2020 Spring Research Interactions in Mathematics - Note for Periodic table for topological insulators and superconductors

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ABSTRACT: In this note, I try to show import ingredients in how Kitaev gave the periodic table of topological insulators and provide relevant mathematical background including Algebraic topology (higher homotopy and Bott periodicity), Clifford algebra, Symmetric space, and K theory. For physics, I mostly followed Martin R. Zirnbauer lecture notes [1]. For Mathematics including Allen Hatcher Algebraic Topology [2] and other notes.

Symmetry Class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Figure 1.

Contents

1 Introduction

The intense research activity on topological insulators started about 10 years ago after the theoretical and experimental discovery of the Quantum Spin Hall Insulator. As the name suggests, this is a close cousin of the standard Quantum Hall Effect, although it differs from it by the presence of spin and by time-reversal symmetry (meaning invariance under the hypothetical operation of inverting the time direction).

This project, our goal is to proof or at least illustrate the periodic table of Topological insulator made by Kitaev [3]. We first explain the meaning of this table and then explain the mathematical theorem called Bott Periodicity and their relationship. Bott periodicity is said to be one of the most surprising phenomena in topology. Perhaps even more surprising is its recent appearance in condensed matter physics. Building on work of Schnyder et al., Kitaev argued that symmetry-protected ground states of gapped free fermion systems, also known as topological insulators and superconductors, organize into a kind of periodic table governed by a variant of the Bott periodicity theorem. In this colloquium, I will sketch the mathematical background, the physical context, and some new results of this ongoing story of mathematical physics

2 Clifford Algebra

It looks like Clifford Algebra is the most relevant to physics. Since the basic algebra behind the fermion Clifford Algebra or one may think there is a Dirac equation which is exactly one of the Clifford Algebra. We first give some definitions of them and classify them in Appendix C. Consider a vector space V of dimension $2n$, and let V carry two structures: a Hermitian scalar product and a (non-degenerate) symmetric bilinear form denoted as bracket:

$$V \otimes V \to \mathbb{C}. \tag{2.1}$$

for any $u, v \in V$:

$$uv + vu = 2\langle u, v\rangle 1 \text{ for all } u, v \in V, \tag{2.2}$$

If the dimension of V over K is n and $e_1, ..., e_n$ is an orthogonal basis of (V, Q), then $Cl(V, Q)$ is free over K with a basis

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n \text{ and } 0 \le k \le n\}. \tag{2.3}$$

Obviously, the total dimension of the Clifford algebra is

$$\dim \mathrm{C}\ell(V, Q) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.. \tag{2.4}$$

Now we will seperate it into two cases. The first is real Clifford Algebra. Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2, \tag{2.5}$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $R^{p,q}$. The Clifford algebra on $\mathbb{R}^{p,q}$ is denoted $Cl_{p,q}(\mathbb{R})$. In short hand, I denoted it as $Cl(p, q)$. On the other hand, every nondegenerate quadratic form on a complex vector space of dimension n is equivalent to the standard diagonal form (since we have i so sign does not matter).

$$Q(z) = z_1^2 + z_2^2 + \cdots + z_n^2. \tag{2.6}$$

Thus, for each dimension n, up to isomorphism there is only one Clifford algebra of a complex vector space with a nondegenerate quadratic form. We will denote the Clifford algebra on Cn with the standard quadratic form by $Cl(n)$. We denote $\mathbb{K}(m)$ to be the $m \times m$ matrix algebra of the field $\mathbb{K}$. There is an amazing identity. In Appendix B. We proved for the real case:

$$\begin{aligned}
&Cl(n+8, 0) \cong Cl(n, 0) \otimes \mathbb{R}(16) \\
&Cl(0, n+8) \cong Cl(0, n) \otimes \mathbb{R}(16) \\
&Cl(s+4, t+4) \cong Cl(s, t) \otimes \mathbb{R}(16).
\end{aligned}$$

For the complex case:

$$Cl(n+2) \cong Cl(n) \otimes \mathbb{C}(2). \tag{2.7}$$

There are period 8 in the real Clifford algebra, on the other hand, there are only period 2 in the complex Clifford algebra. This 'period-8' behavior was discovered by Cartan in 1908, but we will take the liberty of calling it **Bott periodicity**. The proof is given in Appendix C. The proof is not very hard, but actually, this is a very deep result related to K theory and homotopy group of the infinite orthonormal group and infinite unitary group. This fact constructs the periodic table. Let us keep going to different ingredients.

