{1}{r}\left(1 + \alpha_n H(R^{n-1}(.w)_R)\right) \leq \left(\frac{2}{r}\right)^M.$$

□

In analogy with Zygmund's result concerning the original example of Riesz, we shall now establish a dichotomy result for these types of measures. To prove this theorem we shall use a method introduced by Brown [7] (Theorem 1) in much the same way as R.T. Bott did in [5]. Specifically we shall need the following result.

THEOREM 3.8. *(Brown) Let λ_1, λ_2 be probability measures on a measurable space, and let E_i denote expectation with respect to λ_i, $i = 1, 2$. Suppose that there exists a sequence $\{X_n\}$ of complex random variables on the same space, square integrable with respect to λ_1 and λ_2, such that*

$$\sum_{k=1}^{\infty} \sup_{i,n} \left|E_i(X_n \overline{X_{n+k}}) - E_i(X_n)E_i(\overline{X_{n+k}})\right| < \infty$$

and

$$\sum_{n=1}^{\infty} |E_1(X_n) - E_2(X_n)|^2 = \infty.$$

Then λ_1 and λ_2 are mutually singular.

THEOREM 3.9. *If (H, R, α) and (H, R, β) are admissible triples, and the partial Riesz polynomials associated with them converge in the weak-$*$ topology to the measures μ_α and μ_β, respectively, then*

$$\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2 = \infty$$

implies that μ_α and μ_β are mutually singular.

PROOF. To simplify notation, we shall let $\alpha^{(1)} = \alpha$, $\alpha^{(2)} = \beta$, $P_n^{(1)} = P_n(H, R, \alpha)$ and $P_n^{(2)} = P_n(H, R, \beta)$. We shall use the functions $H(R^{n-1}t)$ as our sequence of random variables H_n, and singularity will be proven if we show that

$$\sum_{k=1}^{\infty} \sup_{i,m} \left| \lim_n \int_{\mathbb{T}^d} H_m \overline{H_{m+k}} P_n^{(i)} - \lim_n \int_{\mathbb{T}^d} H_m P_n^{(i)} \cdot \lim_n \int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)} \right| < \infty \tag{13}$$

and

$$\sum_{m=1}^{\infty} \left| \lim_n \int_{\mathbb{T}^d} H_m P_n^{(1)} - \lim_n \int_{\mathbb{T}^d} H_m P_n^{(2)} \right|^2 = \infty. \tag{14}$$

In order to prove (13) we shall actually show that for all k and m,

$$\lim_n \int_{\mathbb{T}^d} H_m \overline{H_{m+k}} P_n^{(i)} = \lim_n \int_{\mathbb{T}^d} H_m P_n^{(i)} \cdot \lim_n \int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)}. \tag{15}$$

We start by fixing $k, m \in \mathbb{N}$. By Proposition 3.6, we have that for all $n > \max\{k, m\}$,

$$\left| \int_{\mathbb{T}^d} H_m \overline{H_{m+k}} P_n^{(i)} - \int_{\mathbb{T}^d} H_m P_{m+k}^{(i)} \cdot \int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1 + \alpha_j^{(i)} H_j\right) \right| \leq 0.$$

Thus, for all $n > \max\{k, m\}$ we have

$$\int_{\mathbb{T}^d} H_m \overline{H_{m+k}} P_n^{(i)} = \int_{\mathbb{T}^d} H_m P_{m+k}^{(i)} \cdot \int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1 + \alpha_j^{(i)} H_j\right). \tag{16}$$

Now, applying Proposition 3.6 once more, we see that

$$\forall\ n > m, \left| \int_{\mathbb{T}^d} H_m P_n^{(i)} - \int_{\mathbb{T}^d} H_m P_m^{(i)} \cdot \int_{\mathbb{T}^d} \prod_{j=m+1}^{n} \left(1 + \alpha_j^{(i)} H_j\right) \right| \leq 0$$

and so

$$\int_{\mathbb{T}^d} H_m P_n^{(i)} = \int_{\mathbb{T}^d} H_m P_m^{(i)} \cdot \int_{\mathbb{T}^d} \prod_{j=m+1}^{n} \left(1+\alpha_j^{(i)} H_j\right) = \int_{\mathbb{T}^d} H_m P_m^{(i)}.$$

