Gauged Linear Sigma Model

Tristan Hübsch
Department of Physics and Astronomy
Howard University, Washington DC

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Gauged Linear σ Model

The truth, nothing but the unvarnished truth,
…but by all means — not *all* of it!

Playbill

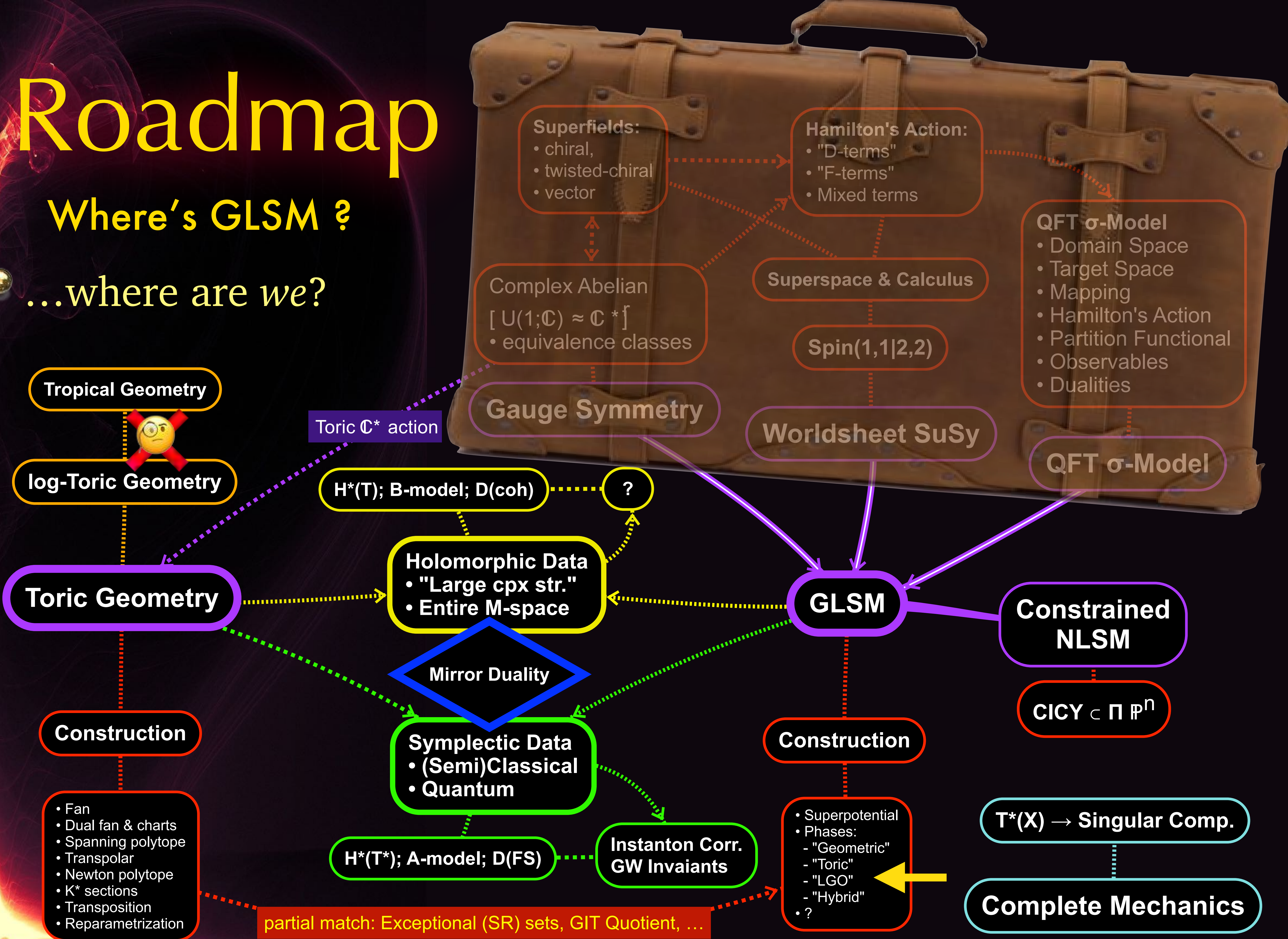
- Roadmap
- QFT σ Models
- Worldsheet SuSy
- GLSM Action
- GLSM → Toric Geometry
- Mirror Symmetry
- Mirror Questions
- …and Amoebas?

Many thanks to Per Berglund and Paul S. Green for many helpful discussions

Roadmap

Where's GLSM ?

…where are *we*?



QFT σ Models

A Bird's-Eye View

in cl.mech.: $\mathbb{R}^1_\tau$

- Domain space: Riemann surface, Σ_g, locally $\sim\mathbb{R}^{1,1}$ w/BC's
- Target space: Lorentzian space(time), such as $\mathbb{R}^{1,9}$ — the target will vary
- Mapping: "coordinate fields," $X^\mu(\xi):\Sigma_g\to\mathbb{R}^{1,9}$
- Hamilton's *action*: "~~energy~~": $S:=\int_{\Sigma_g}L[X;\gamma_{ij}(\xi),G_{\mu\nu}(X),\dots]$
- Classical physics: $\delta S[X;\gamma_{ij}(\xi),G_{\mu\nu}(x),\dots]=0\leftrightarrow$ Euler-Lagrange EoM
- Quantum fluctuations s.t. $\delta S\neq 0$, "deform" $\{\gamma_{ij}(\xi),G_{\mu\nu}(x),\dots\}$
  - Note: Σ_g is not actually *observable* must "sum" over them
  - except for data at "initial" and "final" points $\to\sum_{g=0}^{\infty}\int_{\mathscr{M}_{g;\{\xi\}_i,\{\xi\}_f}}[\cdots]$
  - Whence Feynman's "path integral"

$$Z[G_{\mu\nu},\dots]:=\iint_{X^\mu(\xi):\,\Sigma_{g,\gamma}\to\mathscr{X}}\mathbf{D}[X]\,e^{-iS[X;\gamma_{ij},G_{\mu\nu},\dots]/\hbar}$$

QFT σ Models

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- Domain space: Riemann surface, Σ_g, locally $\sim\mathbb{R}^{1,1}$ w/BC's
- Target space: Lorentzian space(time), ~~such as~~
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"dynamically" determined target space

- Hamilton's *action*: "~~energy~~": $S:=\int_{\Sigma_g}L[X;\gamma_{ij}(\xi),G_{\mu\nu}(X),\dots]$
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Better strategy: specify $S[X;\dots]$ by symmetries & "analytic" properties

QFT σ Models

QFT σ-Model
- Domain Space
- Target Space
- Mapping
- Hamilton's Action
- Partition Functional
- Observables
- Dualities

QFT σ-Model

A Bird's-Eye View

$$Z[G_{\mu\nu},\dots] := \iint \mathbf{D}[X]\, e^{-iS[X;\gamma_{ij},G_{\mu\nu},\dots]/\hbar}$$

