An embedding theorem for automorphism groups of Cartan geometries

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1 Introduction

The main motivation of this paper arises from classical questions about actions of Lie groups preserving a geometric structure: which Lie groups can act on a manifold preserving a given structure, and which cannot? Which algebraic properties of the acting group have strong implications on the geometry of the manifold, such as local homogeneity, or on the dynamics of the action? These questions have been studied extensively by many authors ([AS1], [AS2], [DA], [DAG], [Gr], [Kow], [Ze1], [Ze2], [Zi1], [Zi2] among others). These papers primarily prove results in the setting of either rigid geometric structures of algebraic type, which were defined by M. Gromov in [Gr], or of Q-structures of finite type, which were introduced by Cartan.

Our aim in this paper is to develop new tools for studying the kind of questions mentioned above, in the setting of Cartan geometries. This notion of geometric structures was developed by E. Cartan in the first half of the last century. A Cartan geometry infinitesimally models a manifold M on a homogeneous space G/P, where G is a Lie group and P is a closed subgroup. The geometry is *flat* if and only if M is locally modeled on G/P—that is, M is a $(G, G/P)$-manifold. The framework of Cartan geometries offers a very satisfactory degree of generality, since it essentially comprises all classical rigid geometric structures, including pseudo-Riemannian metrics, conformal

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pseudo-Riemannian structures, non-degenerate CR structures, and projective structures.

The main results of this paper are an embedding theorem (theorem 1.2) comparable to Zimmer's well-known embedding theorem for automorphism groups of Q-structures; its algebraic consequences on automorphism groups (theorem 1.3); its consequences for parabolic geometries admitting automorphism groups of high rank (theorem 1.5); and its consequences for geometries additionally equipped with a finite continuous volume form (theorems 1.6 and 1.7). We now give some background and discuss in detail these results.

1.1 Statement and discussion of main theorem

The Lie algebra of a Lie group G is denoted $\mathfrak{g}$.

Definition 1.1. *Let G be a Lie group and P a closed subgroup. A* Cartan geometry *(M, B, ω) modeled on $(\mathfrak{g}, P)$ is*

1. *a principal P-bundle $\pi : B \to M$*
2. *a $\mathfrak{g}$-valued 1-form ω on B satisfying*
   - *for all $b \in B$, the restriction $\omega_b : T_bB \to \mathfrak{g}$ is an isomorphism*
   - *for all $b \in B$ and $Y \in \mathfrak{p}$, the evaluation $\omega_b(\left.\frac{d}{dt}\right|_0 be^{tY}) = Y$*
   - *for all $b \in B$ and $p \in P$, the pullback $R_p^*\omega = (Ad\ p^{-1}) \circ \omega$*

The automorphism group $\mathrm{Aut}(M, B, \omega)$ consists of the bundle automorphisms of B preserving ω. The diffeomorphisms of M that lift to elements of $\mathrm{Aut}(M, B, \omega)$ will be denoted $\mathrm{Aut}\ M$. When the action of G on G/P has finite kernel, a very mild assumption that we will always make in this paper, it can be shown that the projection from $\mathrm{Aut}(M, B, \omega)$ to $\mathrm{Aut}\ M$ also has finite kernel. The automorphisms of a framing on a manifold form a Lie group (see [Ko] I.3.2), from which it follows that $\mathrm{Aut}\ M$ is a Lie group.

Cartan geometries are closely related to *Q-structures of finite type* (See, for example, [Ko]. Note that the term *G-structures* is more common, but for notational reasons, we use Q). Kobayashi proved that for a linear Lie group Q of *finite type*, the automorphism group of any Q-structure on any M is a Lie group. For such structures, he constructs a principal subbundle B of the frame bundle of some order, equipped with a framing preserved by all automorphisms of the Q-structure. His construction associates to a Q-structure of finite type a Cartan geometry with the same automorphism group. Our theorems on automorphism groups of Cartan geometries thus apply also to automorphism groups of Q-structures of finite type.

For many differential-geometric structures, including pseudo-Riemannian metrics, conformal structures, and, more generally, *parabolic structures* [CS], there is a canonical associated Cartan geometry modeled on some G/P, and in fact a bijective correspondence between Cartan geometries modeled on G/P and such structures. In this case, the automorphism group of the structure coincides with the automorphism group of the associated (M, B, ω). The problem of associating a canonical Cartan geometry to a geometric structure is called the *equivalence problem* (see [Shar]). Many interesting examples can be found in [CS], [CM], [Shar], [Ta]. Some basic examples are given in section 1.3.

Conventions and notations:

The Lie algebra of a Lie group will be denoted by the same symbol printed in the fraktur font; for example, the Lie algebra of G is denoted $\mathfrak{g}$.

Throughout the article, M is a connected manifold, and (M, B, ω) is a Cartan geometry modeled on $\mathbf{X} = G/P$. We will make the following global assumptions.

- *The pair (G, P) is such that the G-action on G/P has finite kernel; in practice this assumption is not restrictive.*
- *The image of P under Ad, denoted $Ad_{\mathfrak{g}}P$, is Zariski closed in Aut $\mathfrak{g}$.*
- *For a Lie subgroup $H < Aut\, M$, the kernel of $Ad : H \to Aut\, \mathfrak{h}$ is amenable; note that for H connected, $\ker Ad = Z(H)$ is always amenable.*

For a subgroup $S < H$, the group $\bar{S}^{\mathrm{Ad}}$ is the Zariski closure of Ad S in Aut $\mathfrak{h}$, and $\bar{S}_d$ is the maximal Zariski closed subgroup of $\bar{S}^{\mathrm{Ad}}$ with no proper cocompact algebraic normal subgroups; see section 3 below for details. The main theorem of the paper is the following.

Theorem 1.2. *Let H be a Lie subgroup of Aut M. Let S be a subgroup of H preserving a finite measure on M. Then there are*

- *$\check{S} < Ad_{\mathfrak{g}}P$ an algebraic subgroup and*
- *$\iota : \mathfrak{h} \to \mathfrak{g}$ a linear monomorphism*

such that $\iota(\mathfrak{h})$ is $\check{S}$-invariant, and ι intertwines $\check{S}\big|_{\iota(\mathfrak{h})}$ and $\bar{S}_d$.

Section 4 contains a more detailed statement, in theorem 4.1.