In particular,

$$\forall\ n > m+k, \quad \int_{\mathbb{T}^d} H_m P_n^{(i)} = \int_{\mathbb{T}^d} H_m P_{m+k}^{(i)}.$$

Therefore,

$$\forall\ n > m+k, \quad \int_{\mathbb{T}^d} H_m P_{m+k}^{(i)} = \lim_n \int_{\mathbb{T}^d} H_m P_n^{(i)}. \tag{17}$$

On the other hand, for all $n > m+k$ we have

$$\int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1+\alpha_j^{(i)} H_j\right) = \lim_n \int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)}. \tag{18}$$

To see this, use Proposition 3.6 to obtain

$$\left| \int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)} - \int_{\mathbb{T}^d} P_{m+k}^{(i)} \cdot \int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1+\alpha_j^{(i)} H_j\right) \right| \leq 0,$$

so that

$$\begin{aligned}\int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)} &= \int_{\mathbb{T}^d} P_{m+k}^{(i)} \cdot \int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1+\alpha_j^{(i)} H_j\right) \\ &= \int_{\mathbb{T}^d} \overline{H_{m+k}} \prod_{j=m+k+1}^{n} \left(1+\alpha_j^{(i)} H_j\right).\end{aligned}$$

Combining (17) and (18) with (16) we see that

$$\forall\ n > \max\{k,m\}, \quad \int_{\mathbb{T}^d} H_m \overline{H_{m+k}} P_n^{(i)} = \lim_n \int_{\mathbb{T}^d} H_m P_n^{(i)} \cdot \lim_n \int_{\mathbb{T}^d} \overline{H_{m+k}} P_n^{(i)},$$

and this establishes (15) and hence (13).

To prove (14), observe that for $n > m$ we have

$$\begin{aligned}\int_{\mathbb{T}^d} H_m P_n^{(i)} &= \int_{\mathbb{T}^d} H_m (1+\alpha_m^{(i)} H_m) \prod_{k\neq m}^{n} (1+\alpha_k^{(i)} H_k) \\ &= \int_{\mathbb{T}^d} H_m \prod_{k\neq m}^{n} (1+\alpha_k^{(i)} H_k) + \alpha_m^{(i)} \int_{\mathbb{T}^d} H_m^2 \prod_{k\neq m}^{n} (1+\alpha_k^{(i)} H_k) \\ &= 0 + \alpha_m^{(i)} \int_{\mathbb{T}^d} H_m^2,\end{aligned}$$

the last equation being justified by Lemma 3.4. We also have $\int_{\mathbb{T}^d} H_m^2 = \int_{\mathbb{T}^d} H^2$ for all $m \in \mathbb{N}$, so that for $i = 1, 2$ we obtain

$$\lim_n \int_{\mathbb{T}^d} H_m P_n^{(i)} = \alpha_m^{(i)} \int_{\mathbb{T}^d} H^2.$$

Thus,

$$\begin{aligned}\left| \lim_n \int_{\mathbb{T}^d} H_m P_n^{(1)} - \lim_n \int_{\mathbb{T}^d} H_m P_n^{(2)} \right|^2 &= \left| \alpha_m^{(1)} \int_{\mathbb{T}^d} H^2 - \alpha_m^{(2)} \int_{\mathbb{T}^d} H^2 \right|^2 \\ &= \left| \int_{\mathbb{T}^d} H^2 \right|^2 \left| \alpha_m^{(1)} - \alpha_m^{(2)} \right|^2.\end{aligned}$$

Since $\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2 = \infty$, we are done. □

THEOREM 3.10. *If (H, R, α) is an admissible triple that converges in the weak-$*$ topology to the measure μ, then μ is singular with respect to Lebesgue measure if $\sum_{n=1}^{\infty} |\alpha_n|^2 = \infty$ and absolutely continuous if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$.*

PROOF. The first part follows immediately from Theorem 3.9, since Lebesgue measure corresponds to the triple $(H, R, \{0\})$. To prove that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ implies that μ is absolutely continuous we proceed as follows.

