Waveform design and quantum detection matched filtering

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Outline and collaborators

1. CAZAC waveforms
2. Frames
3. Matched filtering and related problems
4. Quantum detection
5. Analytic methods to construct CAZAC waveforms

Collaborators: Matt Fickus (frame force), Andrew Kebo (quantum detection), Joseph Ryan and Jeff Donatelli (software).

CAZAC Waveforms

Constant Amplitude Zero Autocorrelation (CAZAC) Waveforms

A K-periodic waveform $u : \mathbb{Z}_K = \{0, 1, \ldots, K-1\} \to \mathbb{C}$ is *CAZAC* if $|u(m)| = 1$, $m = 0, 1, \ldots, K-1$, and the *autocorrelation*

$$A_u(m) = \frac{1}{K} \sum_{k=0}^{K-1} u(m+k)\overline{u}(k) \text{ is 0 for } m = 1, \ldots, K-1.$$

The *crosscorrelation* of $u, v : \mathbb{Z}_K \to \mathbb{C}$ is

$$C_{u,v}(m) = \frac{1}{K} \sum_{k=0}^{K-1} u(m+k)\overline{v}(k) \text{ for } m = 0, 1, \ldots, K-1.$$

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- u CA $\iff$ DFT of u is ZAC off dc. (DFT of u can have zeros)
- u CAZAC $\iff$ DFT of u is CAZAC.
- User friendly software: http://www.math.umd.edu/~jjb/cazac

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- CA allows transmission at peak power. (The system does not have to deal with the suprise of greater than expected amplitude.)
- Distortion amplitude variations can be detected using CA. (With CA amplitude variations during transmission due to additive noise can be theoretically eliminated at the receiver without distorting message.)
- A sharp unique peak in A_u is important because of distortion and interference in received waveforms, *e.g.*, in radar and communications–more later.

Examples of CAZAC Waveforms

$K = 75 : u(x) =$

$$(1, 1, 1, 1, 1, 1, e^{2\pi i \frac{1}{15}}, e^{2\pi i \frac{2}{15}}, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{3}{5}},$$

$$e^{2\pi i \frac{11}{15}}, e^{2\pi i \frac{13}{15}}, 1, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{4}{5}}, 1, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{8}{15}}, e^{2\pi i \frac{4}{5}},$$

$$e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}, e^{2\pi i}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \frac{5}{3}}, 1, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{6}{5}},$$

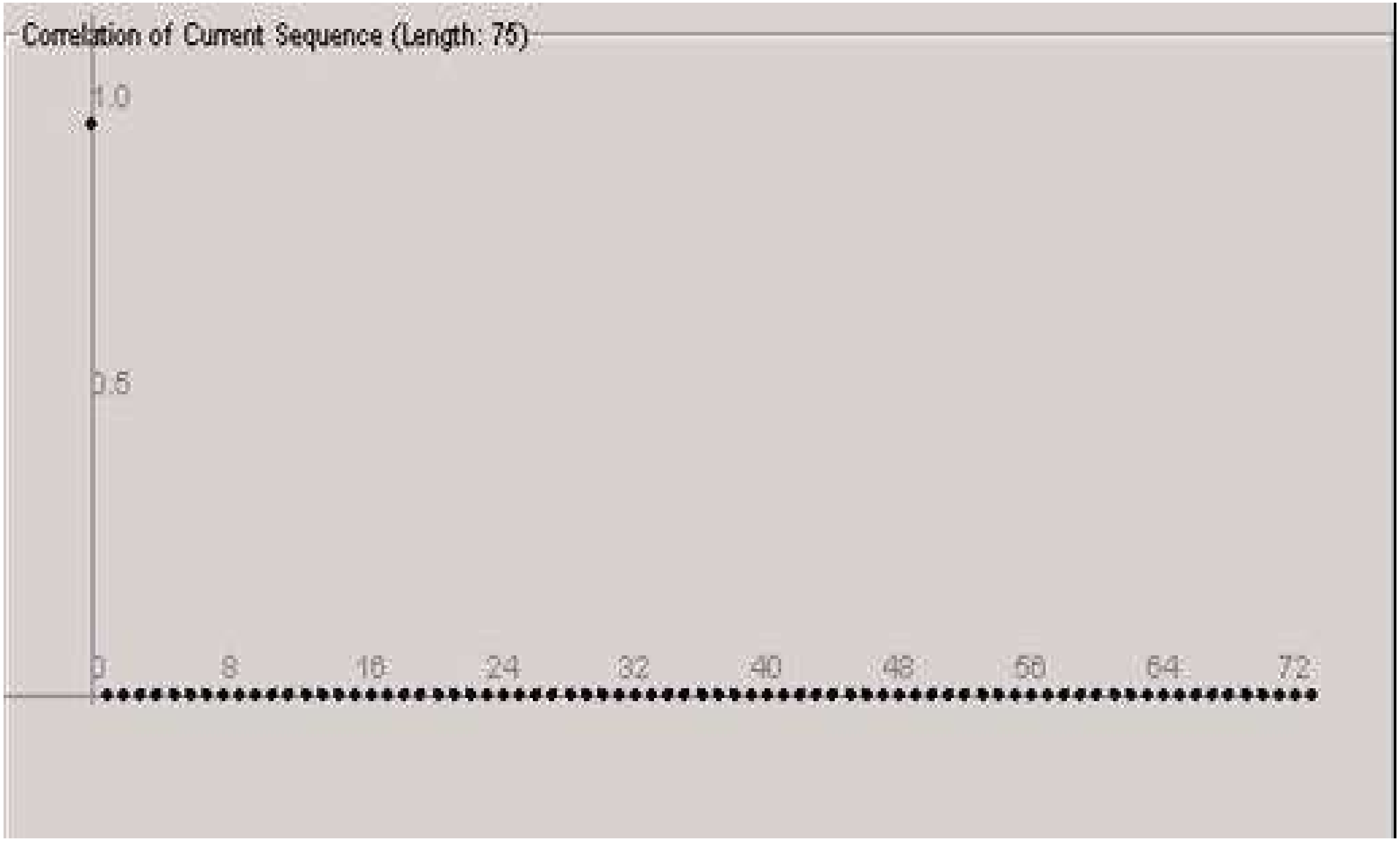
$$e^{2\pi i \frac{8}{5}}, 1, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{28}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{29}{15}},$$

