RECENT PROGRESS IN ALGEBRAIC K–THEORY AND ITS RELATIONSHIP WITH TOPOLOGY AND ANALYSIS

Mini-Course Notes by Jonathan Rosenberg

Brief Outline

Notation

k — a regular commutative ring. (Main case of interest, as we shall see, is $k = \mathbb{Z}$.)
kG — the group ring over k of a group G.
BG — the classifying space of G, a space with contractible universal cover and fundamental group G. Unique up to homotopy equivalence. Has the property that $H_*(BG)$ is the Eilenberg-MacLane homology of G.
$\mathbb{K}(R)$ — the (non-connective) K-theory spectrum of a ring R. If one ignores negative K-theory, basically $K_0(R) \times BGL(R)^+$.
$K_i(R)$ — equal by definition to $\pi_i(\mathbb{K}(R))$.

I. "Assembly, Novikov Conjectures, and Control"

A. Applications of K-theory of group rings

Group rings arise in topology because if X is a space with universal cover $\widetilde{X}$, then $C_*(\widetilde{X};\, k) = C_*(X;\, k\pi_1(X))$, a chain complex of free $k\pi_1(X)$-modules.
Let

$$\mathrm{Wh}_1(G) = K_1(\mathbb{Z}G)/\left(\{\pm 1\} \times G_{\mathrm{ab}}\right),$$

$$\mathrm{Wh}_0(G) = K_0(\mathbb{Z}G)/\mathbb{Z}.$$

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These are related since

$$\mathrm{Wh}_0(G) \hookrightarrow \mathrm{Wh}_1(G \times \mathbb{Z}).$$

Classical applications of $K_1(\mathbb{Z}G)$.

- **Whitehead torsion** — If $X \hookrightarrow Y$ is a homotopy equivalence, its Whitehead torsion in $\mathrm{Wh}(\pi_1(X))$ is the obstruction to "simplicity."
- s-**Cobordism Theorem** — Suppose X^n, X'^n closed connected manifolds, W^{n+1} compact with boundary, $\partial W = X \amalg X'$. W is called an *h-cobordism* if $X \hookrightarrow W$, $X' \hookrightarrow W$ are homotopy equivalences. The Whitehead torsion of $X \hookrightarrow W$ in $\mathrm{Wh}(\pi_1(X))$ vanishes if $W \cong X \times [0, 1]$.
- If $n \geq 5$, "if" can be replaced by "iff," and *all* classes in $\mathrm{Wh}(\pi_1(X))$ can be realized by an h-cobordism.

So Whitehead torsion plays a big role in manifold theory. The oldest example: classification of lens spaces.

Classical applications of $K_0(\mathbb{Z}G)$:

- **Wall obstruction** — If X is a connected CW complex and if X is "finitely dominated," its Wall obstruction in $\mathrm{Wh}_0(\pi_1(X))$ is the obstruction to X being homotopy equivalent to a *finite* complex.
- Arises in space form problem (Swan *et al.*), problem of determining if a non-compact manifold is the interior of a compact manifold with boundary (Siebenmann).

B. Assembly

Out of $\mathbb{K}(k)$ we can make a homology theory $H_*(\,\cdot\,; \mathbb{K}(k))$ such that

$$H_*(\text{point}; \mathbb{K}(k)) = K_*(k).$$

This is close to ordinary homology with coefficients in $K_*(k)$:

$$H_*(X; K_*(k)) \Rightarrow H_*(X; \mathbb{K}(k)).$$

There is a natural map

$$H_*(BG; \mathbb{K}(k)) \xrightarrow{A} K_*(kG),$$

or to be fancier, a map of spectra

$$BG_+ \wedge \mathbb{K}(k) \to \mathbb{K}(kG),$$

called *assembly*. There are many possible definitions; here are a few:

- (Loday) Note that $G \hookrightarrow GL(kG)$, so one obtains an induced map

$$BG \to BGL(kG) \to BGL(kG)^+.$$

  Then use the product $\mathbb{K}(kG) \wedge \mathbb{K}(k) \to \mathbb{K}(kG)$.
- (Weiss-Williams) Any homotopy functor can be shown to have a "best approximation" by a homology functor. Apply this to $X \mapsto \mathbb{K}(k\pi_1(X))$. (To be rigorous one needs groupoids and ringoids because of basepoint problems.)

Examples:

(1) G arbitrary, $K_0(k) = \mathbb{Z}$. In degree 0, assembly is the map $\mathbb{Z} \to K_0(kG)$. In degree 1, assembly is the map
$$K_1(k) \times G_{\mathrm{ab}} \to K_1(kG).$$
So if $\mathrm{Wh}_*(G;\, k) = \operatorname{coker} A$, we recover $\mathrm{Wh}_0(G)$ and $\mathrm{Wh}_1(G)$ if $k = \mathbb{Z}$.