First, we show that $\{\|P_N\|_2\}$ is a convergent sequence. We have

$$\|P_N\|_2^2 = \prod_{n=1}^{N} \left(1 + |\alpha_n|^2 \int_{\mathbb{T}^d} |H|^2 \right),$$

and the product $\prod_{n=1}^{\infty} \left(1 + |\alpha_n|^2 \int_{\mathbb{T}^d} |H|^2 \right)$ converges absolutely whenever

$$\sum_{n=1}^{\infty} \log \left(1 + |\alpha_n|^2 \int_{\mathbb{T}^d} |H|^2 \right) < \infty. \tag{19}$$

Since

$$\log \left(1 + |\alpha_n|^2 \int_{\mathbb{T}^d} |H|^2 \right) \leq |\alpha_n|^2 \int_{\mathbb{T}^d} |H|^2,$$

and, by hypothesis, $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, we obtain (19).

This means that the sequence $\{P_N\}$ is norm bounded in $L^2(\mathbb{T}^d)$ and therefore has a limit point $f \in L^2(\mathbb{T}^d)$ in its weak-$*$ topology, i.e., there exists a subsequence $\{P_{N_j}\}$ such that $P_{N_j} \to f$ in the weak-$*$ topology of $L^2(\mathbb{T}^d)$.

The characters e_k of $\mathbb{T}^d$ belong to $L^2(\mathbb{T}^d)$, so we have

$$\widehat{P_{N_j}}[k] = \int_{\mathbb{T}^d} P_{N_j} e_k \to \int_{\mathbb{T}^d} f e_k = \widehat{f}[k]$$

for all $k \in \mathbb{Z}^d$. Since we also have $\widehat{P_N}[k] \to \widehat{\mu}[k]$, we conclude that $\widehat{f}[k] = \widehat{\mu}[k]$ for all $k \in \mathbb{Z}^d$, $d\mu = f dm$, and μ is absolutely continuous as claimed.

□

4. Support

We now focus our attention on support properties of multiscale Riesz product measures on $G = \mathbb{T}^d$. In general, when the generating function is a trigonometric polynomial, the support of μ is the entire group. As an example of the type of argument involved, we provide the following proof by Brown [6] (Proposition 3).

PROPOSITION 4.1. *Let $G = \mathbb{T}$. Let μ be the multiscale Riesz product associated with the triple (H, T, α), as in Section 2. Then the support of μ is G.*

PROOF. For each $N \in \mathbb{N}$ let $dm_N = \prod_{n=N+1}^{\infty} \left(1 + \alpha_n H \circ T^{n-1}\right) dm$. The sequence $\{m_N\}$ converges weakly to Haar measure m (as in Section 2). Since $\mu = P_N m_N$, μ is absolutely continuous with respect to each m_N. It is actually equivalent to all of them; in fact, the number of zeros of each term of P_N is finite since each P_N is a non-zero trigonometric polynomial on $\mathbb{T}$. Hence, if U is an open set such that $\mu(U) = 0$, we have $m_N(U) = 0$. Thus, $m_N(G) = m_N(G \setminus U)$, and so $\overline{\lim}\, m_N(G \setminus U) = m(G)$. We also have $m(G \setminus U) \geq \overline{\lim}\, m_N(G \setminus U)$, and therefore $m(G \setminus U) = m(G)$. This means that $m(U) = 0$, which can only happen if $U = \emptyset$, i.e., $\mathrm{supp}\mu = G$. $\square$

There do exist, however, Riesz products μ of this type, i.e., $\mathrm{supp}\mu = G$, that bear striking similarities to singular measures, e.g., see Example 4.4 and Figure 1.