3 Symmetric space and Classification of Hamiltonian

Now we move to physics. Most of this part is following [13]. There are ten discrete symmetry classes of topological insulators and superconductors, corresponding to the ten Altland-Zirnbauer classes of random matrices. We will show why there are 10 classes. They are defined by three symmetries of the Hamiltonian $\hat{H} = \sum_{i,j} H_{ij} c_i^\dagger c_j$. Chiral symmetry is a unitary operator S. A Hamiltonian H possesses chiral symmetry when $C\hat{H}C^{-1} = -\hat{H}$. In the Bloch Hamiltonian formalism for periodic crystals, where the Hamiltonian $H(k)$ acts on modes of crystal momentum k, the chiral symmetry, TRS, and PHS conditions become $U_C H(k) U_C^{-1} = -H(k)$, $U_T H(k)^* U_T^{-1} = H(-k)$ and $U_P H(k)^* U_P^{-1} = -H(-k)$. It is evident that if two of these three symmetries are present, then the third is also present, due to the relation $C = PT$.

To start, let us first discuss why there are ten classes. The reason is fairly simple. We know that $C = P \cdot T$. This also means that if a system only has either T or P but not both, it cannot have a chiral symmetry C. In other words, the presence of any two out of the three symmetries implies that the third is also present. On the other hand, if both P and T are absent, then C may or may not be present. This gives us two distinct cases. Note that the eigenvalue for P or T could be ± 1. On the other hand, the chiral symmetry only comes in one flavor, $C^2 = 1$.

Removing symmetries

$$H_{d+1} = H_d cos(k_{d+1}) + C sin(k_{d+1}) \tag{3.1}$$

This Hamiltonian has the same number of bands as H_d, even though the bands are higher-dimensional. Given its simple form, every band $\epsilon^n_{n,d+1}$ is directly related to a band $\epsilon n, d$ of H_d,

$$\epsilon^n_{n,d+1} = \pm\sqrt{(\epsilon^2_{n,d} cos(k_{d+1})^2 + sin(k_{d+1})^2)} \tag{3.2}$$

We can see that C is the only discrete symmetry of H_d. If H_d belongs to class AIII, then H_{d+1} has no symmetry at all. So we have $AIII \rightarrow A$. If instead C is not the only discrete symmetry of H_d, then H_d must have both P, T. With some patience, it is possible to work out the transition exactly, but state that the result is that by removing chiral symmetry and adding one dimension, one obtains that $BDI \rightarrow D, DIII \rightarrow AII, CII \rightarrow C, CI \rightarrow AI$. Fig 2.

Adding symmetries Let's now start from a Hamiltonian without chiral symmetry. Our procedure this time involves a doubling of the number of bands of H_d. We introduce

$$H_{d+1} = H_d cos(k_{d+1})\sigma_x + sin(k_{d+1})\sigma_y. \tag{3.3}$$

Note that just like in our previous argument, the topological invariant of H_{d+1} must be the same as that of H_d. Also, by construction, H_{d+1} has a chiral symmetry given by $C = \sigma_z$ which anti-commutes with all the terms in the Hamiltonian. We make clear statement below:

if H_d has no symmetry at all, then H_{d+1} only has chiral symmetry, meaning that it is in class $AIII$. So we obtain $A \rightarrow AIII$. On the other hand, if H_d has one antiunitary symmetry, then H_{d+1} must have three discrete symmetries. Again, we will not work out the details for each case, but one obtains that $AI \rightarrow BDI, D \rightarrow DIII, AII \rightarrow CII, C \rightarrow CI$. Up to now, we can write down the **Bott Clock**. For two complex classes, we have

$$A \rightarrow AIII \rightarrow A. \tag{3.4}$$

it is also the case for the eight real classes:

$$AI \rightarrow BDI \rightarrow D \rightarrow DIII \rightarrow AII \rightarrow CII \rightarrow C \rightarrow CI \rightarrow AI. \tag{3.5}$$

