MRI SIGNAL RECONSTRUCTION BY FOURIER FRAMES ON INTERLEAVING SPIRALS

John J. Benedetto[*], *Alexander M. Powell*

University of Maryland
Department of Mathematics
College Park, MD 20742

Hui-Chuan Wu

Digital Access Corporation
2359 Research Court
Woodbridge, VA 22192

ABSTRACT

Interleaving spirals are a natural setting for attaining fast MRI signal reconstruction. Using results of Beurling and Landau, as well as quantitative coverings of the spectral domain by translates of the polar set of the target disk space E, harmonics for Fourier frames F are constructed on interleaving spirals to reconstruct signals on E. Because of weak Fourier frame bound estimates by standard calculations, implementation is addressed in terms of £nite frame approximants of F.

1. INTRODUCTION

We address the problem of imaging speed in the data acquisition process of MRI. In particular we consider spiral-scan echo planar imaging (SEPI) for a given closed spatial disk $B(0, R)$, of center 0 and radius R in the plane $\mathbb{R}^2$, and a given Archimedean spiral A_c in the spectral plane $\widehat{\mathbb{R}}^2$. We construct a sequence of points Λ contained in A_c and its interleaves so that every signal f de£ned on $B(0, R)$ can be reconstructed as a linear combination $f = \sum_{\lambda \in \Lambda} a_\lambda(f) e_\lambda$, where $e_\lambda(x) = e^{2\pi i x \cdot \lambda}$. In this situation f is typically a spin density distribution ρ.

Section 2 describes SEPI, beginning with the imaging equation for MR in which the NMR (nuclear magnetic resonance) signal $S(t)$ is obtained by integrating the solution of Bloch's differential equation. The phenomenon of NMR was discovered independently by Felix Bloch and Edward Purcell, see [1] page 13 for historical comments (the word nuclear gives the false impression that nuclear material is used). In Section 3 we provide the mathematical background for our construction of Λ. This includes the theory of frames and a fundamental condition for Fourier frames due to Beurling. We also have an alternative parallel approach depending on a multidimensional version of Kadec's suf£cient condition for Riesz bases in the Fourier frame case.

Using the results from Section 3, we provide the construction of Λ in Section 4. We close in Section 5 with a brief description of the implementation of our construction. The complete implementation and an evaluation of its effectiveness and speed are due to the £rst two authors and Alfredo Nava-Tudela and Yang B. Wang.

2. SPIRAL-SCAN ECHO PLANAR IMAGING (SEPI)

A standard MR imaging equation is a consequence of Felix Bloch's equation for transverse magnetization M_{xy} in the presence of a linear magnetic £eld gradient [1] pages 269-270. In fact, an *MR signal* $S(t)$ is the integration of M_{xy}; and the corresponding imaging equation is

$$\begin{aligned} S(t) &= S(k(t)) = S(k_x(t), k_y(t), k_z(t)) \qquad (1) \\ &= \int\int\int \rho(x, y, z) \exp[-2\pi i < (x, y, z), \\ &\qquad (k_x(t), k_y(t), k_z(t)) >] e^{-t/T_2} dx\, dy\, dz \end{aligned}$$

[2], [3], [1] pages 269-270, [4] Section 16.2 (p344). $S(t)$ is also referred to as an *echo* or FID (free induction decay) and can be measured for the sake of imagining. (1) is a natural physical Fourier transform associated with magnetization, analagous to the natural physical wavelet transform effected by the behavior of the basilar membrane within certain frequency ranges, e.g., [5].

The parameters, variable, and inputs in (1) are the following:

$$k_x(t) = \gamma \int_0^t G_x(u) du \qquad (2)$$

and $G_x(u)$ is an x-directional time varying gradient with similar de£nitions for the y and z variables, T_2 is the transverse relaxation time, the exponential term e^{-t/T_2} representing the T_2 decay appears as a limiting factor in echo planar imaging [6], γ is the gyromagnetic ratio, and $\rho(\overrightarrow{r}) = \rho(x, y, z) = \rho(\overrightarrow{r}, T_2)$ is the spatial spin density distribution from which the spin density image is reconstructed.