Theorem 1.2 is similar to the embedding theorem of Zimmer [Zi2], which says that when a connected simple group H acts on a manifold M preserving a finite volume and a Q-structure, with Q algebraic, then the standard representation of Q contains the adjoint representation of H. In particular, Q contains a subgroup locally isomorphic to H. Zimmer's result is proved using algebraic hulls of measurable cocycles and the Borel density theorem. An important first step in the proof is to show that a finite-measure-preserving action of a simple group has discrete stabilizers almost everywhere. In this case, the Lie algebra $\mathfrak{h}$ linearly embeds in the tangent space at a point, on which Q acts via its standard representation.

Our result here does not require that H be simple, nor be connected, nor preserve a finite measure on M, nor that the action have any discrete stabilizers. The subgroup $S < H$ is not even required to be a Lie subgroup. The Cartan connection gives linear embeddings of $\mathfrak{h}$ in $\mathfrak{g}$ at every point of B. Then measure-theoretic and algebraic arguments, including the Borel density theorem, give the existence of $\check{S} < \mathrm{Ad}_{\mathfrak{g}}P$ restricting to $\bar{S}_d$ on some image of $\mathfrak{h}$. Unfortunately, the theorem gives no information for H discrete.

An embedding theorem of comparable generality in the setting of rigid geometric structures of algebraic type appears in [Gr] 5.2.A. We find that the value of our embedding theorem in the setting of Cartan geometries lies in the several geometric applications detailed below.

Of course, theorem 1.2 has nice consequences when H preserves a finite measure (see theorems 1.6, 1.7, 1.8 and 1.9). Only the subgroup S, however, is required to be measure-preserving. When M is compact, for example, the theorem applies to every amenable subgroup $S < H$. In this way, the theorem yields information on automorphisms of conformal structures, projective structures, and other parabolic structures that do not determine a volume form (see theorems 1.3 and 1.5).

When H is a connected, simple subgroup of Aut M, and H preserves a volume on M and a Q-structure of finite type, then we can obtain the conclusion of Zimmer's theorem from theorem 1.2. On the other hand, Zimmer's embedding theorem deals with arbitrary Q-structures with Q algebraic; in particular, it does not require finite type, so there may not be any associated Cartan geometries. With respect to this comparison, it could be said that we have exchanged the rigidity of simple group actions for rigidity of the structure supporting the action.

1.2 Bounds on automorphism groups

Here we deduce from the main theorem coarse information on the possible groups of automorphisms of (M, B, ω) when M is compact.

For a linear group $R < \mathrm{GL}(\mathbf{R}^n)$, we will define the *real rank* (resp. the *algebraic real rank*), denoted $\mathrm{rk}\ R$ (resp. $\mathrm{rk}^{\mathrm{alg}} R$), to be the maximal dimension of a simultaneously $\mathbf{R}$-diagonalizable subgroup of R (resp. of the Zariski closure of a simultaneously $\mathbf{R}$-diagonalizable subgroup of R). The inequality $\mathrm{rk}\ R \leq \mathrm{rk}^{\mathrm{alg}} R$ always holds. It can be strict, for example, for $R \cong \Gamma \ltimes \mathbf{R}^n$, where Γ is a lattice in a semisimple group $G < GL(\mathbf{R}^n)$ with $\mathrm{rk}\ G > 0$ (see [PR], [Mo]). When R is semisimple, then $\mathrm{rk}(\mathrm{Ad}\ R) = \mathrm{rk}^{\mathrm{alg}}(\mathrm{Ad}\ R)$ is the usual $\mathbf{R}$-rank of R; in fact, for any faithful representation ρ of R, the real rank $\mathrm{rk}\ \rho(R) = \mathrm{rk}^{\mathrm{alg}} \rho(R)$, and agrees with the usual $\mathbf{R}$-rank of R. For R an abstract semisimple group, we will sometimes write $\mathrm{rk}\ R$ below without reference to a specific representation.

For a Lie group R, we define the *nilpotence degree* $n(R)$ to be the maximum nilpotence degree of a connected nilpotent subgroup of R.

Theorem 1.3. *Let (M, B, ω) be a Cartan geometry modeled on G/P with M compact. Let H be a Lie subgroup of Aut M. Then*

1. *$rk^{alg}(Ad\ H) \leq rk(Ad_{\mathfrak{g}} P)$.*

2. *$n(Ad\ H) \leq n(Ad_{\mathfrak{g}} P)$; in particular, if H is connected, then*

$$n(H) \leq n(P) + 1$$

The bound in (1) is comparable to a theorem of Zimmer for automorphisms of Q-structures. He proved in [Zi4] that, for M compact, equipped with a Q-structure, where $Q < \mathrm{GL}(\mathbf{R}^n)$ is algebraic, any simple $H \subset \mathrm{Aut}\ M$ has $\mathrm{rk}\ H \leq \mathrm{rk}\ Q$. Note we assume H is neither simple nor connected.

The bound (1) is tight: if G is a simple noncompact Lie group, and P is a parabolic subgroup of G, then $\mathbf{X} = G/P$ is a compact Cartan geometry modeled on itself, on which G acts by automorphisms, and (1) is an equality in this case.

The bound in (2) is also tight; an example is found among actions of connected nilpotent groups by isometries of a compact Lorentz manifold. Indeed, in this case, $P = O(1, n-1)$, the nilpotent subgroups of which are all abelian. The bound given by the theorem in this case is $n(H) \leq 2$, which is tight since there exist isometric actions of the Heisenberg group on compact Lorentz manifolds. The bound (2) was first proved for Lorentz-isometric actions by Zimmer in [Zi2]. For parabolic Cartan geometries, we conjecture the sharper bound $n(H) \leq n(P)$, and we have proved it in the case of conformal pseudo-Riemannian structures ([FM]).

The above theorem does not require that H be connected. Nevertheless, when H is a discrete group, it yields no information. The techniques of the proof do yield results for semi-discrete groups:

Theorem 1.4. *Let $L < GL(V)$ be an algebraic group with no compact algebraic quotients. Let Γ be a lattice in L, and $H = \Gamma \ltimes V$. Suppose that $H < Aut\ M$ for M a compact Cartan geometry modeled on G/P. Then*

1. *$rk\ L \leq rk(Ad_{\mathfrak{g}} P)$*
2. *$n(L) \leq n(Ad_{\mathfrak{g}} P)$*

1.3 A rigidity result for parabolic geometries

One class of interesting classical geometries for which the equivalence problem has been solved is that of *parabolic geometries* (see for example [CS]). A Cartan geometry (M, B, ω) modeled on $\mathbf{X} = G/P$ is said to be *parabolic* when G is a real semisimple Lie group and P is a parabolic subgroup of G. The main examples of such geometries follow.