Given two Hamiltonians H_1 and H_2, it may be possible to continuously deform H_1 into H_2 while maintaining the symmetry constraint and gap (that is, there exists continuous function $H(t, \vec{k})$ such that for all $0 \leq t \leq 1$ the Hamiltonian has no zero eigenvalue and symmetry condition is maintained,

class	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$
A			
AI			1
AII			-1
AIII	1		
BDI	1	1	1
C		-1	
CI	1	-1	1
CII	1	-1	-1
D		1	
DIII	1	1	-1

Figure 2.

and $H(0, \vec{k}) = H_1(\vec{k})$ and $H(1, \vec{k}) = H_2(\vec{k})$. Then we say that H_1 and H_2 are equivalent. The strong topological invariants of a many-band system in d dimensions can be labeled by the elements of the d $d - th$ **homotopy group** [2] of the symmetric space. These groups are displayed in this table, called the periodic table of topological insulators. Ok, now we can write down the periodic table once we know some algebraic topology and differential geometry!!!. See Appendix A and [2] for the introduction to the homotopy group.

3.1 Symmetry Space

Definition 3.1. A Riemannian manifold, (M, g) is a smooth manifold M together with a bundle metric on the tangent bundle

Definition 3.4. The Levi-Civita connection on a Riemannian manifold is a connection on the tangent bundle, satisfying: In Riemannian geometry there exists something called the Riemann curvature tensor. In a coordinate basis, it has the well-known expression:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \tag{3.6}$$

A (locally) symmetric space is defined to be a Riemannian manifold $X = U/K$ with a Riemann tensor which is covariantly constant:

$$\nabla R = 0. \tag{3.7}$$

A more general example is the Grassmann manifold.

$$X = Gr_n(\mathbb{C}^N) \tag{3.8}$$

An important and well-known fact about symmetric spaces is that they were completely classified by the French geometer Elie Cartan. Aside from a finite number of exceptional spaces, they come in 10 large families, which Cartan called A, AI, AII, AIII, BD, BDI, C, CI, CII, and DIII. Which are exactly the same classes denoted as before.

For each of the symmetry classes, the question can be simplified by deforming the Hamiltonian into a "projective" Hamiltonian and considering the symmetric space in which such Hamiltonians live. These classifying spaces are shown for each symmetry class: For example, a (real symmetric) Hamiltonian in symmetry class AI can have its n positive eigenvalues deformed to $+1$ and its $N-n$ negative eigenvalues deformed to -1; the resulting such matrices are described by the union of real Grassmannian. $\bigcup_{n=0}^{\infty} Gr(n,N) = \bigcup_{n=0}^{\infty} O(N)/(O(n)\times O(N-n))$. So now, we have the statement that: given the symmetric of the Hamiltonian, we can produce the symmetry space. So now, the final piece is that how the symmetric space related to the Clifford algebra and the K theory. We need the "Geometric model of loop spaces".

4 Bott Periodicity and loop space

So now, let explain the story, suppose we have a Hamiltonian with some symmetry. One can construct a symmetric space. But we say one can add or remove a symmetry and go into higher dimension. The operator corresponds to the loop space of that symmetry space. In the original paper in [17]. Bott proved that there are some periods in the these symmetric space. We should start introduce what is the loop space. The loop space ΩX of a pointed topological space X is the space of (based) loops in X, i.e. maps from the circle $S^1 \to X$. In general, give a space X. One can ask the property of $\Omega X, \Omega^2 X, \Omega^3 X....$ The question is hard to answer. But we have a theorem to calculate the homotopy group ot the loop space.

We have the **Eckmann Hilton duality**. The loop space is dual to the suspension of the same space; this duality is sometimes called Eckmann-Hilton duality. The basic observation is that

$$[\Sigma Z, X] \approxeq [Z, \Omega X] \tag{4.1}$$

Thus, setting $Z = S^{k-1}$(the $k-1$ sphere) gives the relationship:

$$\pi_k(X) \approxeq \pi_{k-1}(\Omega X). \tag{4.2}$$

his follows since the homotopy group is defined as $\pi_k(X) = [S^k, X]$ and the spheres can be obtained via suspensions of each-other, i.e. $S^k = \Sigma S^{k-1}$.