$$e^{2\pi i \frac{37}{15}}, 1, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{6}{5}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{12}{5}}, 1, e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \cdot 2}, e^{2\pi i \frac{8}{3}},$$

$$e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{38}{15}}, e^{2\pi i \frac{49}{15}}, 1, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{8}{5}}, e^{2\pi i \frac{12}{5}}, e^{2\pi i \frac{16}{5}},$$

$$1, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{26}{15}}, e^{2\pi i \frac{13}{5}}, e^{2\pi i \frac{52}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{19}{15}}, e^{2\pi i \frac{11}{5}}, e^{2\pi i \frac{47}{15}}, e^{2\pi i \frac{61}{15}})$$

Autocorrelation of CAZAC $K = 75$



Finite ambiguity function

Given K-periodic waveform, $u : \mathbb{Z}_K \to \mathbb{C}$ let $e_m(n) = e^{\frac{-2\pi imn}{K}}$.

- The *ambiguity function* of u, $A : \mathbb{Z}_K \times \mathbb{Z}_K \to K$ is defined as

$$A_u(j,k) = C_{u,ue_k}(j) = \frac{1}{K}\sum_{m=0}^{K-1} u(m+j)\overline{u(m)}e^{\frac{2\pi imk}{K}}.$$

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- Analogue ambiguity function for $u \leftrightarrow U$, $\|u\|_2 = 1$, on $\mathbb{R}$:

$$A_u(t,\gamma) = \int_{\widehat{\mathbb{R}}} U(\omega - \frac{\gamma}{2})\overline{U(\omega + \frac{\gamma}{2})}e^{2\pi it(\omega+\frac{\gamma}{2})}d\omega$$
$$= \int u(s+t)\overline{u(s)}e^{2\pi is\gamma}ds.$$

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- Standard Doppler frequency shift problem: Construct a statistic to determine unknown Doppler frequency shift. Do this for multiple frequencies.
- Provide rigorous justification for CAZAC simulations associated with the Doppler tolerance question and frequency shift problem.