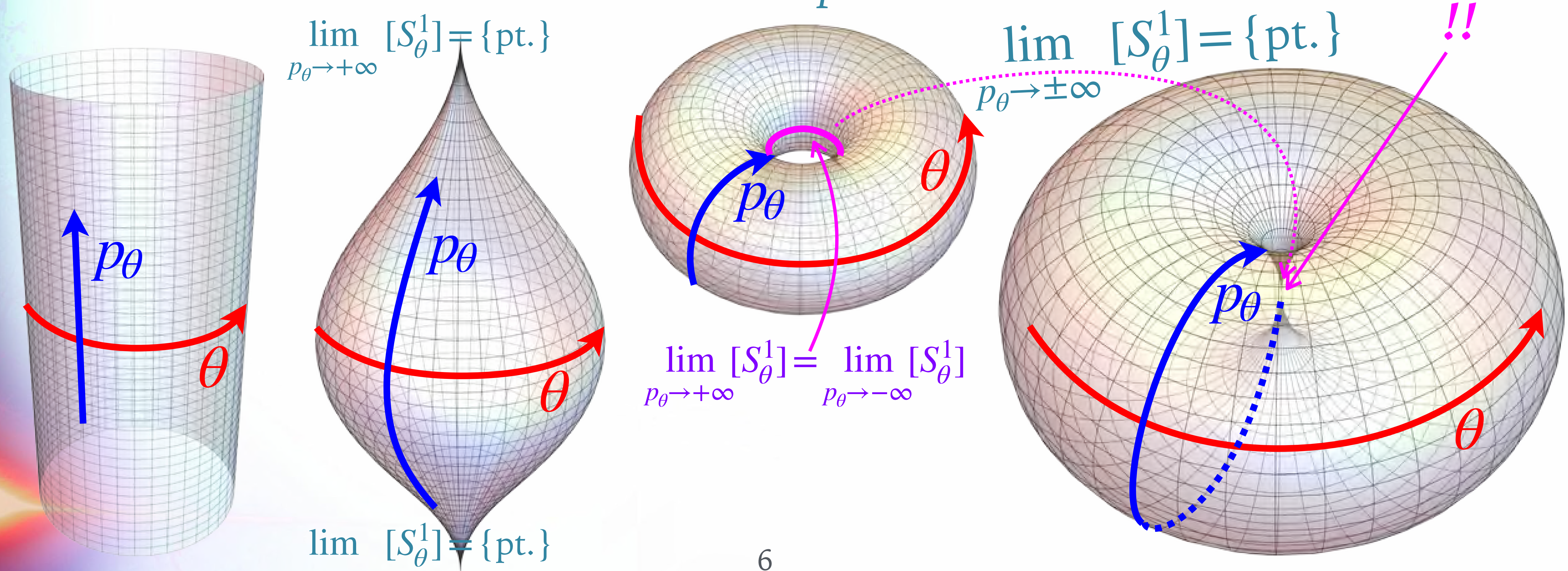
- Generally, $S := \int_{\Sigma_g} L[X;\gamma_{ij}(\xi), G_{\mu\nu}(X),\dots]$
  - has $L[X;\gamma_{ij},G_{\mu\nu},\dots] \propto \gamma^{ij}\frac{\partial X^\mu}{\partial\xi^i}G_{\mu\nu}\frac{\partial X^\nu}{\partial\xi^j}+\dots$: $P_\mu := \frac{\partial L}{\partial(\partial_\tau \xi^j)}$ "conjugate mom." (proper time)
  - The $\Phi := (X^\mu, P_\nu)$ Darboux coordinates ($\{X^\mu, P_\nu\}_{\text{PB}} = \delta^\mu_\nu$) on $T^*(\mathscr{X})$
  - <u>Classical</u> "observables" = real functions over the "phase space," Φ
    - Form an "algebra": $\alpha_1 A + \alpha_2 B = \sum_i \alpha_i C_i$ & $\{A,B\}_{\text{PB}} := \frac{\partial A}{\partial X^\mu}\frac{\partial B}{\partial P_\mu} - \frac{\partial B}{\partial X^\mu}\frac{\partial A}{\partial P_\mu} = \sum_i \alpha'_i C_i$
- "<u>Quantization</u>": "polarization" = "$\mathscr{X}$"$\subset\Phi$ max. PB-Lagrangian
  - and $P_\mu \mapsto \frac{\hbar}{i}\frac{\partial}{\partial X^\mu}$, $A(X,P)\mapsto \hat{A}\big(X, \frac{\hbar}{i}\frac{\partial}{\partial X^\mu}\big)$ & $\{A,B\}_{\text{PB}} \mapsto \frac{1}{i\hbar}([\hat{A},\hat{B}] := \hat{A}\hat{B}-\hat{B}\hat{A})$
  - Ambiguity → "normal ordering," … ⇒ "quantization" = "$\xrightarrow{1-n}$" (inv. img.)
  - In $Z[G_{\mu\nu},\dots] := \iint \mathbf{D}[X]\, e^{-iS[X;\gamma_{ij},G_{\mu\nu},\dots]/\hbar}$, "field redefinitions" & integration are generally "1–1," but "BC" easily make this …*subtle*…

QFT σ Models

A Bird's-Eye View

$$Z[G_{\mu\nu}, \ldots] := \iint \mathbf{D}[X]\, e^{-iS[X;\gamma_{ij}, G_{\mu\nu}, \ldots]/\hbar}$$

- Generally, $S := \int_{\Sigma_g} L[X; \gamma_{ij}(\xi), G_{\mu\nu}(X), \ldots]$
  - where $L[X;\gamma_{ij}, G_{\mu\nu}, \ldots] \propto \gamma^{ij} \frac{\partial X^\mu}{\partial \xi^i} G_{\mu\nu} \frac{\partial X^\nu}{\partial \xi^j} + \ldots$: $P_\mu := \frac{\partial L}{\partial(\partial_\tau \xi^j)}$ "conjugate mom."
  - The (X^μ, P_ν) Darboux coordinates ($\{X^\mu, P_\nu\}_{\text{PB}} = \delta^\mu_\nu$) on $T^*(\mathscr{X})$ *…locally!!*
  - Particle on a circle… *what could be simpler?*



QFT σ Models

A Bird's-Eye View

$$Z[G_{\mu\nu},\ldots] := \iint \mathbf{D}[X]\, e^{-iS[X;\,\gamma_{ij},\,G_{\mu\nu},\ldots]/\hbar}$$

- If $Z[G_{\mu\nu},\ldots] = e^{-iS_{\text{eff}}[\bar{x};\,\bar{\gamma}_{ij}(\xi),\,\widetilde{G}_{\mu\nu}(x),\ldots]/\hbar}$, "renormalized" $G \to \widetilde{G}$
  - "Renormalization" is computed iteratively → iterations = "flow"
  - "Renormalization ~~group~~ flow" has fixed points → "quantum stability"
  - 1979, D. Friedan: 1st order quantum stability → Einstein Eq.'s for $G_{\mu\nu}$
    42 years ago!
    - Subsequently generalized, reproduce all "gauge interactions"
    - Average "vev"s: $x^\mu := \langle X^\mu \rangle := \iint \mathrm{D}[X]\, X^\mu(\sigma)\, e^{-iS[X;\,\gamma_{ij},\,G_{\mu\nu},\ldots]/\hbar}$ = target-space coordinates

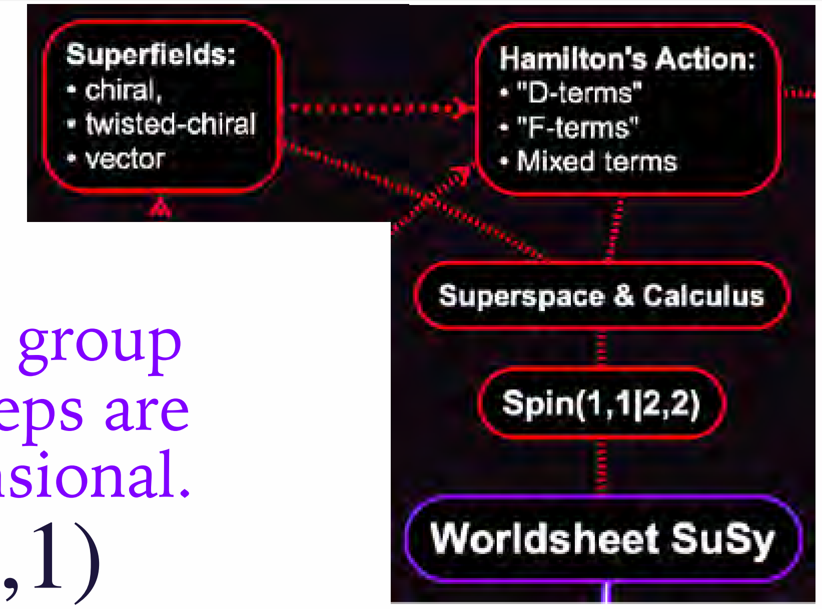
Ehrenfest Theorem

- Renormalizability in 3+1D QFTs strongly restricts $S[\Psi(x);\, A_\mu(x),\,\ldots]$.
  - …tends to be less restrictive for lower-dimensional domain space
  - …is much better behaved in supersymmetric models

e.g., $x^\mu := \langle X^\mu(\sigma) \rangle$ are "0-modes"