Bott's Theorem Bott's original proof (Bott 1959) used Morse theory (tools in geometry), which Bott (1956) had used earlier to study the homology of Lie groups. What Bott's proved that (see Appendix b for the notation):

$$\Omega^2 U \simeq U., \Omega^8 O \simeq O, \tag{4.3}$$

A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Figure 3.

The loop space tower for the complex numbers:

$$U \times U \subset U \subset U \times U. \tag{4.4}$$

Over the real numbers and quaternions:

$$O \times O \subset O \subset U \subset \mathrm{Sp} \subset \mathrm{Sp} \times \mathrm{Sp} \subset \mathrm{Sp} \subset U \subset O \subset O \times O. \tag{4.5}$$

In the appendix B, we will know how to compute the homotopy group of these group, their element are exactly the element in the periodic table. But we have a missing piece we do not explain, why Clifford Algebra has order 8 corresponds to these strange property. One is pure algebra, one side is topology. This is very hard to explain here. You will need to know **Classifying space, k theory**. See Appendix D for more details and reference. Basically, the Clifford algebra indicates some properties of vector bundle. All the vector bundle will form a Abelian group which is called K group. One can see Appendix D for more information. See [18] for the classfying space and fig: 4.

5 Periodic Table

Now, we have basic ingredients to complete the periodic table of Topological insulator. I separated this into three steps.

Why there are period 8 for real case, period 2 for complex case The reason is not easy, but we know the clifford algebra periods lead to the homotopy group of infinite orthonormal groups and infinite unitary group. See Appendix. So we know there must be such period in the two rows of periodic table. The rest of them have same periods. The reason is also simple. One could add symmetry or remove symmetry to get higher dimensional row, so if one this row have this period, then all of them should have the same periods. We can also see this fact by using Bott theorem. The geodesic space of the symmetry space is another symmetry space. Their homotopy group only shifts by one. So they have the same periods.

Fill out the table - complex case Up to now, we can fill out the table:
The second-row AIII, we know its symmetry space is infinite unitary group [15]. See appendix for the calculation, we know $\pi_0(U) = 0, \pi_1(U) = \mathbb{Z}$. It also has period 2. We can use the Bott theorem or loop space tower, or even physics, A class has shifted the homotopy group. So now we solve the complex case. See fig3.

Cartan label	T	C	S	Hamiltonian	G/H (ferm. NLσM)
A (unitary)	0	0	0	$\mathrm{U}(N)$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI (orthogonal)	+1	0	0	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{Sp}(2n)/\mathrm{Sp}(n)\times\mathrm{Sp}(n)$
AII (symplectic)	−1	0	0	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$
AIII (ch. unit.)	0	0	1	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathrm{U}(n)$
BDI (ch. orth.)	+1	+1	1	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
CII (ch. sympl.)	−1	−1	1	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathrm{U}(2n)/\mathrm{O}(2n)$
D (BdG)	0	+1	0	$\mathrm{SO}(2N)$	$\mathrm{O}(2n)/\mathrm{U}(n)$
C (BdG)	0	−1	0	$\mathrm{Sp}(2N)$	$\mathrm{Sp}(2n)/\mathrm{U}(n)$
DIII (BdG)	−1	+1	1	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{O}(2n)$
CI (BdG)	+1	−1	1	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2n)$

Figure 4. Parameter space

Fill out the table - real case The real case is much more complicated. But they also have the period:

$$AI \to BDI \to D \to DIII \to AII \to CII \to C \to CI \to AI. \tag{5.1}$$

That means we only need one to know one of them. If we want to know how to calculate the homotopy of symmetry space without knowing Bott Periodicity. One can see [16]. See Appendix B, then we know how to calculate the homotopy of the infinite orthonormal group.