(2) $G = \mathbb{Z}$, $kG = k[t,\, t^{-1}]$. Assembly A is an isomorphism (recall we're assuming k regular)
$$K_*(k) \oplus K_{*-1}(k) \xrightarrow{\cong} K_*(k[t,\, t^{-1}]),$$
the "Fundamental Theorem."

(3) More generally, A an isomorphism for G free abelian or free (Quillen, Gersten).

(4) G finite. A can be interesting, but is usually very far from being an isomorphism. For example, if $k = \mathbb{Z}$, $H_p(G;\, K_q(\mathbb{Z})) \otimes \mathbb{Q} = 0$ for $p > 0$, so
$$\operatorname{rank} H_j(BG;\, \mathbb{K}(\mathbb{Z})) = \operatorname{rank} K_j(\mathbb{Z}) = 0 \text{ or } 1,$$
while $\operatorname{rank} K_j(\mathbb{Z}G)$ depends on the representation theory of G.

C. Novikov Conjectures

Conjectures: For G torsion-free, A is an isomorphism. (This includes the conjecture that $\mathrm{Wh}_*(G) = 0$.) For general G, A is rationally injective.

Connection with Topology: These are analogues of conjectures of Borel and Novikov (resp.) about "rigidity" of manifolds with large fundamental group. Those conjectures are equivalent to similar statements for the analogous assembly map
$$H_*(BG;\, \mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_*(\mathbb{Z}G).$$

Note an analogy: $\mathrm{Wh}_1(G)$ classifies h-cobordisms modulo trivial ones. Similarly, surgery theory says the homotopy groups of the cofiber of the L-theory assembly map classify homotopy-equivalent manifolds modulo homeomorphism. So we have the dictionary:

K-theory	$\leftrightarrow$	**L-theory**
Wh_*		"Structure sets"
$\mathrm{Wh}_* = 0$		Borel Conj.
$A \otimes \mathbb{Q}$ injective		Novikov Conj.

Note: For groups with torsion, Farrell and Jones conjecture that a more complicated assembly map is an isomorphism. The analogue in analysis is called the Baum-Connes Conjecture.

D. Some Recent Results

Theorem. (Bökstedt-Hsiang-Madsen) *The K-theory Novikov Conjecture is true for $k = \mathbb{Z}$, $H_i(G)$ finitely generated $\forall i$.*

Theorem. (Farrell-Jones) *The K-theory isomorphism conjecture is rationally true for $k = \mathbb{Z}$, G cocompact in a connected Lie group.*

Theorem. (Carlsson-Pedersen) *The K-theory assembly map is split injective for BG compact, EG with a G-equivariant contractible compactification "small at ∞."*

E. K-Theory with Control

Many of the proofs of the results above are based on the idea of *K-theory with control.* This is motivated by *controlled topology*, of which a prototype result is:

Theorem. (Chapman-Ferry) *Given X^n with $n \geq 5$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that any δ-controlled homotopy equivalence $f : X^n \to X'^n$ is ε-homotopic to a homeomorphism. (The condition means that if $g : X' \to X$ is the homotopy inverse, the homotopies from $f \circ g$ to $1_{X'}$ and from $g \circ f$ to 1_X don't move points more than distance δ.)*

Idea of Controlled K-Theory: Given the space X, consider *locally finite* free or projective k-modules $A = \bigoplus_{x \in X} A_x$, with morphisms

$$\phi : A \to B, \quad \phi = \bigoplus_{x,y} \phi_y^x : A_x \to B_y$$

vanishing if x and y "not close." Out of this category, manufacture K-theory $\mathbb{K}(k; X)$ the usual way.

Two main cases:

- **Bounded Case:** X a metric space, and we require that given ϕ, $\exists d_\phi > 0$ such that $\phi_y^x = 0$ if $d(x, y) > d_\phi$.
- **Continuously Controlled Case:** X open dense in $\overline{X}$, and $\forall z \in \overline{X} \smallsetminus X$ and $\forall$ nbhd. V of z in $\overline{X}$, $\exists$ nbhd. U of z in $\overline{X}$ with $\phi_y^x = 0$, $\phi_x^y = 0$ for $x \in V$, $y \notin U$.

Connection with Assembly: For BG finite, the usual assembly map for G is the map on G-fixed points of a G-equivariant bounded assembly map for $X = EG$:

$$\mathbb{H}^{\mathrm{lf}}(X; \mathbb{K}(k)) \to \mathbb{K}(k; X \ (\text{bdd.})).$$

If X has a compactification $\overline{X}$, can study "forget control" map from bounded control on X to continuous control at ∞.