Worldsheet SuSy

A Telegraphic Summary

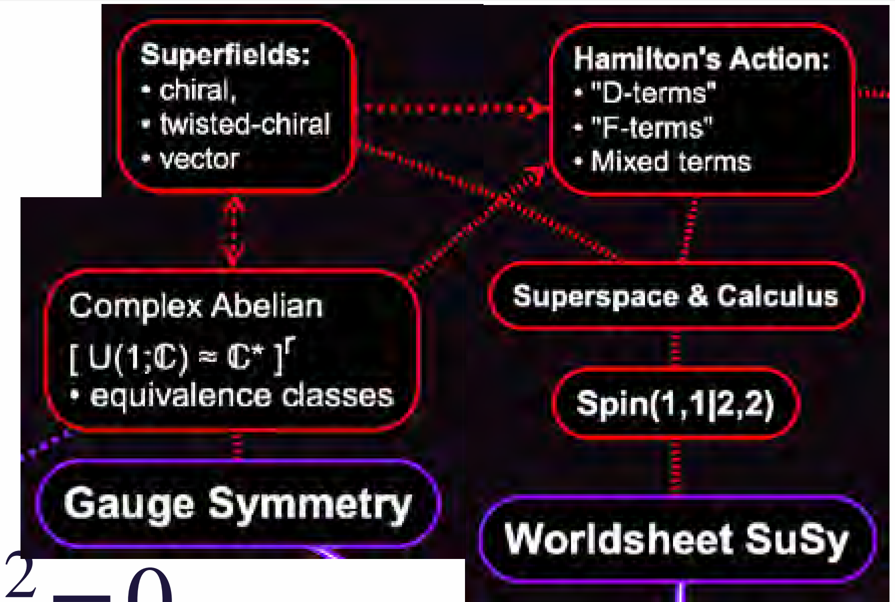


Abelian group ∴ all irreps are 1-dimensional.

- Worldsheet Lorentz group SO(1,1) $\xleftarrow{1\leftarrow 2}$ Spin(1,1)
  - "Tensors" (scalars, vectors, … "*spin*"-n) are faithful representations of SO(1,1)
  - "spinors" ("*spin*"-$\frac{2n+1}{2}$) are not faithful representations of SO(1,1)
  - Extend the worldsheet to a (2,2)-super-Riemann surface
    - so that $(\xi^{++},\xi^{--})\to(\xi^{++},\xi^{--}\,|\,\varsigma^{\pm},\bar\varsigma^{\pm})$ where $\Delta=\partial_{++}\partial_{--}\equiv\frac{\partial}{\partial\xi^{++}}\frac{\partial}{\partial\xi^{--}}$ & $\varsigma^2=0$
    - Def.: $D_{\pm}:=\frac{\partial}{\partial\varsigma^{\pm}}+i\bar\varsigma^{\mp}\partial_{\pm\pm}$ & $\bar D_{\pm}:=\frac{\partial}{\partial\bar\varsigma^{\pm}}+i\varsigma^{\mp}\partial_{\pm\pm}$; $\{D_{\pm},\bar D_{\pm}\}=2i\partial_{\pm\pm}$ "L/R v. fields"
    - Also: $Q_{\pm}:=i\frac{\partial}{\partial\varsigma^{\pm}}+\bar\varsigma^{\mp}\partial_{\pm\pm}$ & $\bar Q_{\pm}:=i\frac{\partial}{\partial\bar\varsigma^{\pm}}+\varsigma^{\mp}\partial_{\pm\pm}$; $\{Q_{\pm},\bar Q_{\pm}\}=2i\partial_{\pm\pm}$ & $\{Q,D\}=0$
    - Note: $Q_{\pm}:=iD_{\pm}+2\bar\varsigma^{\mp}\partial_{\pm\pm}$ & $\bar Q_{\pm}:=i\bar D_{\pm}+2\varsigma^{\mp}\partial_{\pm\pm}$,
  - F. Berezin integration: $\int d\varsigma^{\pm}[\cdots]=[\frac{\partial}{\partial\varsigma^{\pm}}[\cdots]]\equiv[[\underbrace{\frac{\partial}{\partial\varsigma^{\pm}}+i\bar\varsigma^{\mp}\partial_{\pm\pm}}_{D_{\pm}}][\cdots]]_{\varsigma=0}$
    - So, $\int_{\Sigma}\int d^4\varsigma[\cdots]=\int_{\Sigma}[D^4[\cdots]]_{\varsigma,\bar\varsigma=0}=\int_{\Sigma}[\cdots+(\partial\cdots)]$ marked pt's ← "D-terms"
    - So, $[\epsilon\cdot Q=\epsilon^{\pm}Q_{\pm}+\bar\epsilon^{\pm}\bar Q_{\pm}]\int_{\Sigma}\int d^4\varsigma[\cdots]\simeq 0$ b/c $\mathrm{Gr}[\mathfrak{Cl}(D_{\pm},\bar D_{\pm};\partial_{\pm\pm})]\approx\wedge^{*}\,\mathrm{Span}(D_{\pm},\bar D_{\pm})$

Worldsheet SuSy

A Telegraphic Summary



- Superfields = "functions" of $(\xi^{\pm\pm} \,|\, \varsigma^{\pm}, \bar{\varsigma}^{\pm})$, $\varsigma^2 = 0$
  - $\mathbb{Z}$-graded vector(-like) spaces: $\mathbb{U} = u + \varsigma\cdot\psi + \wedge^2\varsigma\cdot U + \wedge^3\varsigma\cdot\Psi + \wedge^4\varsigma\cdot\underline{\mathscr{D}}$
    - BTW, yes: $\int \mathrm{d}^4\varsigma\, \mathbb{U} = \mathscr{D}$, the "D-term" $\mathbb{Z}$: $\{D_\pm, \bar{D}_\pm\} = 2i\partial_{\pm\pm} \Rightarrow 2[D] = 2[\bar{D}_\pm] = [\partial_{\pm\pm}]$
    - Then, w/ $[\partial_{\pm\pm}] := 1 \;\Rightarrow\; [\mathbb{U}] = [u] = [\psi] - \frac{1}{2} = [U] - 1 = [\Psi] - \frac{3}{2} = [\mathscr{D}] - 2$: $\mathbb{Z}$-grading
  - Reducible as SuSy representations: $[\epsilon\cdot Q] \circlearrowleft (u; \psi; U; \Psi; \mathscr{D}; \partial_{\pm\pm}[\cdots])$
    - E.g.: Φ: $\bar{D}_\pm\Phi = 0$ leaves $\Phi = \phi + \varsigma^{\pm}\psi_\pm + \varsigma^{+}\varsigma^{-}\underline{F} + (\partial\cdots)$ — "chiral"
    - E.g.: $V = \overline{V}$ & $V \xrightarrow{\simeq} V + i(\Theta - \bar{\Theta})$, where $\bar{D}_\pm\Theta = 0 = D_\pm\bar{\Theta}$
    - which leaves $V = (0; 0; v_{\pm\pm}, v_1, v_2; \lambda_\pm, \bar{\lambda}_\pm; \underline{\mathscr{D}})$, s.t. $v_{\pm\pm} \simeq v_{\pm\pm} + (\partial_{\pm\pm}\theta)$
    - so $\mathrm{d}\xi\cdot\nabla := \mathrm{d}\xi^{\pm\pm}[\partial_{\pm\pm} + i v_{\pm\pm}]$: $U(1)$ gauge symmetry $\Phi \xrightarrow{\simeq} e^{-iq_\Phi\Theta}\Phi$
    - E.g.: Σ: $\bar{D}_+\Phi = 0 = D_-\Sigma$ leaves $\Sigma = \sigma + \varsigma^{+}\bar{\lambda}_+ + \bar{\varsigma}^{-}\lambda_- + \varsigma^{+}\bar{\varsigma}^{-}(\underline{\mathscr{D} - i\mathscr{F}}) + (\partial\cdots)$
    is "twisted-chiral," e.g. $\Sigma = \{e^{V}\bar{D}_+ e^{-V}, e^{-V}D_- e^{V}\}$, and $\sigma = v_1 + i v_2$