We contrast Proposition 4.1 with the behavior of generalized Riesz products when H is a function of type R, rather than a trigonometric polynomial. Step functions allow for the limit measure μ to have thin support, by virtue of the following result, see Remark 1. Recall that B is defined in Subsection 1.3.

PROPOSITION 4.2. *Let (H, R, α) be an admissible triple with $\overline{\lim}\, \alpha_n < \frac{R-1}{\max|H|}$. Let $\mu_{(H,S)}$ be the weak-$*$ limit-measure associated with it. Assume that for all $b \in B$ and for all $n \in \mathbb{N}$ we have $1+\alpha_n c_b \neq 0$. Then supp $\mu_{(H,S)} = \mathbb{T}^d$.*

PROOF. Let U be an open subset of $\mathbb{T}^d$. There exist $w_M \in W_M$ such that $I(w_M) \subset U$. Thus

$$\begin{aligned}
\left|\mu_{(H,S)}\right|(U) \geq \left|\mu_{(H,S)}\right|(I(w_M)) &= \int_{I(w_M)} d\left|\mu_{(H,S)}\right| \\
&\geq \lim_N \int_{I(w_M)} |P_N| \\
&\geq \lim_N \left|\int_{\mathbb{T}^d} \mathbf{1}(w_M) P_N\right| = \prod_{n=1}^{M} \frac{1}{r} \left|1 + \alpha_n H(R^{n-1}(.w)_R)\right|
\end{aligned}$$

by Lemma 3.5. By hypothesis, none of the terms $1 + \alpha_n H(R^{n-1}(.w)_R)$ is zero, so the finite product

$$\prod_{n=1}^{M} \frac{1}{r} \left|1 + \alpha_n H(R^{n-1}(.w)_R)\right|$$

is also non-zero. Hence,

$$\left|\mu_{(H,S)}\right|(U) > 0$$

for all open sets $U \subset \mathbb{T}^d$.

$\square$

REMARK 1. If for some $n \in \mathbb{N}$ and $b \in B$ we have $\alpha_n = -\frac{1}{c_b}$, then for all $n \geq N$ we have $P_N = 0$ on the set $\frac{1}{R^n}\left([\frac{c_b}{R}, \frac{c_b+1}{R}) + \mathbb{Z}\right)$. Thus, supp P_N is in the complement of $\frac{1}{R^n}\left([\frac{c_b}{R}, \frac{c_b+1}{R}) + \mathbb{Z}\right)$ in $\mathbb{T}$. For example, by making an appropriate choice of $b \in B$ and setting $\alpha_n = -\frac{1}{c_b}$ for all $n \in \mathbb{N}$ we can construct multiscale Riesz product measures supported on Cantor sets, see Examples 4.5 – 4.8.

This point of view leads to the following statement, keeping in mind that the measures of Section 3 are defined on $\mathbb{T}^d$. For a definition of an iterated function system, we refer to Hutchinson's fundamental paper [14], as well as the earlier comparable work by Moran [23].

STATEMENT 4.3. *The periodization of any measure associated with an iterated function system on $\mathbb{R}^d$ can be obtained as a multiscale Riesz product measure associated with a triple (H, T, α) by making an appropriate choice of $b \in B$ and setting $\alpha_n = -\frac{1}{c_b}$ for all $n \in \mathbb{N}$.*

Statement 4.3 and the generalizations and types of examples to which it gives rise reflect ongoing work by the third named author. The following examples are a prelude.

EXAMPLE 4.4. Let $H(t) = 0.7cos(2\pi t) + 0.2cos(4\pi t)$ and $\alpha_n = 1$, and consider the associated multiscale Riesz product μ on $\mathbb{T}$. μ is not a singular measure and supp$\mu = \mathbb{T}$. Figure 1 plots the graph of the antiderivative of P_4.