An Additional Application: Other proofs of the topological invariance of Whitehead torsion (*a priori* a simplicial notion) (Ferry-Pedersen, Ranicki-Yamasaki).

F. Concluding Remarks

- Confluence of very different branches of mathematics
- Still many open questions

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II. "K-Theory, Secondary Invariants, and Differential Geometry"

A. Flat Vector Bundles

Definition: If X is a manifold, a vector bundle E on X is called *flat* if it has a connection with zero curvature. This implies $E = \widetilde{X} \times_\sigma V$, where $\widetilde{X}$ is the universal cover of X and σ is a representation of $G = \pi_1(X)$ on V.

Thus flat bundles correspond to representations σ of G. The stable class of such an E gives a map

$$X \to BG \xrightarrow{B\sigma} BGL(k) \hookrightarrow BGL(k)^+,$$

where $k = \mathbb{R}$ or $\mathbb{C}$ (with the *discrete*) topology. Ignoring the flat structure amounts to looking at the map $X \to BGL(k)^{\text{top}}$.

By Chern-Weil theory, the rational Chern (Pontrjagin) classes of a complex (resp., real) flat bundle must vanish. The bundle can still be non-trivial topologically, but we have:

Proposition. *If E is a flat vector bundle over a finite CW complex X, then $E \oplus E \oplus \cdots \oplus E$ (m summands) is trivial for some m, and $E \otimes E \otimes \cdots \otimes E$ (n factors) is trivial for some n.*

Proof. Say E has rank r. Then $[E] - r$ lies in the augmentation ideal of $K(X)$ and is a torsion element since its rational characteristic classes vanish. The first statement follows from this since $E \oplus E \oplus \cdots \oplus E$ is eventually in the "stable range." As for the second statement, if $r = 1$, then since $c_1(E)$ (or $w_1(E)$ in the real case) is torsion, some tensor power of E is trivial. If $r > 1$, then in K-theory we have

$$[E]^n = (r + ([E] - r))^n = \sum_j \binom{n}{j} r^{n-j}([E] - r)^j,$$

and we can make all the terms with $j > 0$ vanish for suitably high n since $[E] - r$ is both torsion and nilpotent. Again, $E \otimes E \otimes \cdots \otimes E$ is eventually in the stable range, so we can go back from K-theory to bundles. □

All this suggests studying the relationship between $\mathbb{K}(k)^{\mathrm{alg}}$ and $\mathbb{K}(k)^{\mathrm{top}}$.

Theorem. (Suslin) *The natural map $\mathbb{K}(k)^{\mathrm{alg}} \to \mathbb{K}(k)^{\mathrm{top}}$ induces an isomorphism on K-groups with finite coefficients (in positive degrees).*

However, every class in $K_2(k)$ (which is a huge group) arises from some flat bundle over T^2 (topologically trivial except for w_2 when $k = \mathbb{R}$).

Chern-Simons Invariants: If E is a vector bundle over X (say over $k = \mathbb{C}$) with a flat connection θ, then $d\theta + \frac{1}{2}[\theta, \theta] = 0$. Out of $[\theta, \theta]$, Chern-Simons (*et al.*) manufacture *secondary invariants* $\widehat{C}_k(E, \theta) \in H^{2k-1}(X; \mathbb{C}/\mathbb{Z})$ with $\beta\widehat{C}_k(E, \theta) = -c_k(E) \in H^{2k}(X; \mathbb{Z})$.

These can be related to invariants coming from $K_*(\mathbb{C})$. For example:

Theorem. (Dupont) *$2\widehat{C}_2$ coincides with a homomorphism $H_3(SL(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\mathbb{Q}$ given by the dilogarithm (cf. Bloch-Wigner).*

B. Secondary Invariants of Elliptic Operators

Roughly speaking, elliptic operator are to K-homology what vector bundles are to K-cohomology, and there are similar secondary invariants. The most famous are the η and ρ invariants for odd-dimensional manifolds. The η-invariant of a self-adjoint elliptic operator D is a measure of the asymmetry of the spectrum, and is defined by

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}(De^{-tD^2})\, dt.$$

Formally, this is

$$\sum_{\lambda \in \mathrm{Spec}\, D} \mathrm{sign}(\lambda),$$

but the sum is not convergent. The main application is:

Theorem. (Atiyah-Patodi-Singer) *If M^{2n} is compact Riemannian with boundary N^{2n-1} (and product metric near the boundary) and D_M is the Dirac operator on M, then for suitable boundary conditions,*

$$\mathit{Ind}\, D_M = \int_M \widehat{\mathcal{A}}(M) - \tfrac{1}{2}\eta(D_N).$$

The η-invariant is closely related to Chern-Simons classes. One way to see this is to twist D both by a flat bundle (E, θ) and by a trivial flat bundle of the same dimension. If one subtracts the index formulas for the two cases, the integral cancels and one gets an invariant $\eta(E, \theta)$ in $\mathbb{C}/\mathbb{Z}$ ($\mathbb{R}/\mathbb{Z}$ if the associated representation is unitary). This is explained as follows:

Theorem. (Atiyah-Patodi-Singer) *A flat bundle* (E, θ) *on* N^{2n-1} *defines a class in*

$$K^{-1}(N; \mathbb{C}/\mathbb{Z})$$

(basically same as Chern-Simons). The invariant $\eta(E, \theta)$ *is this class paired with the class in* $K_1(N)$ *of the relevant elliptic operator.*

Other Secondary Invariants Related to K-Theory.