$v_{\pm\pm} := v_0 \pm v_3 = 1{+}1\mathrm{D}$ vector

(like $1{+}1$D E&M)

w.l.o.g.

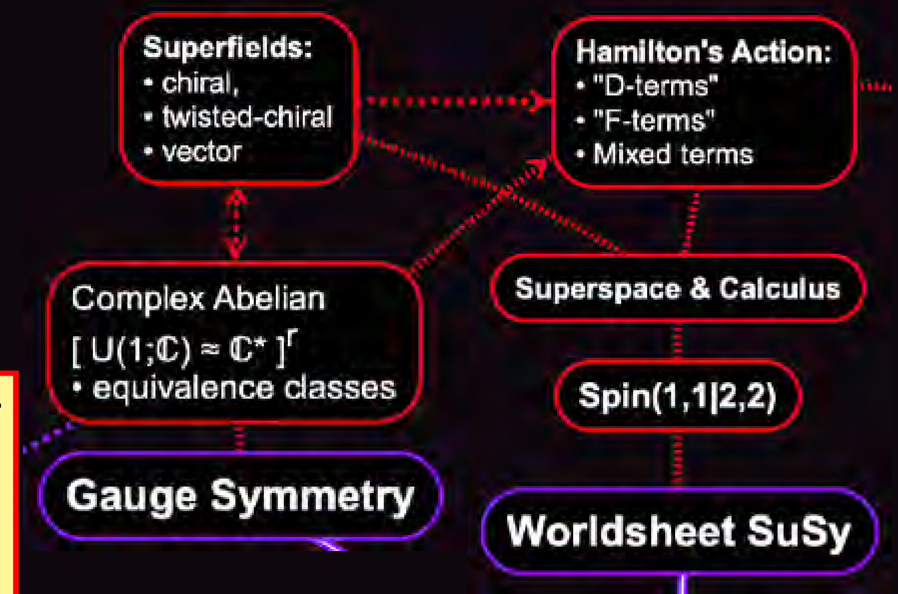
$\mathscr{F} := \partial_{[\mp\mp}v_{\pm\pm]}$

- Chiral Φ's = "matter"; twisted-chiral Σ's = $U(1;\mathbb{C}) \approx \mathbb{C}^*$ gauge symmetry

Worldsheet SuSy

A Telegraphic Summary

$$\Phi \to e^{-iq_\Phi \Theta}\Phi,\ \overline{\Phi} \to e^{iq_\Phi \overline{\Theta}}\overline{\Phi}$$
$$\big(V \to V + i(\Theta - \bar{\Theta})\big) \leftrightarrow \Sigma$$



- One more thing:
  - (Twisted-)chiral superfield close under multiplication
  - If $\bar{D}_\pm \Phi = 0$, then $\bar{D}_\pm(\Phi_1\Phi_2) = 0$; also, $\bar{D}_\pm 1 = 0$; — "chiral ring"
  - If $\bar{D}_+\Sigma = 0 = D_-\Sigma$, then $\bar{D}_+(\Sigma_1\Sigma_2) = 0 = D_-(\Sigma_1\Sigma_2)$; — "twisted-chiral ring"
- And another: $\int \mathrm{d}^2\varsigma\, W(\Phi) + \text{h.c.}$ is supersymmetric — "**F-term**"
  - So is $\int \mathrm{d}^4\varsigma\, \overline{\Phi}\, e^{q_\Phi V}\Phi = \int \mathrm{d}^4\varsigma\, \overline{\Phi}\,\Phi + \ldots$ — "**D-terms**" + $\big(\Phi\text{-}\Sigma$ mixing terms$\big)$.
  - Now, $\int \mathrm{d}^2\varsigma\, W(\Phi) + \text{h.c.} = \underline{F}\, W' + \ldots + \text{h.c.}$ & $\int \mathrm{d}^4\varsigma\, \overline{\Phi}\,\Phi = \underline{\bar{F}}\,\underline{F} + \ldots$
  - So, $\delta_F\big(\int \mathrm{d}^4\varsigma\, \overline{\Phi}\,\Phi + \int \mathrm{d}^2\varsigma\, W + \text{h.c.}\big) = 0 \ \Rightarrow\ \underline{\bar{F}} = -W'$ $\to PE = |W'|^2$
    $\to$ *Morse theory!*
- Similarly: $\int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^-\, \widetilde{W}(\Sigma) + \text{h.c.}$ is supersymmetric — "**twisted F-term**"
  - Simplest: $t \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^-\, \Sigma + \text{h.c.} = t_R \underline{\mathscr{D}} + t_I \underline{\mathscr{F}}$, where $\Sigma = (\sigma; \bar{\lambda}_+, \bar{\varsigma}^- \lambda_-; \underline{\mathscr{D} - i\mathscr{F}})$

$$Z := \iint \mathbf{D}[X]\, e^{-(i/\hbar)\int_\Sigma (\ldots+?)}$$

$$\mathscr{X} = \mathbb{R}^{1,3}_{\text{dS}} \times \mathscr{Y}^6$$

GLSM Action

All Together Now

- $V \simeq (0; 0; v_{\pm\pm}, v_1, v_2; \lambda_\pm, \bar\lambda_\pm; \underline{\mathscr{D}}) \rightsquigarrow \Sigma = \big((v_1 + i v_2); \lambda_\pm, \bar\lambda_\pm; (\underline{\mathscr{D} - i\mathscr{F}})\big)$
  - where $\underline{\mathscr{F}} := (\partial_{--} v_{++} - \partial_{++} v_{--})$; $\delta V = i(\Theta - \overline{\Theta})$ & $\delta\Sigma = 0$
- Hamilton's action for $\Phi_j = (\phi; \psi_\pm; \underline{F})_j$: $\Phi_j \to e^{i q_j \Theta} \Phi_j$-invariant
  - KE: $\int d^4\varsigma \left[\sum_{j=0}^{n} \overline{\Phi}_j e^{q_j V} \Phi_j - \overline{\Sigma}\Sigma\right]$ the "D-terms" contain Φ-Σ mixing
    - So, ϕ_j, σ – 2nd-order, $\psi_\pm, \lambda_\pm$ – 1st order (Dirac) EoM; $F, \mathscr{D}, \mathscr{F}$ – algebraic EoM
  - PE: $\int d^2\varsigma\, W(\Phi) + \int d\varsigma^+ d\bar\varsigma^-\, \widetilde{W}(\Sigma) + \text{h.c.}$
    - $\{D_\pm, \bar D_\pm\} = 2i\partial_{\pm\pm} \rightsquigarrow \{\nabla_\pm, \bar\nabla_\pm\} = 2i\nabla_{\pm\pm}$, ext. aut. ($\supset$ "R-sym"): $\widetilde{W}(\Sigma) \sim \Sigma\big(c + \ln(\Sigma)\big)$
    - Spin(1,1) is abelian, all irreps are 1-dimensional: left- & right-movers are independent
    - External automorphism actions induced *separately* from $\varsigma^+ \to e^{i\alpha}\varsigma^+$ & $\varsigma^- \to e^{i\beta}\varsigma^-$
    - $\Rightarrow$ $U_L(1) \times U_R(1)$: the Hilbert space (and all observables) must be equivariant
    - "ac (sub)ring" → Kähler str., model A, …; twisted-chiral ring — "cc (sub)ring" → cpx str., model B, … chiral ring