Since $S(t)$ is a measurable quantity in the MR process and since precise knowledge of $\rho(x, y, z)$ is desired, it is

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natural to compute the inverse Fourier transform of S, properly adapted to the format in (1). Because of significant issues which arise and goals which must be addressed, the inversion process has to be treated carefully. In particular, there is a significant role for the time-varying gradients. First, the gradients are inputs to the process, and must be designed theoretically in order to be realizable and goal oriented. Once the gradients have been constructed, the imaging data $S(t)$ at time t is really of the form $S(k(t))$ as seen in (1) and (2); and it is usual to refer to the "spectral" domain of S as k-space.

Example 1. Let

$$G_x(t) = \eta \cos \xi t - \eta \xi t \sin \xi t$$

$$G_y(t) = \eta \sin \xi t + \eta \xi t \cos \xi t.$$

By the definition of k_x, k_y, see (2), we compute $k_x(t) = \gamma \eta t \cos \xi t$ and $k_y(t) = \gamma \eta t \sin \xi t$. Combining k_x and k_y we obtain the Archimedean spiral

$$A_c = \{(c\theta \cos 2\pi\theta, c\theta \sin 2\pi\theta) : \theta \geq 0\} \subseteq \widehat{\mathbb{R}}^2,$$

where $\theta = \theta(t) = (1/2\pi)\xi t$ and $c = (1/\theta)\gamma\eta = 2\pi\gamma\eta/\xi$. This idea for formulating time domain gradient pulse forms is due to Ljunggren [7]. They clearly generate a spiral scan in the k-domain and are not difficult to realize.

Remark 1. a. The echo planar imaging (EPI) method, developed by Mansfeld (1977) [1] page 306, theoretically and usually provides high speed data acquisition within the time interval of a few hundreths of seconds. The method utilizes multiple echos by fast gradient alternation. As such, realizable gradient design giving rise to large high speed gradient fields is essential. A solution to this design problem has to be coupled with controlling spatial resolution limits imposed by the T_2-decay in (1), e.g., [1] pages 314-315.

A weakness of this technique as originally fomulated is that the alternating gradient to be applied is a series of rectangular pulses which are difficult to generate for high gradient power and frequency, see [6] pages 2-3 for this and a fuller critique.

b. SEPI ensures rapid scanning for fast data acquisition as well as *relatively* uniform convergence of the entire frequency space of interest after completion of the scan. Spiral scanning simplifies the scanning of data in radial directions once the span is completed. In this regard the inherent T_2 effect appears as an almost circular blurring which is preferable to the one-dimensional blurs observed in earlier EPI. Further, there is a reduction in transient and steady state distortion , since SEPI eliminates the discontinuities of gradient waveforms which arise in uniform rectilinear scanning that proceeds linearly around corners while transversing k-space, e.g., [6], [1].

Interleaving spiraling from rapid spiral scans proceeds from dc levels to high frequencies. Such multiple pulsing can be implemented in SEPI due to its locally circular symmetry property in data acquisition, and the resulting interleaving spirals yield high resolution imaging when accompanied by effective k-space sampling and reconstruction methods, see Section 3-5, cf., [8]. In fact, interleaving spiral scans not only improve k-space sampling strategies, but they also overcome the gradient requirement and T_2-decay limitations for standard EPI.

c. EPI and SEPI are both very fast in terms of image data acquisition but the off resonance and flow properties of the two methods differ; and, in fact, total scan time spiral imaging requires lower gradient power than EPI, e.g., [9]. Further, SEPI has more significantly reduced artifact intensities than the 2-dimensional FFT since its spiral trajectories collect low spatial frequencies with every view; and it also seems to be superior vis a vis motion insensitivity, see [10], [11].