EXAMPLE 4.5. In one dimension, setting $R = 3$, $c_0 = \frac{1}{2}, c_1 = -1$, and $c_2 = \frac{1}{2}$, we recover the classical Cantor middle-thirds measure as the weak-$*$ limit associated with P_N for $\alpha_n = 1$ for all $n \in \mathbb{N}$, see Figure 2. Recall that R, c_0, c_1, and c_2 are needed to construct the generating function H of type R defined in Definition 1.3.

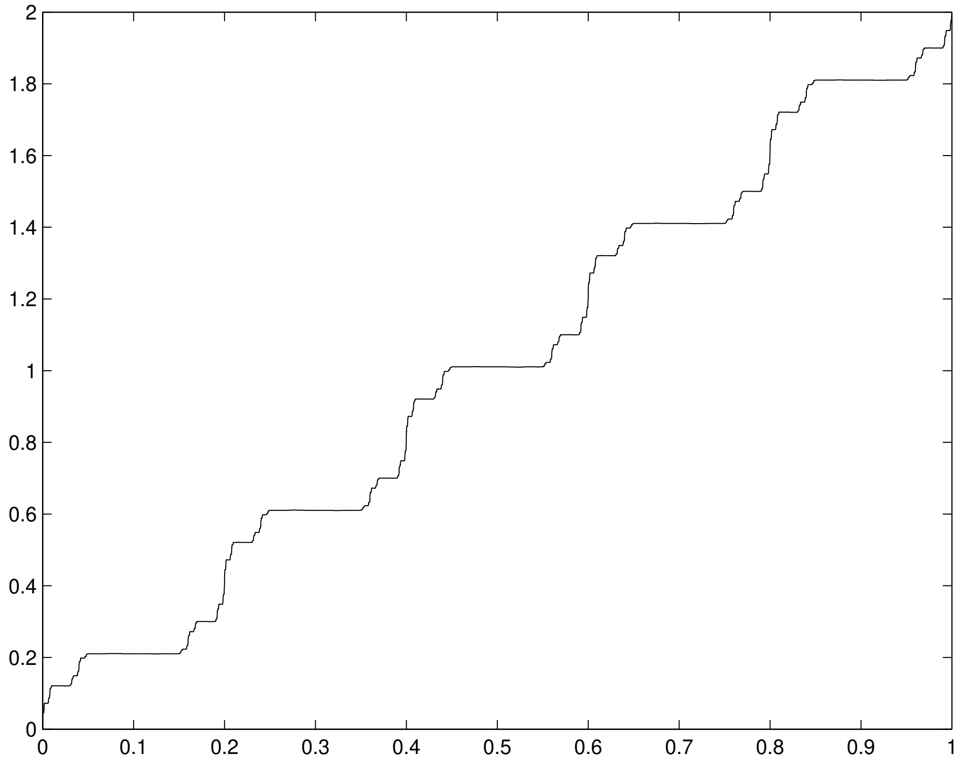

FIGURE 1. Not a Cantor function

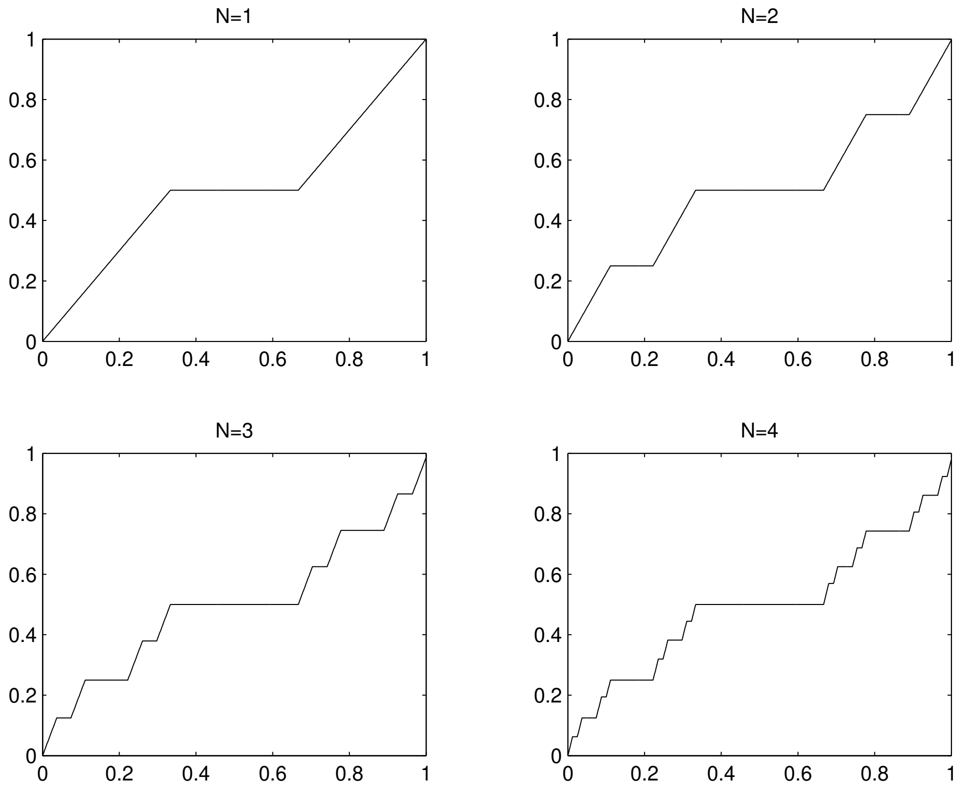