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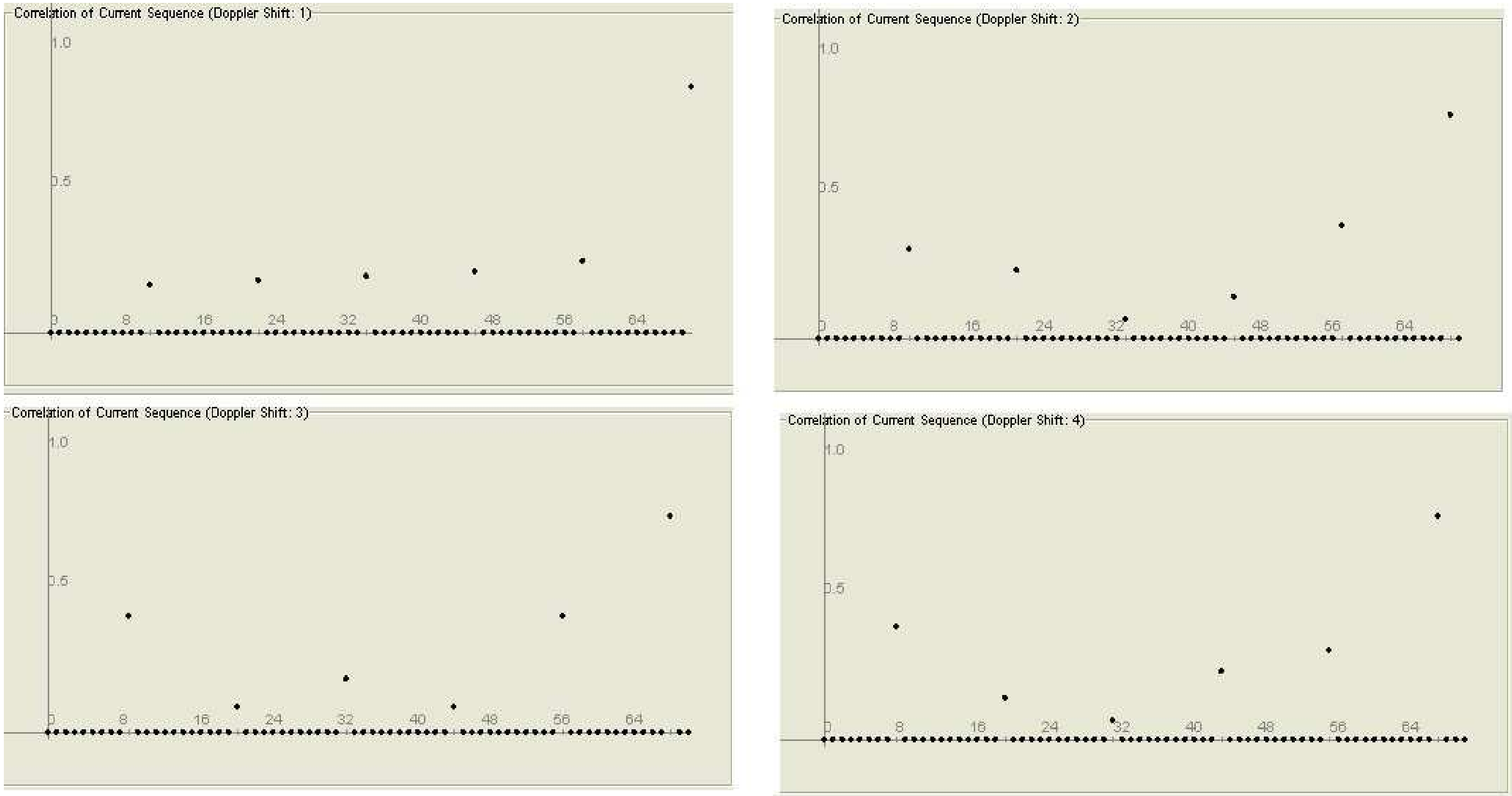
- There are unresolved “arithmetic” complexities which are affected by waveform structure and length.
- Noise analysis is ongoing.

Doppler Statistic



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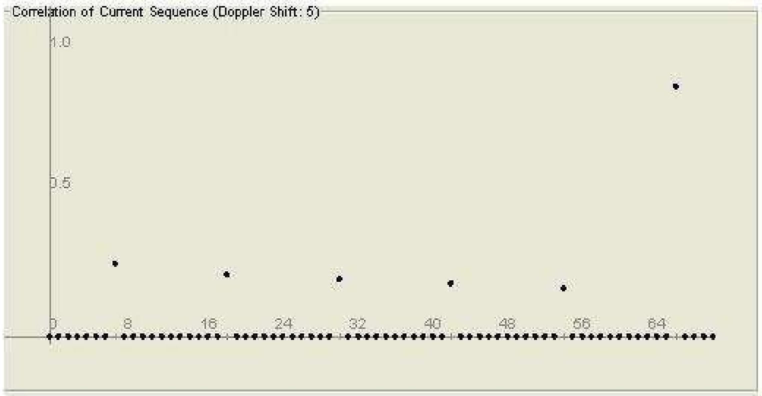

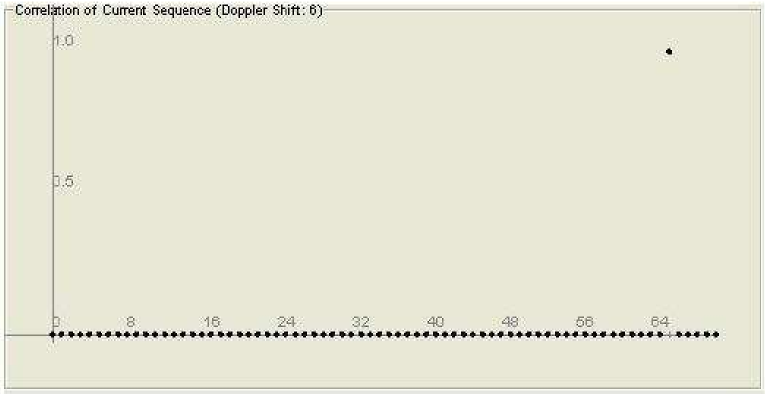

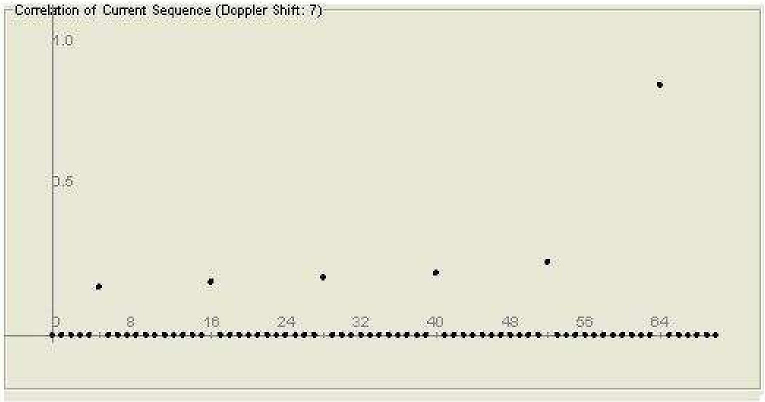

$$\sum_{j=0}^{k-1} |C_{u,ue_k}(j)|^2 = 1$$

Frames

Redundant signal representation

- Given $H = \mathbb{R}_d$ or $H = \mathbb{C}^d$, $N \geq d$. $\{x_n\}_{n=1}^N \subseteq H$ is a *finite unit norm tight frame (FUN-TF)* if each $\|x_n\| = 1$ and, for each $x \in H$,