3. MATHEMATICAL BACKGROUND

Let $\mathbb{R}^d$ be d-dimensional Euclidean space, and let $\widehat{\mathbb{R}}^d$ denote $\mathbb{R}^d$ when it is considered as the domain of the Fourier transforms of signals defined on $\mathbb{R}^d$. The *Paley-Wiener* space PW_E is

$$PW_E = \left\{ \varphi \in L^2(\widehat{\mathbb{R}}^d) : \operatorname{supp} \varphi^\vee \subseteq E \right\},$$

where $E \subseteq \mathbb{R}^d$ is closed, $L^2(\widehat{\mathbb{R}}^d)$ is the space of finite energy signals φ on $\widehat{\mathbb{R}}^d$, i.e.,

$$\|\varphi\|_{L^2(\widehat{\mathbb{R}}^d)} = \left(\int_{\widehat{\mathbb{R}}^d} |\varphi(\gamma)|^2 \, d\gamma \right)^{1/2} < \infty,$$

$\varphi^\vee$ is the inverse Fourier transform of φ defined as

$$\varphi^\vee(x) = \int_{\widehat{\mathbb{R}}^d} \varphi(\gamma) e^{2\pi i x \cdot \gamma} d\gamma,$$

and $\operatorname{supp} \varphi^\vee$ denotes the support of $\varphi^\vee$.

Definition 1. a. Let H be a separable Hilbert space. A sequence $\{x_n : n \in \mathbb{Z}^d\} \subseteq H$ is a *frame* for H if there exist constants $0 < A \leq B < \infty$ such that

$$\forall y \in H, \quad A\|y\|^2 \leq \sum |\langle y, x_n \rangle|^2 \leq B\|y\|^2.$$

There is an extensive literature on frames, e.g., see [12], [13].

b. Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a sequence, and let $E \subseteq \mathbb{R}^d$ be a closed set having finite Lebesgue measure. It is easy to see that $\{e_{-\lambda} : \lambda \in \Lambda\}$ is a frame for the Hilbert space $L^2(E)$ if and only if there exist $0 < A \leq B < \infty$ such that

$$\forall \varphi \in PW_E, \quad A\|\varphi\|_{L^2(\widehat{\mathbb{R}}^d)} \leq \sum_{\lambda \in \Lambda} |\varphi(\lambda)|^2 \leq B\|\varphi\|_{L^2(\widehat{\mathbb{R}}^d)}.$$

In this case, and because of this equivalence, we say that Λ is a *Fourier frame* for PW_E.

Recall that a set Λ is *uniformly discrete* if there is $r > 0$ such that

$$\forall \lambda, \gamma \in \Lambda, \quad |\lambda - \gamma| \geq r,$$

where $|\lambda - \gamma|$ is the Euclidean distance between λ and γ.

Beurling proved the following theorem for the case that E is the closed ball $B(0, R) \subseteq \mathbb{R}^d$ centered at $0 \in \mathbb{R}^d$ and with radius R.

Theorem 1. Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be uniformly discrete, and de£ne

$$\rho = \rho(\Lambda) = \sup_{\zeta \in \widehat{\mathbb{R}}^d} \operatorname{dist}(\zeta, \Lambda),$$

where $\operatorname{dist}(\zeta, \Lambda)$ is the Euclidean distance between the point ζ and the set Λ. If $R\rho < 1/4$, then Λ is a Fourier frame for $PW_{B(0,R)}$.

By the de£nition of Fourier frame the assertion of Beurling's theorem is that every £nite energy signal f de£ned on E has the representation

$$f(x) = \sum_{\lambda \in \Lambda} a_\lambda(f) e^{2\pi i x \cdot \lambda} \tag{3}$$

in L^2-norm on E, where $\sum_{\lambda \in \Lambda} |a_\lambda(f)|^2 < \infty$. Beurling used the term "set of sampling" instead of "Fourier frame".