$\int \mathrm{d}^2\varsigma\, W(\Phi) + t \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^- \,\Sigma + \text{h.c.}$

$\int \mathrm{d}^4\varsigma \,[\sum_{j=0}^n \overline{\Phi}_j e^{q_j V} \Phi_j - \overline{\Sigma}\Sigma]$

$t_R \to r := -t_R$

GLSM Action

in Action

- $\int \mathrm{d}^4\varsigma \,[\sum_{j=0}^n \overline{\Phi}_j e^{q_j V} \Phi_j - \overline{\Sigma}\Sigma] = \sum_i |\underline{F}_i|^2 + \underline{\mathscr{D}}\left(\sum_i q_i |\phi_i|^2\right) + (\underline{\mathscr{D}}^2 + \underline{\mathscr{F}}^2) + \dots$
- $\int \mathrm{d}^2\varsigma\, W(\Phi) + t \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^- \,\Sigma + \text{h.c.} = \sum_i \underline{F}_i\, W_i'(\phi) + t_R \underline{\mathscr{D}} + t_I \underline{\mathscr{F}} + \dots$
- EoM: $\bar{\underline{F}}_i = -W_i'(\phi) \qquad \underline{\mathscr{D}} = \left(\sum_i q_i |\phi_i|^2 - r\right) \qquad \underline{\mathscr{F}} = -t_I$
- So, $\text{PE} = \left[\sum_i q_i |\phi_i|^2 - r\right]^2 + \sum_i |W_i'(\phi)|^2 + t_I{}^2 + |\sigma|^2 \sum_i q_i^2 |\phi_i|^2 + \dots$
  - Ground states @ $\sum_i q_i |\phi_i|^2 \stackrel{!}{=} r$ & $W_i'(\phi) \stackrel{!}{=} 0$ & …
  - But, $q_i \geqslant 0$ makes $W(e^{iq_i\Theta}\Phi_i) \neq W(\Phi_i)$

$\int \mathrm{d}^2\varsigma\, W(\Phi) + t \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^- \,\Sigma + \text{h.c.}$

$\int \mathrm{d}^4\varsigma \,[\sum_{j=0}^{n} \overline{\Phi}_j e^{q_j V} \Phi_j - \overline{\Sigma}\Sigma]$

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GLSM Action

in Action

- $\int \mathrm{d}^4\varsigma \,[\sum_{j=0}^{n} \overline{\Phi}_j e^{q_j V} \Phi_j - \overline{\Sigma}\Sigma] = \sum_i |\underline{F}_i|^2 + \underline{\mathscr{D}}\big(\sum_i q_i |\phi_i|^2\big) + (\underline{\mathscr{D}}^2 + \underline{\mathscr{F}}^2) + \ldots$
- $\int \mathrm{d}^2\varsigma\, W(\Phi) + t \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^- \,\Sigma + \text{h.c.} = \sum_i \underline{F}_i\, W_i'(\phi) + t_R \underline{\mathscr{D}} + t_I \underline{\mathscr{F}} + \ldots$
- EoM: $\underline{\bar{F}}_i = -W_i'(\phi)$ $\quad \underline{\mathscr{D}} = \big(\sum_i q_i |\phi_i|^2 - r\big)$ $\quad \underline{\mathscr{F}} = -t_I$
- So, $\mathrm{PE} = \big[\sum_i q_i |\phi_i|^2 - r\big]^2 + \sum_i |W_i'(\phi)|^2 + t_I{}^2 + |\sigma|^2 \sum_i q_i^2 |\phi_i|^2 + \ldots$
  - Ground states @ $\sum_i q_i |\phi_i|^2 \stackrel{!}{=} r + q_f |p|^2$ & $W_i'(\phi) \stackrel{!}{=} 0$ & … "new" *homogeneous!*
  - But, $q_i \geqslant 0$ makes $W(e^{iq_i\Theta}\Phi_i) \neq W(\Phi_i)$ so, $W(\Phi) = ¶ \cdot f(\Phi)$, $q_P \stackrel{!}{=} -q_f < 0$
  - So, $\{(\phi_0, \ldots, \phi_n) \simeq (e^{iq_0\theta}\phi_0, \ldots, e^{iq_n\theta}\phi_n)\} \stackrel{q_i=1}{=\!=} \mathbb{P}^n$ w/excised $(0,\ldots,0)$
  - Then, $¶ = (p; \pi_\pm; F_p) \to$ fibre of $\mathscr{O}_{\mathbb{P}^n}(q_w)$. $\sum_i |W_i'|^2 \to |f(\phi)|^2 + |p|^2 \sum_i |\frac{\partial f}{\partial \phi_i}|^2$
  - Then: $r > 0 \Rightarrow$ $(\phi_0, \ldots, \phi_n) \neq 0$ $\{f(\phi) = 0\} \subset \mathbb{P}^n$; $f^{-1}(0)$ smooth $\Rightarrow$ $p = 0.$

    ~ temperature
  - and: $r < 0 \Rightarrow$ $p \neq 0$ $f(\phi) = 0 = \frac{\partial f}{\partial \phi_i}$; $f^{-1}(0)$ smooth $\Rightarrow$ $\phi_i = 0;\ p = \sqrt{r/q_W}$