• *Conformal pseudo-Riemannian structures.*

A conformal pseudo-Riemannian structure of signature (p, q) on a manifold M is the data of a conformal class of metrics

$$[g] = \{e^{\sigma} g \mid \sigma \in C^{\infty}(M)\}$$

where g is a metric of signature (p, q). These structures are in canonical correspondence with Cartan geometries modeled on $O(p+1, q+1)/P$, where

$$P \cong (\mathbf{R}^* \times O(p, q)) \ltimes \mathbf{R}^{p+q}$$

is a maximal parabolic subgroup of $O(p+1, q+1)$, and the connection ω satisfies certain normalization conditions.

• *Nondegenerate CR-structures.*

These structures model real hypersurfaces in complex manifolds. A nondegenerate CR-structure of type (p, q) on a $(2m+1)$-dimensional manifold M, where $p+q = m$, is the data of a contact subbundle $E \subset TM$ equipped with an almost-complex structure J and a conformal class of Hermitian metrics of type (p, q). Such a structure is equivalent to a unique normal Cartan geometry modeled on $SU(p+1, q+1)/P$, where P is the parabolic subgroup of $SU(p+1, q+1)$ stabilizing an isotropic complex line in $\mathbf{C}^{p+1,q+1}$. Again, the connection ω must satisfy some normalization conditions. The equivalence problem for strictly pseudo-convex CR-structures was first solved by E. Cartan in dimension 3, and in the general case in [CM], [Ta], [CS].

• *Projective structures.*

A *projective structure* on an n-dimensional manifold M is a family of smooth curves $C : I \to M$, defined locally by an ODE of the form

$$\frac{c_1'' + P_1(c')}{c_1'} = \frac{c_2'' + P_2(c')}{c_2'} = \dots = \frac{c_n'' + P_n(c')}{c_n'}$$

with $P_k(x) = \Sigma a_{ij}^k x_i x_j$ for smooth functions a_{ij}^k. Geodesics of a Riemannian or pseudo-Riemannian metric, for example, satisfy such an equation. To such a projective structure corresponds a unique normalized Cartan geometry modeled on the projective space $\mathbf{RP}^n = PGL(n+1, \mathbf{R})/P$, where P is the stabilizer of a line in $\mathbf{R}^{n+1}$ (see [Shar] ch. 8 for details).

We now come to our main result concerning parabolic geometries. Note that such a geometry (M, B, ω) is modeled on G/P with $\mathrm{rk}(\mathrm{Ad}_{\mathfrak{g}} P) = \mathrm{rk}\ G$. Theorem 1.3 asserts that when a Lie group H acts by automorphisms of (M, B, ω), then $\mathrm{rk}^{\mathrm{alg}}(\mathrm{Ad}\ H) \leq \mathrm{rk}\ G$. When H is connected, and equality $\mathrm{rk}(\mathrm{Ad}\ H) = \mathrm{rk}\ G$ holds, a strong rigidity phenomenon occurs. The universal covers of $\mathbf{X}$ and G are denoted $\widetilde{\mathbf{X}}$ and $\widetilde{G}$, respectively.

Theorem 1.5. *Let (M, B, ω) be a compact parabolic Cartan geometry, modeled on $\mathbf{X} = G/P$, where G has finite center, and ω is regular. Let $H < Aut\ M$ be a Lie subgroup.*

1. *If $rk^{alg}(Ad\ H) = rk\ G$, then there is a Lie algebra embedding $\mathfrak{h} \to \mathfrak{g}$.*
2. *If H is connected, and $rk(Ad\ H) = rk\ G$, then $M \cong \Gamma\backslash\widetilde{\mathbf{X}}$, for some discrete subgroup $\Gamma < \widetilde{G}$.*

The regularity condition on the Cartan connection is very mild. For all parabolic geometries for which the equivalence problem has been solved, regularity is one of the normalization conditions on the Cartan connection. See section 6.1 for the definition of regular connection.

Theorem 1.5 is a wide generalization of previous results obtained in [BN] and [FZ] for conformal actions of connected simple groups on compact pseudo-Riemannian manifolds. For parabolic Cartan geometries modeled on G/P with rk $G = 1$, existence of a nonproper automorphism group implies that M is equivalent to G/P or the complement of a point in G/P. In conformal Riemannian geometry, this statement is a celebrated theorem of Lelong-Ferrand [LF]; it is proved for general rank-one parabolic Cartan geometries in [F].

1.4 Automorphism groups preserving a finite volume

By a finite volume on a manifold M, we mean a continuous volume form ν on M, such that $\int_M \nu < \infty$. This section contains results on actions by automorphisms of a Cartan geometry preserving a finite volume (Theorem 1.6 makes the slightly weaker assumption that the action preserves a finite measure of full support). Isometric actions on pseudo-Riemannian manifolds preserve a volume form, which is always finite if the manifold is compact. Other interesting examples of such actions are affine actions preserving a finite volume. Note that the invariant volume form need only be C^0.

Recall that $\bar{H}_d$ is the discompact radical of $\bar{H}^{\mathrm{Ad}}$ (see §3). In [DAG], it is conjectured that any smooth, faithful, volume-preserving action of a simple Lie group on a compact manifold is everywhere locally free. Here, we prove:

Theorem 1.6. *Let (M, B, ω) be a Cartan geometry, and let $H < Aut\ M$ be a Lie subgroup preserving a finite measure of full support in M. Then for* *almost-every $x \in M$, the stabilizer Lie algebra $\mathfrak{h}_x$ is $\bar{H}_d$-invariant. If there are only finitely-many $\bar{H}_d$-invariant subalgebras of $\mathfrak{h}$, then the H-action is locally free on a dense, open, conull subset of M.*

This theorem is proved in [Zi5] for connected semisimple Lie groups with no compact factor. Our conclusion of local freeness here holds for a wider class of groups, including semidirect products $H = \Gamma \ltimes_\psi R$, where R is a connected Lie group, Γ is a lattice in a semisimple group L, and ψ is the restriction of a nontrivial representation of L in Aut $\mathfrak{r}$.

The next theorem says that when a group of automorphisms of a Cartan geometry preserves a finite volume, then existence of an open orbit implies homogeneity; in this case, the action is everywhere locally free. In general, it is an interesting open question for a geometric manifold when existence of an open locally homogeneous subspace implies local homogeneity everywhere ([DAG], [Du], [Me1]).

Recall that $\bar{H}^{\mathrm{Ad}}$ denotes the Zariski closure of Ad H, and $\bar{H}_d$ its discompact radical.