FIGURE 2. The antiderivatives of the Riesz polynomials from Example 4.5

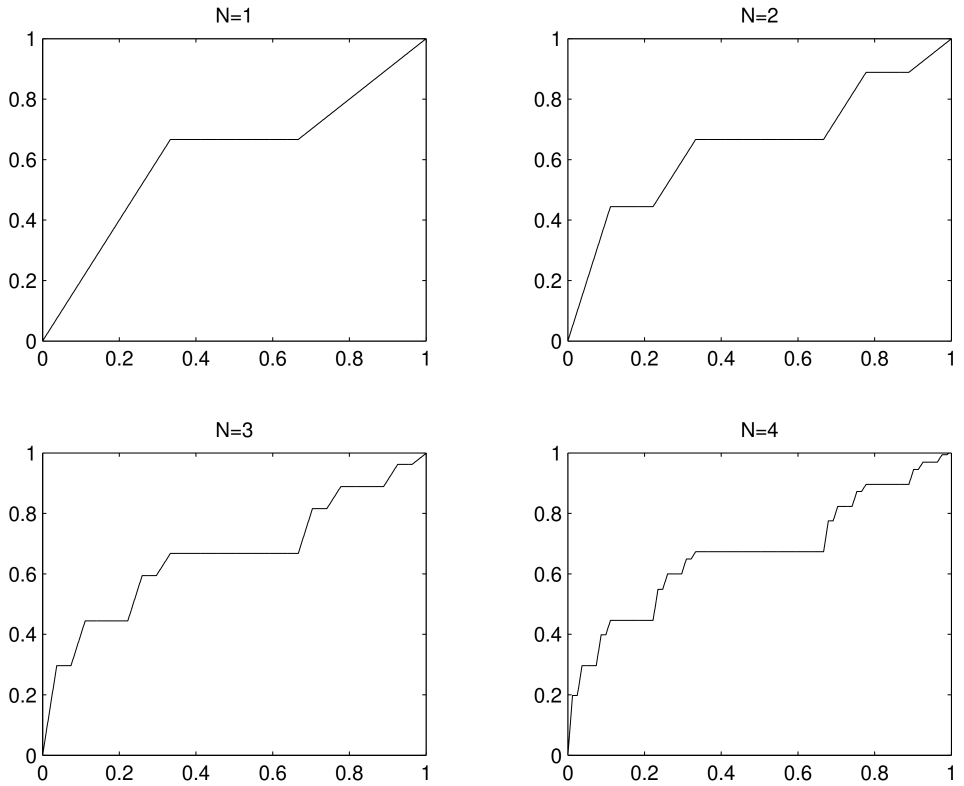

FIGURE 3. The antiderivatives of the Riesz polynomials from Example 4.6

EXAMPLE 4.6. In one dimension, setting $R = 3,\ c_0 = 1,\ c_1 = -1,\ c_2 = 0$, and $\alpha_n = 1$ for all $n \in \mathbb{N}$, we obtain a different measure supported on the Cantor middle-thirds set, see Figure 3.

EXAMPLE 4.7. In one dimension again, setting $R = 4,\ c_0 = c_2 = 1,\ c_1 = c_3 = -1$, and $\alpha_n = 1$ for all $n \in \mathbb{N}$, we obtain a measure supported on a different Cantor set, see Figure 4.