    LGO

GLSM

Bear Essentials

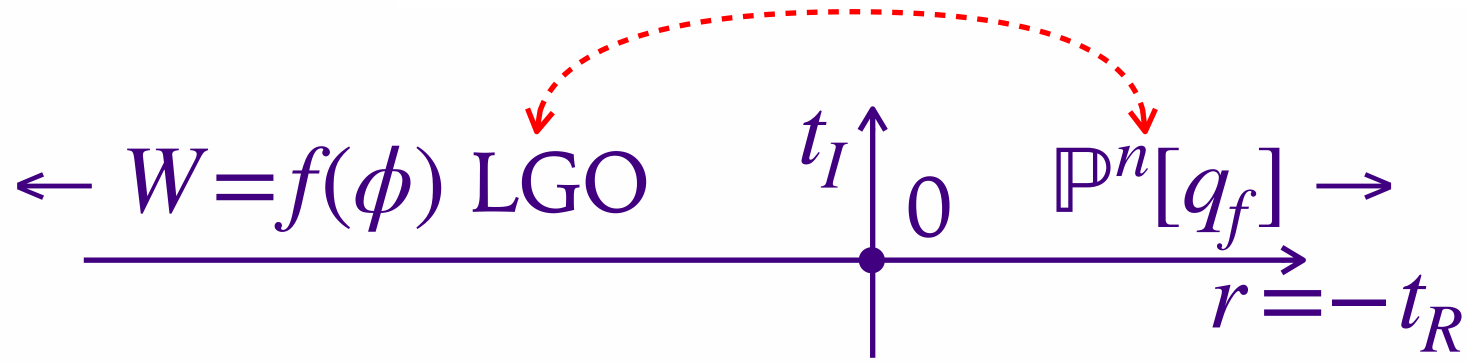
- Chiral "matter": $\Phi_i = (\phi, \psi_\pm, F)_i$ — coordinate fields for X
  - & $\mathscr{P} = (p; \pi_\pm; F_p)$ — fibre coordinate, line bundle $\mathscr{L}_X$
  - $W = \mathscr{P}^a f_a(\Phi)$ — (quasi-)homogeneous, $q_{p_a} = -q_{f_a}$
- Twisted chiral: $\Sigma = \big(\sigma; \lambda_-, \bar{\lambda}_+; (\mathscr{D} - i\mathscr{F})\big)$ — for each $U(1;\mathbb{C}) = \mathbb{C}^\times$
  - Also, separate left and right "R-symmetry," $U_L(1) \times U_R(1)$
- "D-terms": $\sum_i |\underline{F}_i|^2 + \underline{\mathscr{D}}\big(\sum_i q_i |\phi_i|^2\big) + (\underline{\mathscr{D}}^2 + \underline{\mathscr{F}}^2) + \dots$
- "F-terms": $\sum_i \underline{F}_i W_i'(\phi) + t_R \underline{\mathscr{D}} + t_I \underline{\mathscr{F}} + \dots$ — eliminate w/EoM
- $$\mathrm{PE} = \underbrace{\Big[\sum_i q_i |\phi_i|^2 - r\Big]^2}_{=0} + \underbrace{|f(\phi)|^2}_{=0} + \underbrace{|p|^2 \sum_i \Big|\frac{\partial f}{\partial \phi_i}\Big|^2}_{=0} + t_I^2 + \underbrace{|\sigma|^2 \sum_i q_i^2 |\phi_i|^2}_{=0}$$

$$\int \mathrm{d}^2\varsigma\, \mathcal{Y}^\alpha f_\alpha(\Phi) + t^a \int \mathrm{d}\varsigma^+ \mathrm{d}\bar{\varsigma}^- \,\Sigma_a + \text{h.c.}$$

$$\int \mathrm{d}^4\varsigma \left[\sum_{j=0}^{n} \overline{\Phi}_j\, e^{\sum_a q_j^a V_a}\,\Phi_j - \sum_a \overline{\Sigma}_a \Sigma_a\right]$$

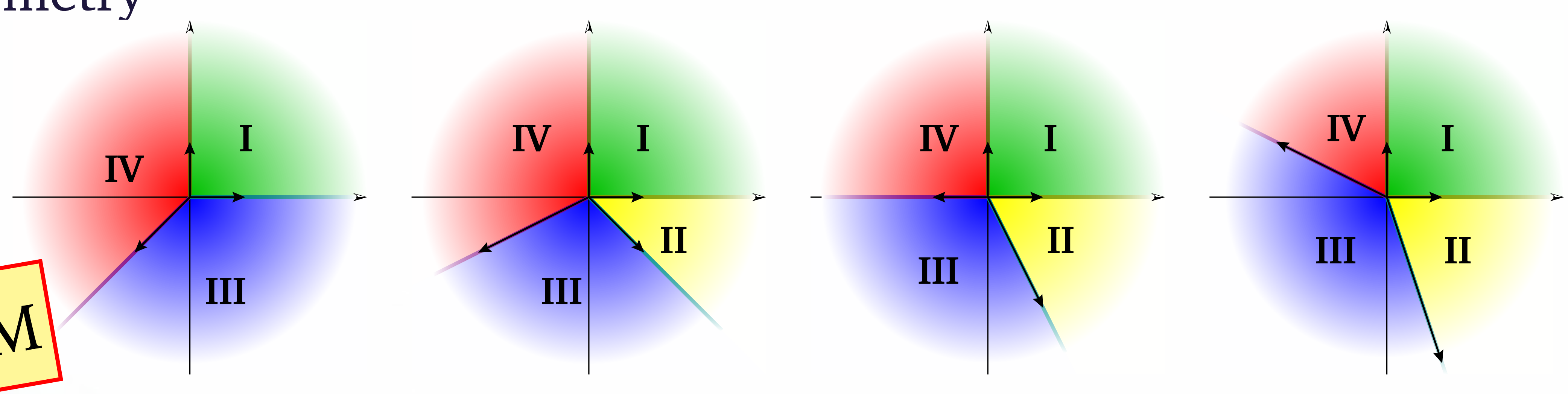
GLSM

→ Toric Geometry



- In pictures:
  - Also: $c_1\big(f^{-1}(0) \subset \mathbb{P}^n\big) = (n+1) - q_f$. For $q_f = n+1$, $(p; \phi_0, \ldots) \in \mathscr{K}^*_{\mathbb{P}^n}$
  - More involved: $F_m^{(n)}[c_1]$ where $F_m^{(n)}$ is the m-twisted $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$
  - After Hirzebruch: $\{x_0 y_0^m + x_1 y_1^m = 0\} \subset \mathbb{P}^n \times \mathbb{P}^1$: $H^2(F_m^{(n)}; \mathbb{Z}) = J_1 \oplus_{\mathbb{Z}} J_2$
  - So, $\vec{q}$: $\left[\begin{array}{c|cccccc} p & \xi_0 & \cdots & \xi_n & \eta_0 & \eta_1 \\ \hline -(n+1) & 1 & \cdots & 1 & 0 & 0 \\ -2 & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{toric}} \left[\begin{array}{cccccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$ & (r_1, r_2)
  - I & II = "geometry"
  - III = LGO
  - IV = hybrid fibre→LGO

LGO ⊂ GLSM



GLSM

→ Toric Geometry

$$\mathbb{P}^n : \left[\begin{array}{c|cccc} p & x_0 & x_1 & \cdots & x_n \\ \hline -(n+1) & 1 & 1 & \cdots & 1 \end{array}\right]$$

$$F_m^{(n)} : \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

- From q_i^a, def. $\vec{\nu}_i \in (N \approx \mathbb{Z}^n)$: $\sum_i q_i^a \vec{\nu}_i = 0$; $\vec{\nu}_i \in \Sigma(1)$ (spanning) fan
  - $\vec{\nu}_i$ up to $\mathrm{GL}(n;\mathbb{Z})$ lattice automorphisms

$\mathbb{P}^4$	ν_0	ν_1	ν_2	ν_3	ν_4
$\Delta^\star_{\mathbb{P}^4}$	-1	1	0	0	0
	-1	0	1	0	0
	-1	0	0	1	0
	-1	0	0	0	1

$F_m^{(4)}$	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
$\Delta^\star_{F_m^{(4)}}$	-1	1	0	0	0	$-m$
	-1	0	1	0	0	$-m$
	0	0	0	1	0	$-m$
	0	0	0	0	1	1

- Cox variables: $\vec{\nu}_i \mapsto x_i$, then $f(x) = \sum_{\vec{\mu}_k \in \Sigma^\circ} \left(a_k \prod_{\vec{\nu}_i \in \Sigma(1)} x_i^{\vec{\nu}_i \cdot \vec{\mu}_k + 1} \right)$
  - where Σ° is spanned by the *polar* of the polytope $\Delta^\star$ spec. by $\vec{\nu}_i \in \Sigma(1)$
  - well defined for "reflexive" $\Delta^\star$; $\Delta := (\Delta^\star)^\circ$ & $\Delta^\circ = \Delta^\star$ (& some details).