6 Conclusion

In this project, we tried to explain the Periodic table for topological insulators and superconductors. The main point is that we proved that how to derive this table from the Clifford Algebra, K theory, and the symmetry space. We also explain the connection between these subjects. We have two approaches, the first approach is we used the Clifford Algebra and symmetry space to show how to construct the table. The second approach is that we can use topological K theory and loop space to calculate the homotopy group of the symmetry space.

Appendix

A Basic introduction to Homotopy

This is quick introduction and give some basic facts about homotopy. One can see [2] for more detail. Formally, a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H\colon X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Given two spaces X and Y, we say they are homotopy equivalent, or of the same **homotopy type**, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$is homotopic to id_Y. Now we can give the definition of homotopy group. In the n-sphere S^n we choose a base point a. For a space X with base point b, we define $\pi_n(X)$ to be the set of homotopy classes of maps $f : S^n \to X$. For $n \geq 1$, the homotopy classes form a group. To define the group operation, the product $f * g$ of two loops $f, g : [0,1]^n \to X$ is defined by setting:

$$(f+g)(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & t_1 \in \left[0, \frac{1}{2}\right] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{A.1}$$

Theorem: The homotopy group for $n \geq 2$ will be the abelian group.

B Tool of calculating homotopy group

In this section, we treat Bott Periodicity as a pure mathematical theorem. The story comes from the fact that homotopy group of sphere $\pi_m(S^n)$ for general $mandn$ are still open. But superisingly, the classical lie group (one can also viewed them as Manifold) $O(\infty)$ and $U(\infty)$ can be computed exactly. We show them here.

$$\begin{aligned} \pi_0(O(\infty)) &\simeq \mathbb{Z}_2 \\ \pi_1(O(\infty)) &\simeq \mathbb{Z}_2 \\ \pi_2(O(\infty)) &\simeq 0 \\ \pi_3(O(\infty)) &\simeq \mathbb{Z} \\ \pi_4(O(\infty)) &\simeq 0 \\ \pi_5(O(\infty)) &\simeq 0 \\ \pi_6(O(\infty)) &\simeq 0 \\ \pi_7(O(\infty)) &\simeq \mathbb{Z} \end{aligned} \tag{B.1}$$

$$\pi_k(O) = \pi_{k+8}(O) \tag{B.2}$$

$$\pi_k(U) = \pi_{k+2}(U) \tag{B.3}$$

$$\pi_0(U) = 0, \pi_1(U) = \mathbb{Z}. \tag{B.4}$$

We will start to proof this via exact sequences [14].

$$O(n) \to O(n+1) \to S^n, U(n) \to U(n+1) \to S^{2n+1} \tag{B.5}$$

On the other hand,from these fibrations,it immediatetly follows that the homotopy groups of $O(n)$ and $U(n)$ stabilize. More precisely, $\pi_i(U(n)) = \pi_i(U(n+1))$ if $n \geq \frac{i}{2}$ and $\pi_i(O(n)) = \pi_i(O(n+1))$. To calculate the unitary group, the process is fairly simple. As $SU(N)$ is simply connected, we see that

$$\pi_1(U(N)) = \pi_1(S^1 * SU(N)) = \pi_1(S^1) \oplus \pi_1(SU(N)) = \mathbb{Z} \tag{B.6}$$

Also $\pi_0(U(N)) = 0$ as $N \to \infty$ Since $U(1) \cong S^1$. S^1 is simply connected. Now we can move how to calculate the homotopy group of the real part [14]. The calculation is much more complicated.

So we leave the details into Reference. But basically, every short exact sequence $F \to E \to B$ will introduce a long exact sequence of homotopy group.

$$\pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \pi_{i-1}(B).. \tag{B.7}$$

By using this tool, one can calculate the homotopy group of infinite orthonormal group.

C Classification of Clifford algebras

In this paragraph, we tried to give the proof of Classification of Clifford Algebras over $\mathbb{R}$ or $\mathbb{C}$. We will show there are period 8 in the real Clifford algebra, on the other hand, there are only period 2 in the complex Clifford algebra. This 'period-8' behavior was discovered by Cartan in 1908, but we will take the liberty of calling it Bott periodicity, since it has a far-ranging set of applications to topology, some of which were discovered by Bott. To the full derivation and all details [7].