Theorem 1.7. *Let (M, B, ω) be a Cartan geometry, and let $H <$ Aut M be a Lie subgroup. If H preserves a finite volume on M and has an open orbit on M, then*

1. *The action of H^o, the identity component of H, is transitive on M.*
2. *If $\bar{H}^{Ad} = \bar{H}_d$, the action of H is everywhere locally free.*
3. *If $\bar{H}^{Ad} = \bar{H}_d$, the stabilizers of points in H^o are lattices $\Gamma \subset H^o$.*

Essentially all classical rigid geometric structures can be described both as Cartan geometries and rigid geometric structures of algebraic type in the sense of Gromov. The interplay of these two points of view can be fruitful, as the following corollary shows. Here we restrict to real-analytic structures and actions, in order to apply certain results of Gromov.

Corollary 1.8. *Let (M, B, ω) be a C^{an} Cartan geometry with M simply connected. Assume that there is a natural associated C^{an} rigid structure $\mathcal{S}$ of algebraic type on M with $Aut_{loc}M = Aut_{loc}(M, \mathcal{S})$. Let ν be a C^{an} finite volume form on M. Let $H <$ Aut M be a Lie subgroup such that H preserves ν and has a dense orbit in M. Then the geometry is homogeneous: there exists $H' <$ Aut M acting transitively on M and preserving ν.*

1.5 Homogeneous reductive geometries

Finally, we record a corollary for those Cartan geometries called reductive. A Cartan geometry (M, B, ω) modeled on G/P is *reductive* when there is an $\mathrm{Ad}_{\mathfrak{g}}P$-invariant decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$, in which the restriction of $\mathrm{Ad}_{\mathfrak{g}}P$ to $\mathfrak{n}$ is faithful. Note that $\mathfrak{n}$ is in general a subspace of $\mathfrak{g}$, not a subalgebra. Classical examples of reductive geometries are pseudo-Riemannian metrics, for which

$$(\mathfrak{g}, \mathfrak{p}) = (\mathfrak{so}(p,q) \ltimes \mathbf{R}^n, \mathfrak{so}(p,q))$$

or linear connections, for which

$$(\mathfrak{g}, \mathfrak{p}) = (\mathfrak{gl}(\mathbf{R}^n) \ltimes \mathbf{R}^n, \mathfrak{gl}(n, \mathbf{R}))$$

Corollary 1.9. *Let (M, B, ω) be a reductive Cartan geometry, and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ an $Ad_{\mathfrak{g}}P$-invariant decomposition. Let $H < Aut\ M$ be a Lie subgroup such that $\bar{H}^{Ad} = \bar{H}_d$. If H acts transitively on M preserving a finite volume, then*

1. *The group $\bar{H}^{Ad}$ is isomorphic to a subgroup $\check{H} < P$.*
2. *There is a linear isomorphism $\iota : \mathfrak{h} \to \mathfrak{n}$ with $Ad\,\check{H}$-invariant image, which intertwines the representations of $\bar{H}^{Ad}$ on $\mathfrak{h}$ and of $\check{H}$ on $\iota(\mathfrak{h})$.*
3. *The identity component H^o is isomorphic to a central extension of a subgroup $H' < \check{H}$.*

2 Automorphisms of a Cartan geometry

Let (M, B, ω) be a Cartan geometry modeled on $\mathbf{X} = G/P$, and $H < \mathrm{Aut}\ M$ a Lie subgroup. The H-action on B is free because it preserves a framing, given by the pullback by ω of any basis of $\mathfrak{g}$ (see [Ko] I.3.2).

This section introduces several important H-equivariant maps from B to algebraic varieties. These maps will be central in the proof of the main theorem.

2.1 The curvature map of a Cartan geometry

The simplest example of a Cartan geometry is the triple $(\mathbf{X}, G, \omega^G)$, where ω^G is the Maurer-Cartan form on G, which evaluates to X on the left-invariant vector field corresponding to $X \in \mathfrak{g}$. In addition to fulfilling the definition of a Cartan geometry, the Maurer-Cartan form on G satisfies the *structural equation*:

$$d\omega^G(X,Y) + [\omega^G(X), \omega^G(Y)] = 0$$

for every pair of vector fields X and Y on G.

For a general Cartan geometry (M, B, ω), the 2-form

$$K(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$$

is not zero, and is called *the curvature form* of ω. It is a fundamental fact that K vanishes if and only if (M, B, ω) is locally isomorphic to $(\mathbf{X}, G, \omega^G)$ (see [Shar]). Thus, the manifolds M *locally* modeled on $\mathbf{X}$, also known as manifolds with $(G, \mathbf{X})$-structures, correspond exactly to *flat* Cartan geometries (M, B, ω), modeled on $\mathbf{X}$.

The associated *curvature map* is

$$\begin{aligned} \kappa \quad &: \quad B \to \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) \\ \kappa_b(\sigma(X), \sigma(Y)) \quad &= \quad K_b(\omega_b^{-1}(X), \omega_b^{-1}(Y)) \end{aligned}$$

Where $X, Y \in \mathfrak{g}$, and σ is the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$. This map is well-defined because K_b vanishes as soon as one of its argument is tangent to a fiber of $B \to M$. The map κ is P-equivariant, where P acts on $\mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ via the adjoint: for any $\check{p} \in \mathrm{Ad}\ P$ and $\beta \in \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$,

$$(\check{p}.\beta)(u,v) = \check{p}^{-1}.(\beta(\check{p}.u, \check{p}.v))$$

For any $b \in B$, the Cartan connection induces a natural linear map

$$\begin{aligned} \iota_b \quad &: \quad \mathfrak{h} \to \mathfrak{g} \\ \iota_b(X) \quad &= \quad \omega_b\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \varphi_X^t.b\Big) \end{aligned}$$

where φ_X^t is the flow along X for time t. The freeness of the H-action implies that ι_b is injective. In general, ι_b is not a Lie algebra homomorphism. It nearly is, however, when $\kappa_b = 0$.