GLSM

& Mirror Symmetry

$$\mathbb{P}^n: \left[\begin{array}{c|cccc} p & x_0 & x_1 & \cdots & x_n \\ \hline -(n+1) & 1 & 1 & \cdots & 1 \end{array}\right]$$

$$F_m^{(n)}: \left[\begin{array}{cc|ccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

- So, $\Sigma \in (N \otimes_{\mathbb{Z}} \mathbb{R})$ and $\Delta^\star \ni 0$ w/vertices $= \vec{\nu}_i$ define $V(\Sigma)$
  - Compute $\Sigma^\circ(k) \ni \sigma^\circ \ni \vec{\mu}: \ \vec{\mu}\cdot\vec{\nu}(+1)=0, \ \ \vec{\nu} \in \sigma \in \Sigma(n+1-k)$
  - $^\circ$ is <u>*inclusion-reversing*</u> : if $\partial\sigma = \tau_1 \cup \cdots \cup \tau_r$ then $\sigma^\circ = \tau_1^\circ \cap \cdots \cap \tau_r^\circ$
- For the anticanonical section $f(x) = \sum_{\vec{\mu}_k \in \Sigma^\circ} \left(a_k \prod_{\vec{\nu}_i \in \Sigma(1)} x_i^{\vec{\nu}_i \cdot \vec{\mu}_k (+1)}\right)$
- Mirror model from $f^\top(y) = \sum_{\vec{\nu}_i \in \Sigma} \left(b_i \prod_{\vec{\mu}_k \in \Sigma^\circ(1)} y_k^{\vec{\mu}_k \cdot \vec{\nu}_i (+1)}\right)$ — $\Sigma \leftrightarrow \Sigma^\circ$ transposition
  - "Transposition method": judicious reductions of $f(x)$ and $f^\top(y)$ to "invertible" poly's $\Rightarrow$ $f^{-1}(0)/H$ and $(f^\top)^{-1}(0)/H'$ are mirror models, $H \times H'$ diagonally acting discrete symmetry, spec'd by rows/columns of $[\![\vec{\mu}_k \cdot \vec{\nu}_i + 1]\!]^{-1}$
  - BTW, the "(+1)" in the exponent $\Leftarrow f(x) \in \Gamma(\mathscr{K}^{*\,+1}_{V(\Sigma)})$ & $f^\top(y) \in \Gamma(\mathscr{K}^{*\,+1}_{V(\Sigma^\circ)})$.
  - Multiple "judicious reductions" $\Rightarrow$ "multiple mirrors" ↺ fract. transf.

LGO ⊂ GLSM

Mirror Questions

…about the mirror of $\mathbb{P}^2$

- Claim: "Mirror of $\mathbb{P}^2$ is the Landau-Ginzburg $W_0 = \frac{1}{xy} + x + y$."
  - See: arXiv-abstract of Gross' $\mathbb{P}^2$-paper [P²P=aXiv:0903.1378 → book]
  - The mirror of a manifold is a… *polynomial?!* *a physicist being such a fuddy-duddy 'bout "what are we talking about?!"*
    - Details: "certain computations on $\mathbb{P}^2$ match certain other computations on W_0"
  - **Q.**: LGO in a GLSM over which (toric?) space $V(\Sigma)$, so $W\in\Gamma(\mathscr{L}_{V(\Sigma)})$?
  - **Also**, which bundle $\mathscr{L}_{V(\Sigma)}$? Surely, not the anticanonical one… ← I'll be back.
  - Clearer [P²P, p.3] (paraphrase TH): $\mathbb{P}^2 \xrightarrow{\text{mirror}} (\mathbb{C}^\times)^2 \approx \{x_0 x_1 x_2 = 1\} \subset \mathbb{C}^3$
  - $\mathbb{P}^2$ is rigid: no complex structure deformation; cpx str. = {pt}.
  - $\mathbb{P}^2$ has dim $H^{1,1}(\mathbb{P}^2) = 1$: space of Kähler classes 1-dim.
  - $(\mathbb{C}^\times)^2$ has a 1-dim. moduli space of complex structures? ← I'll be back.
  - $(\mathbb{C}^\times)^2$ has a rigid Kähler class? $J = \mathrm{d}x\wedge\mathrm{d}\bar{x} + \mathrm{d}y\wedge\mathrm{d}\bar{y}$, up to coord. reparam.

GLSM over $V(\Sigma)$, w/ $W \in \Gamma\big(\mathscr{L}_{V(\Sigma)}\big)$

$\mathbb{P}^2 \xrightarrow{\text{mirror}} (\mathbb{C}^\times)^2 \approx \{x_0 x_1 x_2 = 1\} \subset \mathbb{C}^3$

[P²P = aXiv:0903.1378]

LGO ⊂ GLSM

Mirror Questions

…about the mirror of $\mathbb{P}^2$

¡¿ $W_0 = \frac{1}{xy} + x + y$?!

- Claim: "Mirror of $\mathbb{P}^2$ is $(\mathbb{C}^\times)^2 \approx \{x_0 x_1 x_2 = 1\} \subset \mathbb{C}^3$." *"best written as"*
  - Also [P²P]: $W_0 = y_0 + x_0 + x_1 + x_2$, "$y_0$" = "flat coordinate"
  - These $x_i \in \mathbb{C}^3$, so $x_0 x_1 x_2 = 1 \;\Rightarrow\; x_0 = \frac{1}{x_1 x_2}$, so $W_0 = y_0 + \frac{1}{x_1 x_2} + x_1 + x_2$
  - Still, not homogeneous; no consistent $U(1;\mathbb{C}) \approx \mathbb{C}^\times$ (gauge) charges…
  - *Homogenize*: add x_3 so $\{(x_0 x_1 x_2 - 1) = 0\} \subset \mathbb{C}^3 \to \{(x_0 x_1 x_2 - x_3{}^3) = 0\} \subset \mathbb{P}^3|_{x_3 \neq 0}$
    - where $\mathbb{P}^3|_{x_3\neq 0} = (\mathscr{U}_3 \approx \mathbb{C}^3)$ is one of four affine charts; so consider $\mathbb{P}^3[3]$
  - …it's a Fano 2-fold, & $(\mathbb{C}^\times)^2 = \mathbb{P}^3[3]_{x_3\neq 0} = \mathbb{P}^3[3] \smallsetminus \mathbb{P}^3[3]_{x_3=0}$
  - …and $\mathbb{P}^3[3]_{x_3=0} \equiv \mathbb{P}^3[1,3] \equiv \mathbb{P}^2[3]$! $\therefore$ $\mathbb{P}^3[3]|_{x_3\neq 0}$ is a non-compact CY2-fold [Tian-Yau, '91 & '92]
    - Yes, we knew that, since $\mathbb{P}^3[3]_{x_3\neq 0} = (\mathbb{C}^\times)^2$ is *flat*.
    - But: $\mathbb{P}^3[1,3] \equiv \mathbb{P}^2[3]$ has a 1-dim cpx. str. moduli space.

🤨🧐🤔 …*any independent proof that* $\dim \mathscr{M}_{(\mathbb{C}^\times)^n} = 1$?