$$x = \frac{d}{N} \sum_{n=1}^{N} \langle x, x_n \rangle x_n.$$

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- $\{x_n\}_{n=1}^N \subseteq H$ is an *A-tight frame* if $\{x_n\}_{n=1}^N$ spans H and $A\|x\|^2 = \sum_{n=1}^{N} |\langle x, x_n \rangle|^2$ for each $x \in H$.

Recent applications of FUN-TFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]

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- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]

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- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.
- Thus, if certain types of noises are known to exist, then the FUN-TFs are constructed using this information.
- Orthonormal bases, vertices of Platonic solids, kissing numbers (sphere packing and error correcting codes) are FUN-TFs.

DFT FUN-TFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUN-TFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\, d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

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- “Sigma-Delta” Super Audio CDs - but not all authorities are fans.

Frame force

The *frame force:* $F : S^{d-1} \times S^{d-1} \setminus D \to \mathbb{R}^d$ is defined as $F(a,b) = \langle a,b\rangle(a-b)$, S^{d-1} is the unit sphere in $\mathbb{R}^d$.

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- Compute these frames.

Multifunction vector-valued frame waveforms

Problem: Construct, code, and implement (user-friendly) N-periodic waveforms $(N \geq d)$

$$u : \mathbb{Z}_N \rightarrow S^{d-1} \subseteq \mathbb{R}^d \text{ (or } \mathbb{C}^d),$$
$$n \rightarrow u_n = (u_n(1), u_n(2), \ldots, u_n(d)), n = 0, 1, \ldots, N-1$$

which are FUN-TFs (for redundant signal representation) and CAZAC (zero or low correlation off dc), i.e.,

$$x = \frac{d}{N} \sum_{n=0}^{N-1} \langle x, u_n \rangle u_n \text{ and } A_u(m) = \frac{1}{N} \sum_{j=0}^{N-1} \langle u_{m+j}, u_j \rangle = 0,$$

$$m = 1, \ldots N-1.$$

Techniques for multifunction CAZAC waveforms

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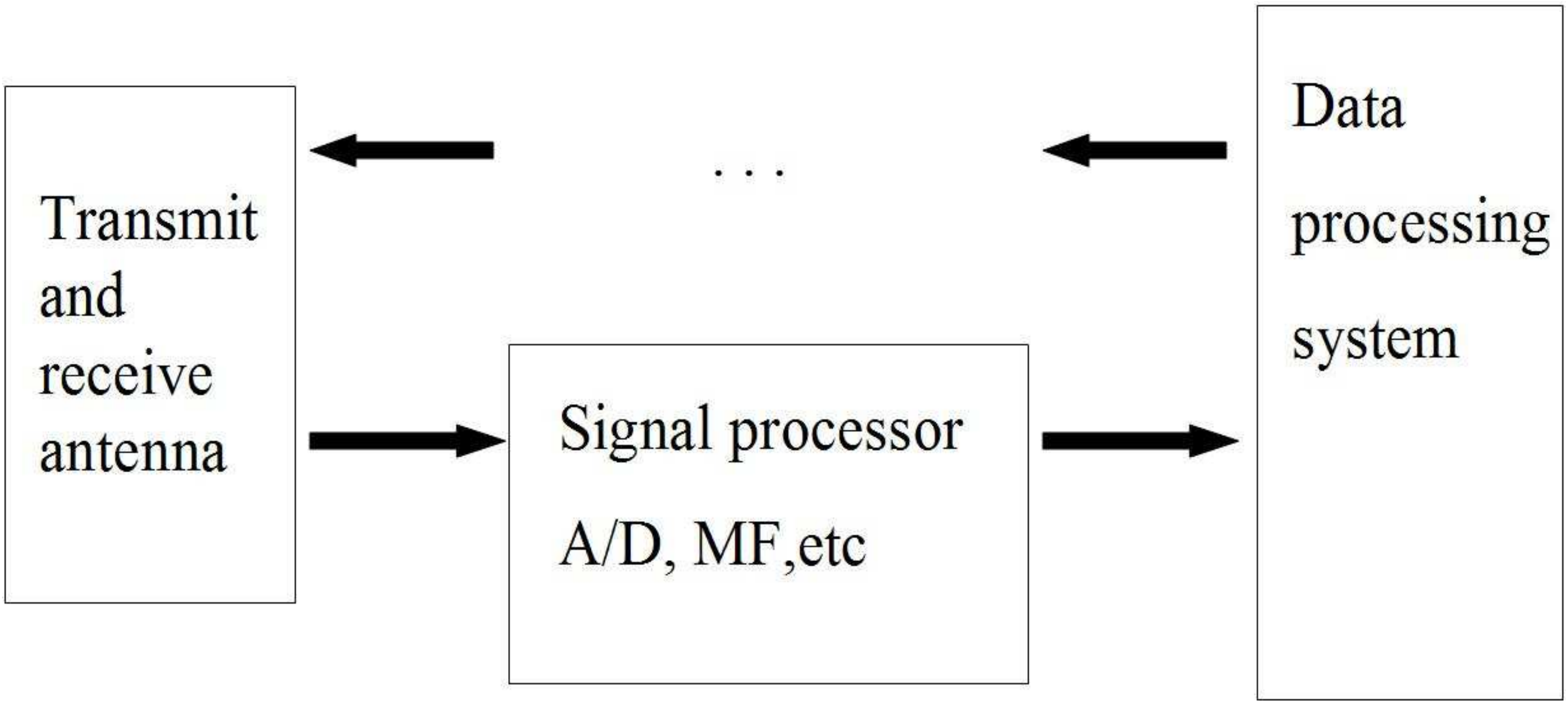
- Grassmannian "min-max" waveforms [Calderbank, Conway, Sloane, *et al.*, Kolesar, B]
- Grassmannian analysis gives another measure of the crosscorrelation. A FUN frame $\{u_n\}_{n=1}^N \subseteq H$ is *Grassmannian* if $\max_{k\neq l} |\langle u_k, u_l\rangle| = \inf \max_{k\neq l} |\langle x_k, x_l\rangle|$, where the infimum is over all FUN frames.