Lemma 2.1. *Let (M, B, ω) be a Cartan geometry modeled on G/P. If for some $b \in B$, the curvature κ_b vanishes on the image of $\sigma \circ \iota_b$, then $-\iota_b : \mathfrak{h} \to \mathfrak{g}$ is an embedding of Lie algebras.*

Proof: Let $X, Y \in \mathfrak{h}$, viewed as Killing fields on B. Recall that

$$d\omega(X,Y) = X.\omega(Y) - Y.\omega(X) - \omega([X,Y])$$

Since X is a Killing field for ω,

$$\omega(Y(b)) = \omega((\varphi_X^t)_{*b} Y(b))$$

Now

$$\begin{aligned}
(X.\omega(Y))(b) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \omega(Y(\varphi_X^t b)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left[\omega(Y(\varphi_X^t b)) - \omega((\varphi_X^t)_{*b} Y(b))\right] \\
&= \omega([X,Y](b))
\end{aligned}$$

Similarly,

$$(Y.\omega(X))(b) = \omega([Y,X](b))$$

Then

$$\begin{aligned}
\kappa_b(\sigma \circ \iota_b(X), \sigma \circ \iota_b(Y)) &= d\omega(X(b), Y(b)) + [\omega(X(b)), \omega(Y(b))] \\
&= \omega([X,Y](b)) - \omega([Y,X](b)) - \omega([X,Y](b)) + [\omega(X(b)), \omega(Y(b))] \\
&= \omega([X,Y](b)) + [\omega(X(b)), \omega(Y(b))] \\
&= \iota_b([X,Y]) + [\iota_b(X), \iota_b(Y)]
\end{aligned}$$

Therefore, vanishing of κ_b implies

$$-\iota_b([X,Y]) = [\iota_b(X), \iota_b(Y)] = [-\iota_b(X), -\iota_b(Y)]$$

as desired. $\diamondsuit$

2.2 Natural equivariant maps to varieties

To an action of H by automorphisms of a Cartan geometry M modeled on G/P can be associated various equivariant maps to algebraic varieties built from the adjoint representations of H and G.

The maps $\iota_b : \mathfrak{h} \to \mathfrak{g}$ for $b \in B$ give a map

$$\iota : \; B \to \mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$$

where $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$ is the variety of all injective linear maps from $\mathfrak{h}$ to $\mathfrak{g}$. The map ι is continuous, since ω is so. Observe that H and P have commuting actions on B, as well as on $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$, by pre- and post-compositions with the adjoint actions. It will be crucial in the sequel that the latter $(\mathrm{Ad}\, H \times \mathrm{Ad}_{\mathfrak{g}} P)$-action is the restriction of an algebraic action, and that $\mathrm{Ad}_{\mathfrak{g}} P$ is algebraic. To a Lie subgroup $H < \mathrm{Aut}\, M$ can be associated a real-algebraic space

$$U = \mathrm{Mon}(\mathfrak{h}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$$

where H acts trivially on $\mathrm{Hom}(\wedge^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$, and a continuous map $\phi : \; B \to U$, which is continuous and $(H \times P)$-equivariant.

We denote by $\check{P}^b$ the algebraic subgroup of $\mathrm{Ad}_{\mathfrak{g}} P$ consisting of those $\check{p}$ such that $\check{p}.\iota_b(\mathfrak{h}) = \iota_b(\mathfrak{h})$. There is a natural algebraic homomorphism

$$\begin{aligned} \rho_b &: \; \check{P}^b \to \mathrm{GL}(\mathfrak{h}) \\ \rho_b(\check{p}) &= \iota_b^{-1} \circ \check{p} \circ \iota_b \end{aligned}$$

Now, if $S < H$ is a subgroup, and $b \in B$, define

$$\check{S}^b = \{\check{p} \in \check{P}^b \mid \rho_b(\check{p}) \in \bar{S}_d \text{ and } \check{p}.\kappa_b = \kappa_b\}$$

The group $\check{S}^b$ may not be algebraic in general, but it is if $\rho_b : \check{S}^b \to \bar{S}_d$ is *surjective*, as in this case it is the preimage in $\check{P}^b$ of the algebraic group $\bar{S}_d$ by the algebraic homomorphism ρ_b. The restriction of ρ_b to $\check{S}_b$ is then an algebraic epimorphism onto $\bar{S}_d$.

The main point in the proof of theorem 1.2 will be precisely to show the existence of $b \in B$ such that $\rho_b : \check{S}_b \to \bar{S}_d$ is a surjection.

3 Background on measure theory

In this section we present the version of the Borel density theorem that figures in the proof of the main theorem 1.2. Primary references for material here on invariant measures for real-algebraic actions are [Zi3, ch 2, 3], [D], and [Shal]. We will be considering real-algebraic groups and varieties, and all results will be stated in this setting; in fact, we will use the term “algebraic” to mean real-algebraic below.

For G an algebraic group and V an algebraic variety, an algebraic action of G on V is an action given by an algebraic morphism $\varphi : G \times V \to V$. For a locally compact group S and a homomorphism $\psi : S \to G$, where G is an algebraic group, we denote the real points of the Zariski closure of $\psi(S)$ by $\bar{S}^{\psi}$ or $\bar{S}$.

The *discompact radical* G_d of an algebraic group G is the maximal algebraic subgroup of G with no cocompact, algebraic, normal subgroups. It is the minimal algebraic subgroup Q containing all algebraic subgroups with no compact algebraic quotients, which exists by the Noetherian property of algebraic subgroups. By convention, we will put $G_d = \{e\}$ when $G = \{e\}$. Note that G_d is characteristic. See [Shal] for more details. Given $\psi : S \to G$, we denote by $\bar{S}_d^{\psi}$ or $\bar{S}_d$ the discompact radical of $\bar{S}^{\psi}$.

The formulations of theorems 1.2 and 1.6 in terms of $\bar{H}_d$ and $\bar{S}_d$ apply to a wider class of groups than those for which the Zariski closures $\bar{H}^{\mathrm{Ad}}$ or $\bar{S}^{\mathrm{Ad}}$ have no compact algebraic quotients. Consider the following group. Let

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & e^{\lambda} \end{pmatrix} \in GL(3, \mathbf{R})$$

for some $\theta \neq 0$ modulo 2π, $\lambda \neq 0$. Let $H = \langle g \rangle \ltimes \mathbf{R}^3 \cong \mathbf{Z} \ltimes \mathbf{R}^3$. Then $\bar{H}_d^{\mathrm{Ad}} \cong \mathbf{R}^*$, but for any $S < H$, the group $\bar{S}^{\mathrm{Ad}}$ is either trivial or has S^1 as a direct factor.