In theory, for the case $E = B(0, R)$, we can not expect to construct either tight or exact Fourier frames for the spiral in Section 4, see [14].

It is possible to make a quantitative comparison between Kadec's $\frac{1}{4}$ theorem and Beurling's theorem. For now we provide the following remark which shows that the construction of Section 4 can also be achieved by use of Kadec's theorem.

Remark 2. a. Kadec (1964) proved that *if* $\Lambda = \{\lambda_m : m \in \mathbb{Z}\} \subseteq \widehat{\mathbb{R}}$ *and* $\sup_{m \in \mathbb{Z}} |\lambda_m - \frac{m}{2R}| < 1/4$, *then* Λ *is a Riesz basis for* $PW_{[-R,R]}$. This means that $\{e^{2\pi i \lambda_m / R}\}$ is an exact frame for $PW_{[-R,R]}$, which, in turn, means it is a bounded unconditional basis for $PW_{[-R,R]}$ or, equivalently, that it is a frame which ceases to be a frame if any of its elements is removed, e.g., [12].

b. There have been re£nements and generalizations of Kadec's theorem which are relevant to our discussion, e.g., Sun and Zhou [15]. In fact, given $R > 0$, the Sun and Zhou result gives rise to exact frames for $L^2([-R, R]^d)$ which become frames for $L^2(B(0, R))$. For $d = 2$, the corresponding set $\Lambda \subseteq \widehat{\mathbb{R}}^2$ can be chosen on interleaves of a given spiral $A \subseteq \widehat{\mathbb{R}}^2$. This allows us to proceed as in section 4, replacing the application there of Beurling's theorem by a multidimensional Kadec theorem.

4. FOURIER FRAMES ON INTERLEAVING SPIRALS

We shall now use Beurling's theorem stated in Section 3 to give a constructive non-uniform sampling signal reconstruction method in the setting of spirals and their interleaves. The method is much more general than the geometry of interleaving spirals.

For the case of spirals there are three cases: given an Archimedean spiral A, to show there is $R > 0$, generally small, and a Fourier frame $\Lambda \subseteq A$ for $PW_{B(0,R)}$ (the calculation for this case uses techniques from the following case); given an Archimedean spiral A and $R > 0$, to show there are £nitely many interleaves of A and a Fourier frame Λ, contained in their union, for $PW_{B(0,R)}$ (Example 2); given $R > 0$, we note there is an Archimedean spiral A and a Fourier frame $\Lambda \subseteq A$ for $PW_{B(0,R)}$ (Example 3).

Example 2. Given *any* $R > 0$ and $c > 0$. We shall show how to construct a £nite interleaving set $B = \cup_{k=1}^{M-1} A_k$ of spirals

$$A_k = \left\{ c\theta e^{2\pi i(\theta - k/M)} : \theta \geq 0 \right\}, \quad k = 0, 1, \ldots, M-1,$$

and a uniformly discrete set $\Lambda_R \subseteq B$ such that Λ_R is a Fourier frame for $PW_{B(0,R)}$. Thus, all of the elements of $L^2(B(0,R))$ will have a decomposition in terms of the Fourier frame $\{e_\lambda : \lambda \in \Lambda_R\}$.

We begin by choosing M such that $cR/M < 1/2$. We shall write any given $\xi_0 \in \widehat{\mathbb{R}}^2$ as $\xi_0 = r_0 e^{2\pi i \theta_0}$, where $r_0 \geq 0$ and $\theta_0 \in [0, 1)$. Then either $0 \leq r_0 < c\theta_0 < c$ or there is $n_0 \in \mathbb{N} \cup \{0\}$ for which

$$c(n_0 + \theta_0) \leq r_0 < c(n_0 + 1 + \theta_0).$$

In this latter case, we can £nd $k \in \{0, \ldots, M-1\}$ such that

$$c(n_0 + \theta_0 + \frac{k}{M}) \leq r_0 < c(n_0 + \theta_0 + \frac{k+1}{M}).$$

Thus,

$$\operatorname{dist}(\xi_0, B) \leq \frac{c}{2M}.$$

Next, we choose $\delta > 0$ such that $R\rho < 1/4$, where $\rho = (c/2M + \delta)$.