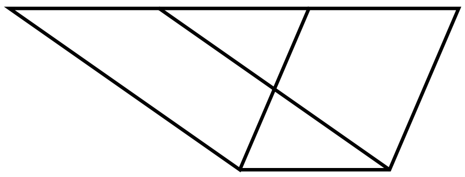
Helton-Howe invariants of almost-commuting operators, Connes-Karoubi multiplicative character of a p-summable Fredholm operator.

C. Invariants of Hyperbolic 3-Manifolds

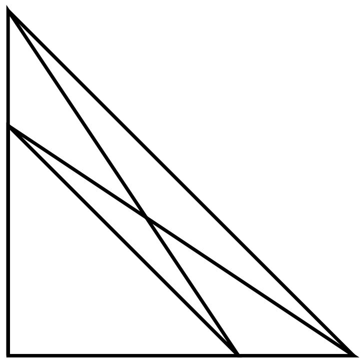
Hilbert's Third Problem: If P_1 and P_2 are polyhedra of the same volume, can P_1 be cut into finitely many pieces congruent to pieces that can be reassembled to give P_2?

Equivalence of Two-Dimensional Polygons

1. **Two parallelograms of the same base and height are scissors-congruent.**



2. **Two triangles of the same area are scissors-congruent.**



Answer: In dimension 3, Hilbert conjectured the answer was "no," and this was proven by Dehn (1900). But the negative answer itself raises a question, to describe the equivalence classes for *scissors congruence.* This is especially interesting for polyhedra in hyperbolic 3-space.

Theorem. (Dupont-Sah) *The scissors congruence group of polyhedra in hyperbolic 3-space is divisible without 2-torsion. It is the (-1)-eigenspace for complex conjugation* *acting on $\mathcal{P}(\mathbb{C})$, where $\mathcal{P}(\mathbb{C})$ fits into the exact sequence*

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL(2,\mathbb{C})^d;\mathbb{Z}) \to \mathcal{P}(\mathbb{C}) \to \mathbb{C}^\times \wedge \mathbb{C}^\times \xrightarrow{\text{Steinberg symbol}} K_2(\mathbb{C}) \to 0.$$

Aspherical Manifolds and K-Theory: Now suppose G is a group for which BG can be taken to be a compact oriented manifold M^n, and one has a representation $G \xrightarrow{\sigma} GL(k)$. This induces a map $BG \xrightarrow{B\sigma} BGL(k)$ and thus a class $B\sigma_*([M])$. We can detect it via any invariant of $K_*(k)$ that factors through the Hurewicz homomorphism.

Application to Hyperbolic 3-Manifolds: If M^3 is a compact oriented hyperbolic 3-manifold, its fundamental group G comes with map to $PSL(2,\mathbb{C})$. Thus basically one gets a class in $H_3(GL(\mathbb{C}))$. Recall:

Theorem. (Suslin) *For a field k,*

$$K_n^M(k) \cong H_n(GL(n,k))/H_n(GL(n-1,k)).$$

Modulo torsion, $K_3(k)$ is built out of $K_3^M(k)$ and the *Bloch group* $\mathcal{B}(k)$.

Since it comes from $H_3(GL(2))$, the homology invariant of a hyperbolic 3-manifold should basically live in the Bloch group.

This is confirmed by:

Theorem. (Neumann-Yang) *An oriented finite-volume hyperbolic 3-manifold M^3 has an invariant $\beta(M) \in \mathcal{B}(\mathbb{C})$ (roughly speaking, its scissors congruence class). In fact, $\beta(M) \in \mathcal{B}(k)$ for an associated number field $k(M)$. Under the (normalized) Bloch regulator $\mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$, $\beta(M)$ goes to $\mathrm{vol}(M) + iCS(M)$.*

If $k(M)$ is embedded in $\mathbb{C}$ as an imaginary quadratic extension of a totally real number field, then $CS(M)$ is rational.

(Conjecturally, $CS(M)$ is irrational if $k(M) \cap \overline{k(M)} \subset \mathbb{R}$. This seems to be backed by numerical evidence.)

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