Theorem 3.1 (see [D, 2.6], [Shal, 3.11])**.** *Let $\psi : S \to Aut\, V$, for S a locally compact group and V an algebraic variety. Assume S preserves a finite measure μ on V. Then μ is supported on the set of $\bar{S}_d$-fixed points in V.*

Proof: [sketch] By an ergodic decomposition argument, it is enough to assume that the measure μ is S-ergodic. It is well known that $\bar{S}$-orbits in V are locally closed [Zi3, 3.1.1], and it follows that μ is supported on a single $\bar{S}$-orbit. We claim that μ is $\bar{S}$-invariant. Indeed, Zimmer proved that the stabilizers of probability measures are Zariski closed when V is a quasi-projective variety [Zi3, 3.2.4], but by a theorem of Chevalley every $\bar{S}$-orbit is $\bar{S}$-equivariantly quasi-projective. It remains to show that every $\bar{S}$-invariant measure is supported on the set of $\bar{S}_d$-fixed points. The latter group is generated by algebraic 1-parameter groups, isomorphic to either the additive or the multiplicative group of $\mathbf{R}$. The statement is true for these groups, again by applying an ergodic decomposition and restricting to one orbit. The proof follows. $\diamondsuit$

Corollary 3.2. *Let $\psi : S \to Aut\ V$, for S a locally compact group and V an algebraic variety. Suppose S acts continuously on a topological space M preserving a finite Borel measure μ. Assume $\phi : M \to V$ is an S-equivariant measurable map. Then $\phi(x)$ is fixed by $\bar{S}_d$ for μ-almost-every $x \in M$.*

Proof: Follows from the previous theorem, with the measure $\phi_*(\mu)$. $\diamondsuit$

4 Proof of the main theorem 1.2

This section will be devoted to the proof of theorem 1.2, or more precisely to that of theorem 4.1 below which is a detailed version of it.

We consider (M, B, ω), a fixed Cartan geometry modeled on G/P. Recall the global assumption that $\mathrm{Ad}_{\mathfrak{g}} P < \mathrm{Aut}\ \mathfrak{g}$ is Zariski closed. The notations are those of section 2.2. Recall that $\pi : B \to M$ is the bundle projection.

Theorem 4.1. *Let $H < Aut\ M$ be a Lie subgroup and $S < H$. Suppose there is an S-invariant finite measure μ on M. Then there exists $\Lambda \subset B$ such that $\pi(\Lambda)$ is μ-conull and, for every $b \in \Lambda$,*

$$\rho_b : \check{S}^b \to \bar{S}_d$$

is an algebraic epimorphism. In particular, $\check{S}^b$ preserves κ_b and ρ_b intertwines the representation of $\check{S}^b$ on $\iota_b(\mathfrak{h})$ and that of $\bar{S}_d$ on $\mathfrak{h}$.

Proof: Observe first of all that the main point is to prove the surjectivity of ρ_b. The two other assertions of the theorem follow by the definitions of $\check{S}^b$ and ρ_b.

As in section 2.2, consider the real-algebraic variety

$$U = \mathrm{Mon}(\mathfrak{h}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$$

and the $H \times P$-equivariant map $\phi = \iota \times \kappa : B \to U$.

The map ϕ descends to an H-equivariant map between the spaces of P-orbits:

$$\bar{\phi} : M = B/P \to V = U/P$$

The $\mathrm{Ad}_{\mathfrak{g}}P$-action on U is algebraic, and there is a $\mathrm{Ad}_{\mathfrak{g}}P$-invariant stratification

$$U = U'_0 \cup \cdots \cup U'_r$$

such that, for each $j \geq 0$, the stratum U'_j is Zariski open and dense in $\cup_{i \geq j} U'_i$, and each quotient space $V'_i = U'_i/\mathrm{Ad}_{\mathfrak{g}}P$ is a smooth algebraic variety (see [Ro] or [Gr] 2.2). Let us call $U_0 = \bigcup_{h \in H} h.U'_0$. This is a Zariski open set, which is in fact a finite union $h_1.U'_0 \cup ... \cup h_s.U'_0$. Now, each quotient $h_j.U'_0/\mathrm{Ad}_{\mathfrak{g}}P$ has the structure of a smooth algebraic variety. As a consequence, $U_0/\mathrm{Ad}_{\mathfrak{g}}P$ has itself the structure of a smooth algebraic variety, and since the actions of H and $\mathrm{Ad}_{\mathfrak{g}}P$ commute, H acts algebraically on $U_0/\mathrm{Ad}_{\mathfrak{g}}P$. We can then write U as $U = U_0 \cup F_0$. Let us apply the previous argument to the smooth part of the Zariski closed set F_0. We get a decomposition $U = U_0 \cup U_1 \cup F_1$, where U_1 and F_1 are $H \times \mathrm{Ad}_{\mathfrak{g}}P$-invariant, and $U_1/\mathrm{Ad}_{\mathfrak{g}}P$ is a smooth algebraic variety. After finitely many steps, we thus obtain a decomposition:

$$U = U_0 \cup \cdots \cup U_r$$

where each U_i is $H \times \mathrm{Ad}_{\mathfrak{g}}P$-invariant, and the quotient $V_i = U_i/\mathrm{Ad}_{\mathfrak{g}}P$ is a smooth algebraic variety, on which H acts algebraically.

Denote by μ_i the restriction of μ to $M_i = \bar{\phi}^{-1}(V_i)$. It suffices to prove that ρ_b is surjective for μ_i-almost-every $\pi(b) \in M_i$, for all i. Because M_i is S-invariant for all i, each measure μ_i is S-invariant. Given i such that $\mu_i \neq 0$, corollary 3.2 applied to the S-equivariant map $\bar{\phi} : (M_i, \mu_i) \to V_i$ gives that $v = \bar{\phi}(m)$ is an $\bar{S}_d$-fixed point for μ_i-almost-every $m \in M_i$. For such an m, choose $b \in \pi^{-1}(m) \subset B$, and set $u = \phi(b)$. Then $\mathrm{Ad}_{\mathfrak{g}}P.u$ is stabilized by $\bar{S}_d$. This means that for any $s \in \bar{S}_d$, there exists $\check{p}_s \in \mathrm{Ad}_{\mathfrak{g}}P$, not necessarily unique, such that $s.(\iota_b, \kappa_b) = \check{p}_s.(\iota_b, \kappa_b)$, or in other words, $\iota_b \circ s = \check{p}_s \circ \iota_b$, and $\check{p}_s.\kappa_b = \kappa_b$. Thus, $\check{p}_s \in \check{S}^b$, and $\rho_b(\check{p}_s) = s$. We thus get that $\rho_b : \check{S}^b \to \bar{S}_d$ is a surjection, and theorem 4.1 is proved. $\diamondsuit$

5 Proofs of bounds on automorphism groups

Recall that $\bar{H}_d$ denotes the discompact radical of the Zariski closure of Ad H. A Lie subalgebra of End(V) will be called *algebraic* when it is the Lie algebra of a Zariski closed subgroup of GL(V). Also, throughout this section, M is compact.