Matched Filtering

Processing



Natural problems associated with multifunction frame waveforms (1)

- Implement FUN-TF $\Sigma\Delta$ A/D converters to take advantage of proven improved error estimates for linear reconstruction over PCM and comparable to MSE-PCM. (MSE-PCM is based on Bennett's white noise assumption which is not always valid. With consistent reconstruction, and its added numerical complexity, MSE-PCM is comparable to FUN-TF MSE-$\Sigma\Delta$.)

Natural problems associated with multifunction frame waveforms (2)

- Distinguish multiple frequencies and times (ranges) in the ambiguity function,

$$A(\text{``}t\text{''}, \text{``}\gamma\text{''}) = \int_{\hat{R}} U(\omega)(\sum \alpha_j \overline{U(\omega + \gamma_j)} e^{2\pi i t_j \omega}) d\omega,$$

by means of multifunction frame waveforms.

Natural problems associated with multifunction frame waveforms (3)

- Compute optimal 1-tight frame CAZAC waveforms, $\{e_n\}_{n=1}^N$, using quantum detection error:

$$P_e = \min_{\{e_n\}}(1 - \sum_{i=1}^N \rho_n |\langle u_n, e_n\rangle|^2), \ \sum_{n=1}^N \rho_n = 1, \rho_n > 0,$$

where $\{u_n\}_{n=1}^N \subseteq S^{d-1}$ is given. This is a multifunction matched filtering.

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- If $v(t) = \overline{au(t - t_0)}$ for some t_0, then

$$C_{v,u}(t_0) = \sup_t |C_{v,u}(t)|.$$

$C_{v,u}$ is $L^2(\mathbb{R})$ crosscorrelation and maximum system response is given by *matched filter* $a\overline{\hat{u}(\gamma)}e^{-2\pi i t_0 \gamma}$

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- In digital case, CAZACs arise since travel time depends on crosscorrelation peak, and sharp peaks obviate distortion and interference in received waveform.

Outline of multifunction matched filtering problem

- QM formulates concept of measuring a dynamical quantity (e.g., position of an electron in $\mathbb{R}^3$) and the probability p that the outcome is in $U \subseteq \mathbb{R}^3$.

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- In $H = \mathbb{C}^d$, POVMs and 1-tight frames are equivalent.
- Given $\{u_n\}_{n=1}^N \subseteq S^{d-1}$. Compute/construct a 1-tight frame minimizer $\{e_n\}_{n=1}^N$ of quantum detection (QD) error P_e.

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- Show that the QD error is a potential energy function of frame force in $\mathbb{C}^N$.
- Use the orthogonal group and the Euler-Lagrange equation for the potential P_e to *compute* equations of motion and a minimal energy solution $\{e_n\}_{n=1}^N$.

Quantum Detection

Positive-operator-valued measures

Let $\mathcal{B}$ be a σ-algebra of sets of X. A *positive operator-valued measure* (POM) is a function $\Pi : \mathcal{B} \to \mathcal{L}(H)$ such that

1. $\forall U \in \mathcal{B},\ \Pi(U)$ is a positive self-adjoint operator,
2. $\Pi(\emptyset) = 0$ (zero operator),
3. $\forall$ disjoint $\{U_i\}_{i=1}^{\infty} \subset \mathcal{B}$ and $x, y \in H$,

$$\left\langle \Pi\left(\bigcup_{i=1}^{\infty} U_i\right) x, y \right\rangle = \sum_{i=1}^{\infty} \langle \Pi(U_i)x, y\rangle,$$

4. $\Pi(X) = I$ (identity operator).

