Consider the Lie group $\text{SL}_2 \mathbb{R}$ of linear transformations of $\mathbb{R}^2$ with determinant 1, which we can represent via matrices as

$$\text{SL}_2 \mathbb{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\},$$

and the Borel subgroup

$$B := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

By definition, we have a natural action of $\text{SL}_2 \mathbb{R}$ on $\mathbb{R}^2$, and this induces an action on the projective line $\mathbb{RP}^1$, the space of 1-dimensional linear subspaces of $\mathbb{R}^2$. Under this action, the subgroup $B$ is the stabilizer of the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{RP}^1$ corresponding to the line generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so since $\text{SL}_2 \mathbb{R}$ acts transitively on $\mathbb{RP}^1$, we may identify the homogeneous space $\text{SL}_2 \mathbb{R}/B$ with $\mathbb{RP}^1$.

Each 1-dimensional subspace of $\mathbb{R}^2$ intersects the unit circle at a unique pair of antipodal points, so we can think of $\mathbb{RP}^1$ as the circle with antipodal points identified. Moreover, all but one point of $\mathbb{RP}^1$ is
of the form \((\frac{1}{t})\) for some \(t \in \mathbb{R}\), so we can also think of \(\mathbb{RP}^1\) as a copy of the affine line together with a “point at infinity” \((\frac{0}{1})\).

As transformations, the one-parameter subgroup \(\exp(s\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\) acts by translations on the affine line and fixes the point at infinity:

\[
\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ s + t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

**Figure 2.** The transformation \(\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\) acts by translating along the copy of the affine line

**Figure 3.** A depiction of the left-action of \(\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\) on the circle with antipodal points identified
The one-parameter subgroup \( \exp(s[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]) = [\begin{smallmatrix} e^s & 0 \\ 0 & e^{-s} \end{smallmatrix}] \), on the other hand, acts by rescaling the affine line by \( e^{-2s} \), and also fixes the point at infinity:

\[
\begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} e^s \\ e^{-s}t \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-2st} \end{bmatrix}
\]

and

\[
\begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-s} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

\[\text{Figure 4. The transformation } \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \text{ acts by rescaling the copy of the affine line by } e^{-2s}.\]

\[\text{Figure 5. A depiction of the left-action of } \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \text{ on the circle with antipodal points identified.}\]
Finally, we have the one-parameter subgroup \( \exp \left( s \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \), which is a bit of an oddity. The transformation \( \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \) fixes the point \( \left( \frac{1}{0} \right) \), as the rescaling matrix did, but not the "point at infinity". Instead, it acts by translations along another copy of the affine line, given by \( \{ (t) : t \in \mathbb{R} \} \); with respect to this new affine line, the point \( \left( \frac{1}{0} \right) \) would be thought of as the "point at infinity":

\[
\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} s + t \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Figure 6. A depiction of the left-action of \( \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \) on the circle with antipodal points identified.

On the original copy of the affine line, we have

\[
\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 + st \\ t \end{pmatrix},
\]

which corresponds to the point \( \frac{t}{1+st} \in \mathbb{R} \) when \( 1 + st \neq 0 \). I’ve gotten in the habit of calling these things “(unipotent) tilts”, for reasons that will be clearer in a moment, though there isn’t really a standard terminology for these transformations.

An additional, and convenient, one-parameter subgroup for this geometry is \( \exp \left( \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \), which corresponds to just rotating the circle (while keeping antipodal points identified). The reason that this one-parameter subgroup is convenient here is that it happens to act transitively on \( \text{SL}_2 \mathbb{R}/B \), since it just rotates along the circle.
As I explained in the last lecture, we think of SL$_2\mathbb{R}$ as the space of configurations of an observer over SL$_2\mathbb{R}/B \cong \mathbb{RP}^1$. Each matrix $A \in$ SL$_2\mathbb{R}$ uniquely determines a pair of column vectors $A([1 \ 0])$ and $A([0 \ 1])$. The vector $A([1 \ 0])$ generates the line $\langle A([1 \ 0]) \rangle$ corresponding to $q_B(A)$, and $A([0 \ 1])$ determines the copy of the affine line along which we move when we right-translate by $[1 \ 0]$. Indeed, for $[u \ v] \in$ SL$_2\mathbb{R}$, we have

$$[u \ v] \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = [u + sv \ v],$$

so when we right-translate $[u \ v]$ by $[1 \ 0]$, we move from the point $q_B([u \ v])$ corresponding to the line $\langle u \rangle$ to the point $q_B([u + sv \ v])$ corresponding to the line $\langle u + sv \rangle$.

![Figure 7](image)

**Figure 7.** In the matrix $[u \ v] \in$ SL$_2\mathbb{R}$, the vector $u$ determines the line $\langle u \rangle$ corresponding to the point $q_B([u \ v])$ and $v$ determines a copy of the affine line along which we move when we right-translate by $[1 \ 0]$. The rescalings, tilts, and $-1$ together generate the subgroup $B$, so given a configuration $g \in$ SL$_2\mathbb{R}$ over a point $q_B(g) \in$ SL$_2\mathbb{R}/B$, changing to a different configuration over that point corresponds to right-translating by compositions of these tilts, rescalings, and $-1$. As with affine geometry, it is worth taking a moment to imagine what this looks like.

As we might expect, right-translating $A$ by a rescaling just rescales ourselves along the affine line determined by $A([1 \ 0])$. The unipotent tilts, on the other hand, *tilt* the affine line determined by $A([0 \ 1])$ along the line corresponding to $q_B(A)$:

$$[u \ v] \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = [u \ su + v].$$
Figure 8. Right-translating $[u \ v] \in \text{SL}_2 \mathbb{R}$ by a unipotent tilt takes the affine line determined by $v$ and tilts it along the line determined by $u$. 