Then, for each $k = 0, 1, \ldots, M-1$, we choose a uniformly discrete set Λ_k of points along the spiral A_k having curve distance between consecutive points less than 2δ, and beginning within 2δ of the origin. The curve distance, and consequently the ordinary distance, from any point on the spiral A_k to Λ_k is less than δ. Finally, we set $\Lambda_R = \cup_{k=0}^{M-1} \Lambda_k$. Thus, by the triangle inequality,

$$\begin{aligned} \forall \xi \in \widehat{\mathbb{R}}^2, \quad \operatorname{dist}(\xi, \Lambda_R) &\leq \operatorname{dist}(\xi, B) + \operatorname{dist}(B, \Lambda_R) \\ &\leq \frac{c}{2M} + \delta = \rho. \end{aligned}$$

Hence, $R\rho < 1/4$ by our choice of M and δ; and so we invoke Beurling's theorem to conclude that Λ_R is a Fourier frame for $PW_{B(0,R)}$.

Example 3. Note that since we are reconstructing signals on a space domain having area about R^2, we require essentially R interleaving spirals. On the other hand, if we are allowed to choose the spiral(s) after we are given $PW_{B(0,R)}$, then we can choose Λ_R contained in a *single* spiral A_c for $c > 0$ small enough.

5. IMPLEMENTATION

We close by indicating our setup for implementing the construction of Section 4, see [16] for implementation.

Given $N > 0$, e.g., $N = 256$, let $f \in L^2([-\frac{1}{2}, \frac{1}{2}]^2)$ be considered an element of $L^2(\mathbb{R}^2)$ vanishing outside of $[-\frac{1}{2}, \frac{1}{2}]^2$; and further assume that f is piecewise constant taking the value $f_{m,n} \in \mathbb{C}$ for all $x \in (-\frac{1}{2}, \frac{1}{2}) + [\frac{m}{N}, \frac{m+1}{N+1}) \times [\frac{n}{N}, \frac{n+1}{N})$, where $(-\frac{1}{2}, \frac{1}{2}) \in \widehat{\mathbb{R}}$ and $m, n \in \{0, 1, \cdots, N-1\}$. For convenience, we write f lexicographically as $\{f_{a_k} : k = 0, 1, \cdots, N^2 - 1\}$. The Fourier transform of $f \in L^2(\mathbb{R}^2)$ is

$$\widehat{f} = \sum_{n=0}^{N^2-1} f_{a_n} H_{a_n},$$

where $H_{m,n}(\lambda, \gamma) = \frac{-1}{4\pi^2 \lambda\gamma} e(\frac{m\lambda + n\gamma}{N})(e(\frac{\lambda}{N}) - 1)(e(\frac{\gamma}{N}) - 1)$ and $e(\lambda) = e^{-2\pi i \lambda}$.

We designate $\alpha = (\lambda, \gamma) \in \widehat{\mathbb{R}}^2$, choose α_k, $k = 0, 1, 2, \cdots, M-1$, for $M \geq N^2$, and de£ne the $M \times N^2$ matrix $L = (H_{a_n}(\alpha_m))$. With the appropriate choices based on Section 4, L is a Bessel map and $S = L^* L$, an $N^2 \times N^2$ matrix where L^* is the adjoint of L, is the associated frame operator. As such, the £nite frame decomposition of f, corresponding to the Fourier frame signal reconstruction method of Section 3 and 4, is $f = S^{-1} L^*(Lf)$.

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