- A POM Π on $\mathcal{B}$ has the property that given any fixed $x \in H$, $p_x(\cdot) = \langle x, \Pi(\cdot)x\rangle$ is a measure on $\mathcal{B}$. (Probability if $\|x\| = 1$).

Quantum Detection

Positive-operator-valued measures

Let $\mathcal{B}$ be a σ-algebra of sets of X. A *positive operator-valued measure* (POM) is a function $\Pi : \mathcal{B} \to \mathcal{L}(H)$ such that

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- A dynamical quantity Q gives rise to a measurable space $(X, \mathcal{B})$ and POM. When measuring Q, $p_x(U)$ is the probability that the outcome of the measurement is in $U \in \mathcal{B}$.

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$$\Pi(U) = \mathbb{1}_U.$$

- Suppose the state of the electron is given by $x \in H$ with unit norm. Then the probability that the electron is found to be in the region $U \in \mathcal{B}$ is given by

$$p(U) = \langle x, \Pi(U)x \rangle = \int_U |x(t)|^2 \, dt.$$

Parseval frames correspond to POMs

- Let $F = \{e_n\}_{n=1}^{N}$ be a Parseval frame for a d-dimensional Hilbert space H and let $X = \mathbb{Z}_N$.

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- Clear that Π satisfies conditions (1)-(3) for a POM. Since F is Parseval, we have condition (4) ($\Pi(X)x = \sum_{i \in X} \langle x, e_i \rangle e_i = x$). Thus Π defines a POM.

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- Conversely, let $(X, \mathcal{B})$ be a measurable space with corresponding POM Π for a d-dimensional Hilbert space H. If X is countable then there exists a subset $K \subseteq \mathbb{Z}$, a Parseval frame $\{e_i\}_{i \in K}$, and a disjoint partition $\{B_j\}_{j \in X}$ of K such that for all $j \in X$ and $y \in H$,

$$\Pi(j)y = \sum_{i \in B_j} \langle y, e_i \rangle e_i.$$

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$$p_{x_i}(j) = \langle x_i, \Pi(j) x_i \rangle \approx \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

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- Since $\langle x_i, \Pi(i) x_i \rangle$ is the probability of a successful detection of the state x_i, then the probability of a detection error is given by

$$P_e = 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \Pi(i) x_i \rangle.$$

Quantum detection problem

- If we construct our POM using Parseval frames, the error becomes

$$\begin{aligned} P_e &= 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \Pi(i) x_i \rangle \\ &= 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \langle x_i, e_i \rangle e_i \rangle \\ &= 1 - \sum_{i=1}^{N} \rho_i \; |\langle x_i, e_i \rangle|^2 \end{aligned}$$

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- *Quantum detection problem:* Given a unit normed set $\{x_i\}_{i=1}^N \subset H$ and positive weights $\{\rho_i\}_{i=1}^N$ that sum to 1. Construct a Parseval frame $\{e_i\}_{i=1}^N$ that minimizes

$$P_e = 1 - \sum_{i=1}^{N} \rho_i \, |\langle x_i, e_i \rangle|^2$$

over all N-element Parseval frames. ($\{e_i\}_{i=1}^N$ exists by a compactness argument.)

Naimark theorem

Naimark Theorem Let H be a d-dimensional Hilbert space and let $\{e_i\}_{i=1}^N \subset H$, $N \geq d$, be a Parseval frame for H. Then there exists an N-dimensional Hilbert space H' and an orthonormal basis $\{e_i'\}_{i=1}^N \subset H'$ such that H is a subspace of H' and

$$\forall i = 1, \ldots, N, \ \mathcal{P}_H e_i' = e_i,$$

where $\mathcal{P}_H$ is the orthogonal projection $H' \to H$.

- Given $\{x_i\}_{i=1}^N \subset H$ and a Parseval frame $\{e_i\}_{i=1}^N \subset H$. If $\{e_i'\}_{i=1}^N$ is its corresonding orthonormal basis for H', then, for all $i = 1, \ldots, N$, $\langle x_i, e_i \rangle = \langle x_i, e_i' \rangle$.

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- Minimizing P_e over all N-element Parseval frames for H is equivalent to minimizing P_e over all N-element orthonormal bases for H'.
- Thus we simplify the problem by minimizing P_e over all N-element orthonormal sets in H'.