C.1 Classification of Real Clifford algebras

In this section, we tried to classify there are period 8 in the real Clifford algebra. We will first construct some of them and proof the main theorem. The real case is significantly more complicated, exhibiting a periodicity of 8 rather than 2, and there is a 2-parameter family of Clifford algebras.

Theorem: For all $n, s, t \geq 0$ we have the following isomorphisms:

$$Cl(0, n+2) \cong Cl(n, 0) \otimes Cl(0, 2), Cl(n+2, 0) \cong Cl(0, n) \otimes Cl(2, 0), \tag{C.1}$$

and

$$Cl(s, t) \cong Cl(s, t) \otimes Cl(1, 1) \tag{C.2}$$

As the three cases are very similar, we prove the second equation.Let us write $\mathbb{R}^{n+2,0} = \mathbb{R}^{n,0} \oplus \mathbb{R}^{2,0}$. Let e_1, e_2be an orthonormal basis for $\mathbb{R}^{2,0}$ and let us denote by the same symbols their image in $Cl(2, 0)$. This means that $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$. We write:

$$\phi : \mathbb{R}^{n+2,0} \to Cl(0, n) \otimes Cl(2, 0) \tag{C.3}$$

by $\phi(x) = x \otimes e_1 e_2$, and $\phi(e_i) = 1 \otimes e_i$ for $x \in \mathbb{R}^{n,0}$. This map is Clifford by virtue of the identities satisfied by $e_1 e_2$; indeed.

$$\begin{aligned} \phi(x + ae_1 + be_2)^2 &= (x \otimes e_1 e_2 + a1 \otimes e_1 + b1 \otimes e_2)^2 \\ &= -x^2 - a^2 - b^2 = -Q(x^2 + a^2 + b^2)1 \otimes 1. \end{aligned}$$

Hence ϕ extends uniquely to an algebra homomorphism $\phi : Cl(n+2, 0) \to Cl(0, n) \otimes Cl(2, 0)$ which is injective on generators and by dimension must be an isomorphism.

Up to now, we are ready to finish the proof. We denote $\mathbb{K}(m)$ to be the $m \times m$ matrix algebra of the field $\mathbb{K}$. We need the lemma $\mathbb{K}(m) \otimes \mathbb{R}(n) \cong \mathbb{K}(mn)$.

Corollary. For all $n, s, t \geq 0$, the following are isomorphisms of real algebras:

$$
\begin{aligned}
Cl(n+8,0) &\cong Cl(n,0) \otimes \mathbb{R}(16) \\
Cl(0,n+8) &\cong Cl(0,n) \otimes \mathbb{R}(16) \\
Cl(s+4,t+4) &\cong Cl(s,t) \otimes \mathbb{R}(16)
\end{aligned}
$$

Proof. This follows directly from repeated application of Theorem 2.3 and the following isomorphisms $Cl(1,1)^{\otimes^4} = \mathbb{R}(16), Cl(2,0)^{\otimes^2} = Cl(0,2)^{\otimes^2} = \mathbb{R}(16)$.

Now, we can list the complete classifications about the real Clifford algebra.

Corollary: For every $n \geq 0$, the complex Clifford algebra $Cl(n)$ is isomorphic to

$$
\begin{aligned}
&Cl(n) \cong \mathbb{C}(2^{\frac{n}{2}}) \text{ if n is even.} \\
&Cl(n) \cong \mathbb{C}(2^{\frac{n-1}{2}}) \oplus \mathbb{C}(2^{\frac{n-1}{2}}) \text{ if n is odd.}
\end{aligned}
$$

Proof. This follows easily from complex Bott periodicity and the "initial conditions" $Cl(0) \cong \mathbb{C}$ and $Cl(1) \cong \mathbb{C} \oplus \mathbb{C}$. We also list the first 8 real Clifford Algebra.

$$
\begin{aligned}
C_0 &= \mathbb{R} \\
C_1 &= \mathbb{C} \\
C_2 &= \mathbb{H} \\
C_3 &= \mathbb{H} \otimes \mathbb{H} \\
C_4 &= \mathbb{H}(2) \\
C_5 &= \mathbb{C}(4) \\
C_6 &= \mathbb{R}(8) \\
C_7 &= \mathbb{R}(8) \otimes \mathbb{R}(8) \\
C_8 &= \mathbb{R}(16)
\end{aligned}
$$

