MATH 141, Review Sheet on Trig Substitutions

Prof. Jonathan Rosenberg

October 17, 2012

The following table summarizes what subsitution to use, depending on what appears in the integrand.

combination in integrand	**substitution**	dx	u **as a function of** x
$\sqrt{a^2 - x^2} = a\ \cos u$	$x = a\ \sin u$	$dx = a\ \cos u\, du$	$u = \sin^{-1}(x/a)$
$\sqrt{a^2 + x^2} = a\ \sec u$	$x = a\ \tan u$	$dx = a\ \sec^2 u\, du$	$u = \tan^{-1}(x/a)$
$\sqrt{x^2 - a^2} = a\ \tan u$	$x = a\ \sec u$	$dx = a\ \sec u \tan u\, du$	$u = \sec^{-1}(x/a)$

We skipped hyperbolic functions (section 7.4 in Ellis and Gulick), but they provide an optional alternative for the last two rows of the above table, since $\sinh^2 u + 1 = \cosh^2 u$. So if you prefer to use hyperbolic functions, the following substitutions are also recommended:

combination in integrand	**substitution**	dx	u **as a function of** x
$\sqrt{a^2 + x^2} = a\ \cosh u$	$x = a\ \sinh u$	$dx = a\ \cosh u\, du$	$u = \sinh^{-1}(x/a)$
$\sqrt{x^2 - a^2} = a\ \sinh u$	$x = a\ \cosh u$	$dx = a\ \sinh u\, du$	$u = \cosh^{-1}(x/a)$

Sometimes, depending on your choice of substitution, the answer can appear to vary quite a bit (though all the answers are equivalent). Example:

$$\begin{aligned}
\int \sqrt{a^2+x^2}\,dx &= \text{(with substitution } x = a\tan u\text{)} \\
&= \int (a\ \sec u)(a\ \sec^2 u\,du) = a^2 \int \sec^3 u\,du \\
&\text{(by the method of Ellis and Gulick, p. 518)} \\
&= \frac{a^2}{2}\Big(\sec u \tan u + \ln|\sec u + \tan u|\Big) + C \\
&= \frac{a^2}{2}\Big(\frac{x\sqrt{a^2+x^2}}{a^2} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\Big) + C.
\end{aligned}$$

$$\begin{aligned}
\int \sqrt{a^2+x^2}\,dx &= \text{(with substitution } x = a\sinh u\text{)} \\
&= \int (a\ \cosh u)(a\ \cosh u\,du) = a^2 \int \cosh^2 u\,du \\
&= \frac{a^2}{2}\int (1+\cosh 2u)\ du \\
&= \frac{a^2}{2}\Big(u + \frac{1}{2}\sinh 2u\Big) + C \\
&= \frac{a^2}{2}\Big(u + \sinh u \cosh u\Big) + C \\
&= \frac{a^2}{2}\Big(\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\Big) + C.
\end{aligned}$$