Quantum detection error as a potential

- Treat the error term as a potential.

$$P = P_e = \sum_{i=1}^{N} \rho_i (1 - |\langle x_i, e_i' \rangle|^2) = \sum_{i=1}^{N} P_i.$$

where we have used the fact that $\sum_{i=1}^{N} \rho_i = 1$ and each

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- For $H' = \mathbb{R}^N$, we have the relation,

$$\|e_i' - x_i\|^2 = 2 - 2\langle x_i, e_i' \rangle$$

where we have used the fact that $\|e_i'\| = \|x_i\| = 1$. We can rewrite the potential P_i as

$$P_i = \rho_i \left(1 - \left[1 - \frac{1}{2} \|x_i - e_i'\|^2 \right]^2 \right).$$

A central force corresponds to quantum detection error

- Given P_i, define the function $p_i : \mathbb{R} \longrightarrow \mathbb{R}$ by

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- Thus P_i is a potential corresponding to a central force in the following way:

$$-xf_i(x) = p_i'(x) = 2\rho_i \left(1 - \frac{1}{2}x^2\right) x$$

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- Hence, the force $F_i = -\nabla P_i$ is

$$F_i(x_i, e_i') = f_i(\|x_i - e_i'\|)(x_i - e_i') = -2\rho_i \langle x_i, e_i' \rangle (x_i - e_i'),$$

a multiple of the frame force! The total force is given by

$$F = \sum_{i=1}^{N} F_i.$$

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- The equilibrium position of the points $\{e'_i\}_{i=1}^N$ is the position where all the forces produce no net motion. In this situation, the potential P is minimized.
- For the remainder, let $\{e'_i\}_{i=1}^N$ be an ONB for $\mathbb{R}^N$ that minimizes P. Recall that $\{e'_i\}_{i=1}^N$ exists by compactness. The *quantum detection problem* is to construct or compute $\{e'_i\}_{i=1}^N$.

A parameterization of $O(N)$

- Consider the orthogonal group

$$O(N) = \{\Theta \in GL(N, \mathbb{R}) : \Theta^\tau \Theta = I\}.$$

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- Since $O(N)$ is an $N(N-1)/2$-dimensional smooth manifold, we can locally parameterize $O(N)$ by $N(N-1)/2$ variables, i.e., $\Theta = \Theta(q_1, \ldots, q_{N(N-1)/2})$ for each $\Theta \in O(N)$.

  Hence, for all $\theta \in O(N)$ there is a surjective diffeomorphism b_θ

$$\begin{array}{lcl} & O(N) & \\ & \cup & \\ b_\theta : & \mathcal{U}_\theta & \longrightarrow \mathcal{U} \subset \mathbb{R}^{N(N-1)/2} \end{array}$$

  for relatively compact neighborhoods $\mathcal{U}_\theta \subseteq O(N)$ and $\mathcal{U} \subseteq \mathbb{R}^{N(N-1)/2}$, $\theta \in \mathcal{U}_\theta$.

A parameterization of ONBs

- Let $\{w_i\}_{i=1}^N$ be the standard ONB for $H' = \mathbb{R}^N$: $w_i = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}}}, 0, \ldots, 0)$.

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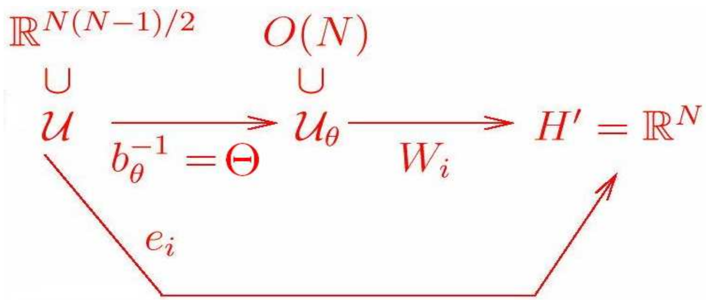
- Since any two orthonormal sets are related by an orthogonal transformation, we can smoothly parameterize an orthonormal set $\{e_i\}_{i=1}^N$ with N elements by $N(N-1)/2$ variables, i.e.,

$$\{e_i(q_1, \dots, q_{N(N-1)/2})\}_{i=1}^N = \{\Theta(q_1, \dots, q_{N(N-1)/2})w_i\}_{i=1}^N \subset H'.$$



where for all $\Psi \in O(N)$, $W_i(\Psi) = \Psi w_i$.

$$e_i(\vec{q}) = e_i(q_1, \dots, q_{N(N-1)/2}) = W_i \circ b_\theta^{-1}(\vec{q}) = (b_\theta^{-1}(\vec{q}))w_i \in \mathbb{R}^N.$$

Lagrangian dynamics on $O(N)$

- We now convert the frame force F acting on the orthonormal set $\{e_i\}_{i=1}^N$ into a set of equations governing the motion of the parameterization points $\vec{q}(t) = (q_1(t), \ldots, q_{N(N-1)/2}(t))$, see (1). We define the Lagrangian L and total energy E defined for $\vec{q}(t)$ by:

$$L = T - P_e, \quad E = T + P_e,$$

where

$$T = \frac{1}{2} \sum_{j=1}^{N(N-1)/2} \left(\frac{d}{dt} q_j(t) \right)^2 .$$

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- Using the Euler-Lagrange equations for the potential P_e

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

we obtain the equations of motion

$$\frac{d^2}{dt^2} q_j(t) = -2 \sum_{i=1}^{N} \rho_i \langle x_i, e_i(\vec{q}(t))\rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j}(\vec{q}(t)) \right\rangle . \tag{1}$$

Point of view

- Choose $\vec{q}\,' \in \mathbb{R}^{N(N-1)/2}$ such that $e_i(\vec{q}\,') = e_i' \in \mathbb{R}^N$ for all $i = 1, \ldots, N$.