5.1 Proof of theorem 1.3

To prove (1) of theorem 1.3, choose an $\mathbf{R}$-split subgroup of Ad H with Zariski closure $\bar{S}$ of dimension $\mathrm{rk}^{\mathrm{alg}}(\mathrm{Ad}\ H)$. Let $S = \mathrm{Ad}^{-1}(\bar{S}) < H$. The kernel of Ad is amenable by assumption (§1.1), and $\bar{S}$ is abelian, so S is amenable. There is a finite S-invariant measure on M, and theorem 4.1 gives an algebraic subgroup $\check{S}^b < \mathrm{Ad}_{\mathfrak{g}} P$ such that $\rho_b(\check{S}^b) = \bar{S}_d = \bar{S}$.

The Lie algebra of $\check{S}^b$ admits a Levi decomposition into algebraic subalgebras $\check{\mathfrak{s}}^b = \mathfrak{r} \ltimes \mathfrak{u}$, where $\mathfrak{u}$ consists of nilpotent endomorphisms, and $\mathfrak{r}$ is a reductive Lie algebra (see [WM1] ch.4). The homomorphism ρ_b induces an epimorphism of algebraic algebras $d\rho_b : \check{\mathfrak{s}}^b \to \bar{\mathfrak{s}}$. The image of $\mathfrak{u}$ consists of elements that are both $\mathbf{R}$-split and nilpotent, so that $\mathfrak{u} \subset \ker d\rho_b$. The algebra $\mathfrak{r}$ splits into algebraic subalgebras $\mathfrak{r} = \mathfrak{z} \oplus \mathfrak{l}$ where $\mathfrak{l}$ is semisimple, and $\mathfrak{z}$ is an abelian algebra of $\mathbf{C}$-split endomorphisms. Because $\bar{\mathfrak{s}}$ is abelian, $\mathfrak{l} \subset \ker d\rho_b$. The $\mathbf{C}$-split algebraic subalgebra $\mathfrak{z}$ decomposes as a sum $\mathfrak{z} = \mathfrak{z}^s \oplus \mathfrak{z}^e$ where $\mathfrak{z}^s$ consists of $\mathbf{R}$-split elements, and $\mathfrak{z}^e$ of elements with purely imaginary eigenvalues. Since $d\rho_b$ is a morphism of real algebraic Lie algebras, it respects this decomposition. Therefore, $\mathfrak{z}^e \subset \ker d\rho_b$, and $d\rho_b(\mathfrak{z}^s) = \bar{\mathfrak{s}}$. Then we have

$$\mathrm{rk}(\mathrm{Ad}_{\mathfrak{g}} P) \geq \dim \mathfrak{z}^s \geq \dim \bar{\mathfrak{s}} = \mathrm{rk}^{\mathrm{alg}}(\mathrm{Ad}\ H)$$

To prove (2), let N be a connected nilpotent subgroup of Ad H of maximal degree, and let $\bar{N}$ be the Zariski closure. Let $S = \mathrm{Ad}^{-1}(\bar{N}) \subset H$. As above, S is amenable, so theorem 1.2 gives an algebraic subgroup $\check{S}^b$ such that $\rho_b(\check{S}^b) = \bar{S}_d$. The right-hand side is the discompact radical of $\bar{N}$.

Lemma 5.1. *Let N be a connected nilpotent group in $GL(\mathbf{R}^k)$. Let $\bar{S}_d$ be the discompact radical of the Zariski closure $\bar{N}$. Then $n(\bar{\mathfrak{s}}_d) = n(\mathfrak{n})$.*

Proof: Let $n = n(\mathfrak{n})$. Any n-tuple $(u_1, \ldots, u_n) \in \mathfrak{n}^n$ satisfies

$$[u_n, \cdots [u_2, u_1] \cdots] = 0$$

Because this is an algebraic condition, it is satisfied by all n-tuples of elements of $\bar{\mathfrak{n}}$. Therefore $n(\mathfrak{n}) = n(\bar{\mathfrak{n}})$.

Of course $n(\bar{\mathfrak{s}}_d) \leq n(\bar{\mathfrak{n}})$, so it suffices to show $n(\bar{\mathfrak{s}}_d) \geq n(\bar{\mathfrak{n}})$. Write $\bar{\mathfrak{n}} = \mathfrak{r} \ltimes \mathfrak{u}$ where $\mathfrak{u}$ is composed of nilpotents, and $\mathfrak{r}$ is reductive. The algebra $\mathfrak{r}$ is reductive and nilpotent, so it is abelian. The adjoint representation of $\mathfrak{r}$ on $\mathfrak{u}$ is both nilpotent and reductive, so it is trivial. Therefore $\mathfrak{r}$ is abelian and centralizes $\mathfrak{u}$, so that $n(\bar{\mathfrak{n}}) = n(\mathfrak{u})$. But $\mathfrak{u} \subset \bar{\mathfrak{s}}_d$, so $n(\bar{\mathfrak{n}}) \leq n(\bar{\mathfrak{s}}_d)$. $\diamondsuit$

Now as above $d\rho_b$ is a surjective morphism of algebraic Lie algebras $\check{\mathfrak{s}}^b \to \bar{\mathfrak{s}}_d$. Take an algebraic Levi decomposition

$$\check{\mathfrak{s}}^b = \mathfrak{r} \ltimes \mathfrak{u}$$

and write $\mathfrak{r} = \mathfrak{z} \oplus \mathfrak{l}$ as above. Since $\bar{\mathfrak{s}}_d$ is nilpotent, and $\mathfrak{l}$ is semisimple, again $\mathfrak{l} \subset \ker d\rho_b$. The kernel of $d\rho_b$ in $\mathfrak{u}$ is ad $\mathfrak{r}$-invariant, so there is an ad $\mathfrak{r}$-invariant complementary subspace V in $\mathfrak{u}$. The subspace $\mathfrak{z} \oplus V$ is mapped surjectively on $\bar{\mathfrak{s}}_d$ by $d\rho_b$. Let $\mathfrak{z}'$ be the kernel in $\mathfrak{z}$ of the adjoint restricted to V, and let $\mathfrak{z}''$ be a complementary subalgebra in $\mathfrak{z}$. Then $\mathfrak{z} = \mathfrak{z}' \oplus \mathfrak{z}''$. Because elements of $\mathfrak{z}''$ act via the adjoint on V by nontrivial $\mathbf{C}$-split endomorphisms, none of them is nilpotent. Therefore $\mathfrak{z}'' \subset \ker d\rho_b$. Let $\mathfrak{u}'$ be the Lie algebra generated by V. Then the subalgebra $\mathfrak{z}' \oplus \mathfrak{u}'$ of $\check{\mathfrak{s}}^b$ is nilpotent and maps onto $\bar{\mathfrak{s}}_d$ via $d\rho_b$. By lemma 5.1, we conclude:

$$n(\mathrm{Ad}\ H) = n(\mathfrak{n}) = n(\bar{\mathfrak{s}}_d) \leq n(\mathfrak{z}' \oplus \mathfrak{u}') \leq n(\mathrm{Ad}_{\mathfrak{g}} P)$$