We can also obtain measures supported on two-dimensional self-similar sets, as the following example shows.

EXAMPLE 4.8. In the case $d = 2$, again using $R = 3$ and letting $c_b = \frac{1}{8}$ for all b except $c_{(1,1)} = -1$, we obtain the Sierpinski carpet as the support of $\mu_{(H,3,1)}$.

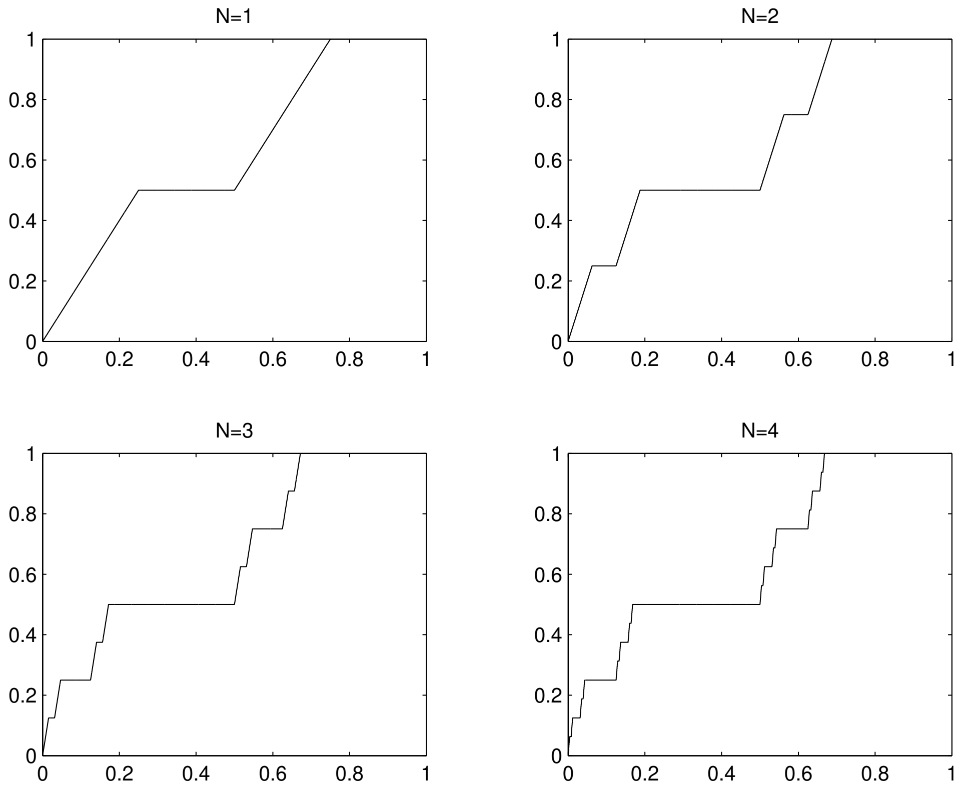

FIGURE 4. The antiderivatives of the Riesz polynomials from Example 4.7

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