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- Define $\tilde{q} : \mathbb{R} \to \mathbb{R}^{N(N-1)/2}$ such that $\tilde{q}(t) = \vec{q}\,'$ (a constant function).

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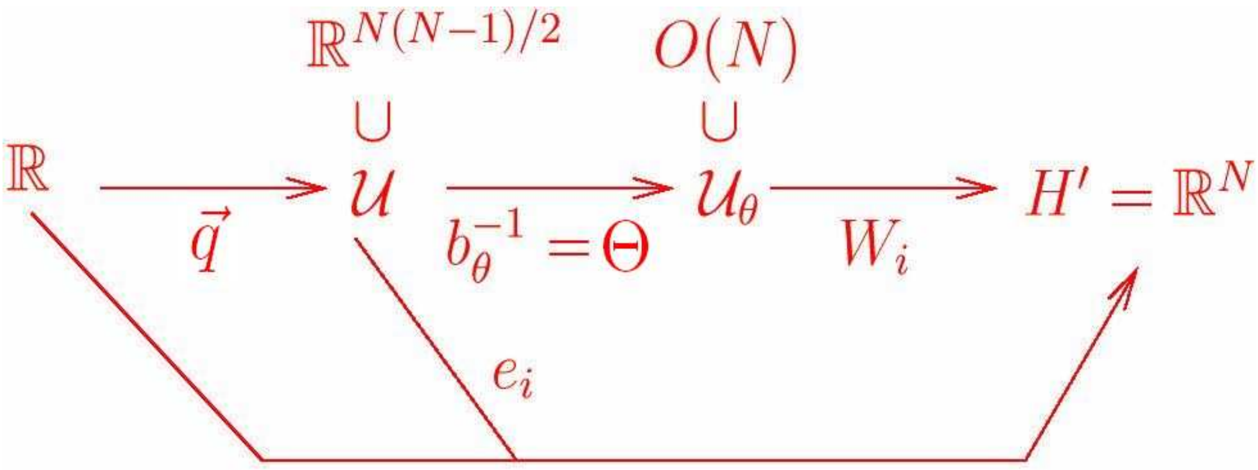
- Recall

$$(1) \qquad \frac{d^2}{dt^2} q_j(t) = -2 \sum_{i=1}^{N} \rho_i \langle x_i, e_i(\vec{q}(t)) \rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j}(\vec{q}(t)) \right\rangle .$$

Remark The definition of $\tilde{q}$ and equation (1) introduce t into play for solving the quantum detection problem.



Theorem Constant function $\tilde{q} : \mathbb{R} \to \mathbb{R}^{N(N-1)/2}$ is a minimum energy solution of (1).

Results

It can be shown that

- **Theorem** Denote by $\vec{q}(t) = (q_1(t), \ldots, q_{N(N-1)/2}(t))$ a solution of the equations of motion that minimizes the energy E and denote by $\mathcal{P}_H$ the orthogonal projection from H' into H. Then $\vec{q}(t)$ is a constant solution and the set of vectors

$$\{\mathcal{P}_H e_i(\vec{q}(t))\}_{i=1}^{N} \subset H$$

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- **Theorem** A minimum energy solution, a minimizer of P_e, satisfies the expression

$$\sum_{i=1}^N \rho_i \langle x_i, e_i \rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j} \right\rangle = 0.$$

Numerical problems

- The use of Lagrangia provides a point of view for computing the TF minimizers of P_e. (Some independent, direct calculations are possible (Kebo), but not feasible for large values of d and N.)

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$$\sum_{i=1}^{N} \rho_i \langle x_i, e_i \rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j} \right\rangle .$$

- With the parameterization of $SO(N)$, the error P_e is a smooth function of the variables $(q_1, \ldots, q_{N(N-1)/2})$, that is,

$$P_e(q_1, \ldots, q_{N(N-1)/2}) = 1 - \sum_{i=1}^{N} \rho_i \left| \langle x_i, e_i(q_1, \ldots, q_{N(N-1)/2}) \rangle \right|^2 .$$

A conjugate gradient method can be used to find the minimum values of P_e.

Analytical methods

- **Problem:** Let $p = \{p_k\}_{k\in \mathcal{Z}}$ be positive definite, *i.e.*, for any finite set $F \subseteq \mathbb{Z}$ and any $\{c_j\}_{j\in F} \subseteq \mathbb{C}$:

$$\sum_{j,k\in F} c_j \bar{c}_k p(j-k) \geq 0$$

  Suppose $p = 0$ on a given $F \subseteq \mathbb{Z}$. When can we construct unimodular $u : \mathbb{Z} \to \mathbb{C}$ such that:

$$p(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|j|\leq N} u(j+k)\overline{u(j)}?$$

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- This is the same problem for $\mathbb{Z}$ that we have been addressing for $\mathbb{Z}_N$ in the one-dimensional CAZAC case.

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- Finite approximation and software as with algebraic CAZACs.

That's all folks!