"flat coordinate" = coeff. of the "universal monomial" $x_0 x_1 x_2 \in \Gamma(\mathscr{K}^*_{\mathbb{P}^2}) \mapsto 1$ on $\mathbb{P}^3[3]|_{x_3\neq 0}$

GLSM over $V(\Sigma)$, w/ $W \in \Gamma\big(\mathscr{L}_{V(\Sigma)}\big)$

$\mathbb{P}^2 \xrightarrow{\text{mirror}} (\mathbb{C}^\times)^2 \approx \{x_0x_1x_2 = 1\} \subset \mathbb{C}^3$

[P²P = aXiv:0903.1378]

LGO ⊂ GLSM

Mirror Questions

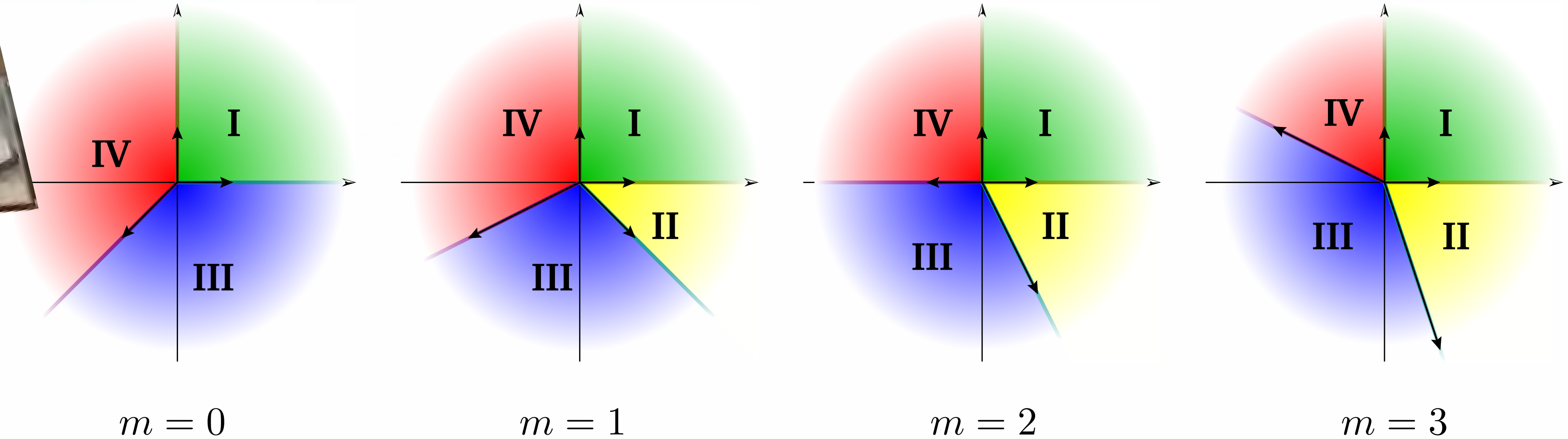
…about the mirror of $\mathbb{P}^2$

- So, "Mirror of $\mathbb{P}^2 \equiv \mathbb{P}^3[1]$ is $(\mathbb{C}^\times)^2 \approx \{x_0x_1x_2 = x_3{}^3\} \in \mathbb{P}^3[3]_{x_3 \neq 0}$."
  - Barannikov, '01, [math/0010157]: "$\mathbb{P}^n \xrightarrow{\text{mirror}} (\mathbb{C}^\times)^n \approx \{(x_0 \cdots x_n = 1)\} \subset \mathbb{C}^{n+1}$"
- There is a "general" pattern here: $\mathbb{P}^{n+1}[n+1] \leftrightarrow \mathbb{P}^n \equiv \mathbb{P}^{n+1}[1]$
  - $c_1(\mathbb{P}^{n+1}[n+1]) + c_1(\mathbb{P}^n \equiv \mathbb{P}^{n+1}[1]) = c_1(\mathbb{P}^{n+1})$
  - & $(\mathbb{P}^n = \mathbb{P}^{n+1}[1]) \xrightarrow{\text{mirror}} \{(x_0 \cdots x_n = x_{n+1}{}^{n+1})\} \subset \mathbb{P}^{n+1}|_{x_{n+1} \neq 0} = \mathbb{P}^{n+1}[n+1] \smallsetminus \mathbb{P}^{n+1}[1, n+1]$
  - More generally: $A^n[d] \xrightarrow{\text{mirror}} \big(A^n[(c_1 - d)] \smallsetminus A^n[d, (c_1 - d)]\big)$ ⁉︎ comp. CY$(n-2)$-fold; non-comp. CY$(n-1)$-fold

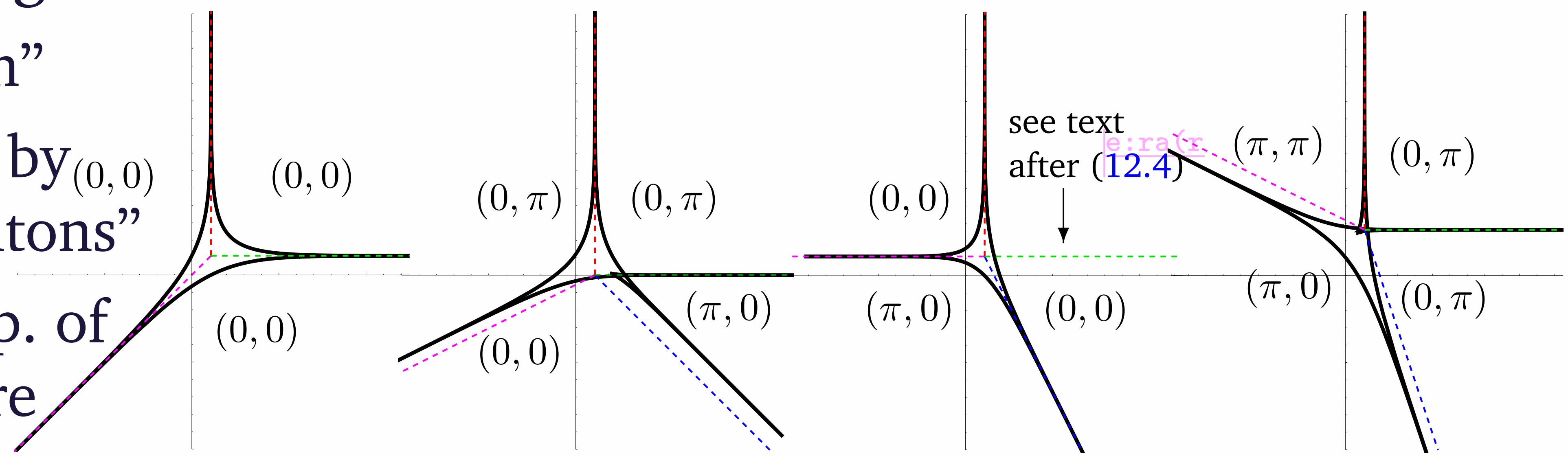
    — *false for $d = 0$* —
  - And, for $d = c_1$: $A^n[c_1] \xrightarrow{\text{mirror}} \big(A^n \smallsetminus A^n[c_1]\big)$ ⁉︎ $\overset{?}{\rightsquigarrow}$ "transposition mirror" ⁉︎
- BTW, in $\mathbb{P}^n$-GLSM, $W \equiv 0$: $W \in \Gamma(\mathscr{O}_{\mathbb{P}^n})$, $W = \text{const.} \mapsto 0$
- One more thing: Hori-Vafa [hep-th/0002222]: $(e^{Y_i} \neq 0) \leftrightarrow \mathbb{C}^\times$

GLSM

…and Amoebas?



- *One more thing…*
  - The "secondary fan"
  - becomes modified by "worldsheet instantons"
  - This is the toric rep. of the Kähler structure
  - …with shifts and "thickening" → "amoebas" ← log-geometry $\lim_{t\to\infty} \ln(t^a + t^b) = \max(a, b)$



  - Tropical geometry ⇔ "large cpx str." $\overset{\text{mirror}}{\longleftrightarrow}$ "large Kähler class/radius"
  - "log-geometry" ⇔ "smallish cpx str." $\overset{\text{mirror}}{\longleftrightarrow}$ "smallish Kähler class/radius"

Thank You!

Tristan Hübsch
Department of Physics and Astronomy
Howard University, Washington DC