C.2 Classification of complex Clifford algebras

The complex case is particularly simple. We first introduce the **bicomplex** number which is isomorphic $\mathbb{C} \oplus \mathbb{C}$. A bicomplex number is a pair (w, z) of complex numbers. the product of two bicomplex numbers as

$$(u, v)(w, z) = (uw - vz, uz + vw). \tag{C.4}$$

One can easily show that bicomplex numbers is isomorphic to Cl(2). Now we can proof the main theorem in this paragraph.

Proposition 2.1. For all $n \geq 0$ there is an isomorphism of complex associative algebras

$$Cl(n+2) \cong Cl(n) \otimes C(2). \tag{C.5}$$

proof:
we need to construct map $\Phi : \mathbb{C}^{n+2} \to \mathbb{C}^n \otimes C(2)$. If we can show that this map is isomorphism not only in the vector space sense but have the same Clifford algebra structure. Then we finished the proof. We write $\mathbb{C}^{n+2} = \mathbb{C}^n \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$

$$\Phi(1) = x \otimes \sigma_y, \Phi(e_1) = 1 \otimes i\sigma_x, \Phi(e_2) = 1 \otimes i\sigma_z. \tag{C.6}$$

for $x \in Cn(n)$. One checks that Φ is Clifford and that the induced map $\Phi : \mathbb{C}^{n+2} \to \mathbb{C}^n \otimes C(2)$, being injective on generators and mapping between equidimensional spaces, is an isomorphism. Now we have all the clifford algebra defined on the complex number. If we know $C(0) \cong \mathbb{C}, C(1) \cong \mathbb{C} \oplus \mathbb{C}$.

D Little K theory Explained: why Clifford Algebra leads to K theory

The most difficult part of this project or this subject is the rigorous connection between the Clifford Algebra and the Homotopy group of the infinite orthonormal group. If someone goes to google this subject or see Kiteav's paper, one will hear a phrase called "Topological K-theory". If you are the first time meeting with algebraic topology, K theory will look notorious difficult to understand. Also, K theory is a vast subject it's impossible to explain any details in this paper. But I still want to give some basic ideas and tell how things get together. The K theory aims to study the vector bundle on the manifold X. This problem is very hard, so one gives some algebraic constructions to all the vector bundle and construct an abelian group called the K group. One can think K group is just like a cohomological functor on the category of the topological space to an Abelian group. To be more precisely, If we take the collection of isomorphism classes of real vector bundles over X and two to be equivalent if they become isomorphic after adding (direct sum) on trivial bundles, we get something called KX, the "real K-theory of X". I also give some notation, given $[X, Y]$ means the homotopy map from the topological map from $X \to Y$.

This space BG is called the classifying space of G because it has a principal $G - bundle$ over it, and given any decent topological space X (say a CW complex) you can get all principal G-bundles over X (up to isomorphism) by taking a map $f : X \to BG$ and pulling back this principal G-bundle over BG. Moreover, homotopic maps to BG give isomorphic G-bundles over X this way.

$$KX = [X, BO(\infty)] \tag{D.1}$$

where the right-hand side means "homotopy classes of maps from X to $BO(\infty)$. If we take X to be S^{n+1}, we see

$$KS^{n+1} = \pi_{n+1}(BO(\infty)) = \pi_n(O(\infty)). \tag{D.2}$$

It follows that we can get all elements of π_n of $O(\infty)$ from real vector bundles over S^{n+1}. If for any pointed space we define $K^n(X) = K(S^n \text{smash} X)$. We get a cohomology theory called K-theory, and it turns out that $K^{n+8}(X) = K(X)$ which is an isomorphism of groups - namely, Bott periodicity.

Ok, good, we spent a lot of time giving some basic idea of these ingredients. Now we should put them together. The Atiyah-Bott-Shapiro Theorem says that there's a natural map. To see all the details. [8]

$$M_{n-1}/i * M_n \to KO^{-n}(*) \cong K\tilde{O}(S^n) \tag{D.3}$$

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