$\diamondsuit$

5.2 Semi-discrete groups: proof of 1.4

Recall that Γ is a lattice in an algebraic group $L < \mathrm{GL}(V)$ with $L = L_d$, and the group $H = \Gamma \ltimes V$ acts by automorphisms of the compact Cartan geometry M. Note that L is the Zariski closure of Ad H on $V = \mathfrak{h}$, by the Borel density theorem.

Let $U = \mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$, and let $\iota : B \to U$ be the $H \times P$-equivariant map defined in section 2.2. Let $Q < L$ be an amenable, algebraic subgroup with $Q_d = Q$. Much as in section 4, we seek a Q-fixed point in $U/\mathrm{Ad}_{\mathfrak{g}} P$.

The Zariski closure L commutes with $\mathrm{Ad}_{\mathfrak{g}} P$ on U because H does. Let $B' = L \times_\Gamma B$ be the L-space of equivalence classes of pairs $(h, b) \in L \times B$, where $(h, b) \sim (h\gamma^{-1}, \gamma.b)$. Define an $L \times P$-equivariant extension of ι

$$\begin{aligned} \iota' \quad : \quad & B' \to U \\ & [(h, b)] \mapsto h.\iota(b) \end{aligned}$$

Let

$$M' = L \times_\Gamma M = B'/P$$

Note that M' is an M-bundle over L/Γ, which has an L-invariant probability measure ν. The space of probability measures on M' projecting to ν is a compact convex L-space, so it contains a fixed point for Q; in other words, there is a finite Q-invariant measure μ on M'.

Denote by η the H-equivariant map $M \to U/\mathrm{Ad}_{\mathfrak{g}}P$ covered by ι. There is an L-equivariant continuous lift

$$\eta' : M' \to U/\mathrm{Ad}_{\mathfrak{g}}P$$

The L-action on U preserves the stratification as in the proof of 4.1 in section 4, so it acts on the smooth quotient varieties $V_i = U_i/\mathrm{Ad}_{\mathfrak{g}}P$, for each stratum $U_i \subset U$. Let μ_i be the restriction of $\eta'_*(\mu)$ to V_i. Each μ_i is Q-invariant. The corollary 3.2 of the Borel density theorem gives a Q-fixed point $v = \eta'([(h, x)]) = h.\eta(x)$ for some $h \in L$ and $x \in M$. Then $h^{-1}Qh$ fixes $\eta(x)$. In other words, for any $b \in B$ lying over $x \in M$, the algebraic subgroup $\check{P}^b < \mathrm{Ad}_{\mathfrak{g}}P$, when restricted to $\iota_b(V)$, contains a subgroup isomorphic to $h^{-1}Qh$. Let $\check{S}^b$ be the algebraic subgroup $\rho_b^{-1}(h^{-1}Qh)$, so ρ_b is an algebraic epimorphism from $\check{S}^b$ onto $h^{-1}Qh$.

Now take $Q < L$ to be an $\mathbf{R}$-split algebraic subgroup of dimension $\mathrm{rk}\ L$. Then following the proof of 1.3 (1) gives

$$\mathrm{rk}\ L = \dim Q \leq \mathrm{rk}(\mathrm{Ad}_{\mathfrak{g}}P)$$

Similarly, there is a connected, nilpotent, algebraic $Q < L$ of degree $n(L)$, and following the proof of 1.3 (2) gives $n(L) \leq n(\mathrm{Ad}_{\mathfrak{g}}P)$.

6 A rigidity result for parabolic geometries

6.1 Parabolic subalgebras, grading, regular connections

In this section, G is a connected semisimple Lie group. Let $\mathfrak{a} \subset \mathfrak{g}$ be a maximal $\mathbf{R}$-split abelian subalgebra, and $\Pi \subset \mathfrak{a}^*$ the root system. This set of roots admits a basis $\Phi = \{\alpha_1, ..., \alpha_r\}$ of *simple roots*, such that any root can be written uniquely as a sum $n_1\alpha_1 + ... + n_r\alpha_r$, where $n_1, ..., n_r$ are integers of the same sign. The simple roots determine a decomposition

$$\Pi = \Pi^+ \cup \{0\} \cup \Pi^-$$

where Π^+ (resp. Π^-) denotes the positive (resp. negative) roots—those for which all the integers n_i are positive (resp. negative).

For $\alpha \in \Pi$, the root space

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid (\text{ad } X)(Y) = \alpha(X) \cdot Y \qquad \forall X \in \mathfrak{a}\}$$

The associated root space decomposition is

$$\mathfrak{g} = \Sigma_{\alpha\in\Pi^-}\mathfrak{g}_\alpha \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \Sigma_{\alpha\in\Pi^+}\mathfrak{g}_\alpha$$

The subalgebra $\mathfrak{m}$ is a compact subalgebra, and $\mathfrak{a} \oplus \mathfrak{m}$ is exactly the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. We denote by A the connected subgroup of G with Lie algebra $\mathfrak{a}$, and by M_G the subgroup with Lie algebra $\mathfrak{m}$; it is compact if and only if $Z(G)$ is finite.

Let $\Theta \subset \Phi$, and

$$\Theta^- = \{\alpha = \Sigma n_i\alpha_i \mid \alpha_i \in \Theta \text{ and } n_i \in \mathbf{Z}^{\leq 0}\}$$

The *standard parabolic subalgebra associated to* Θ is

$$\mathfrak{p} = \Sigma_{\alpha\in\Theta^-}\mathfrak{g}_\alpha \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \Sigma_{\alpha\in\Pi^+}\mathfrak{g}_\alpha$$

For example, taking $\Theta = \emptyset$ yields the *minimal parabolic subalgebra*

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \Sigma_{\alpha\in\Pi^+}\mathfrak{g}_\alpha$$

For $\Theta = \Phi$, one gets $\mathfrak{p} = \mathfrak{g}$.

Let Θ be a proper subset of Φ. Reordering the elements of Φ if necessary, we can assume $\Phi \setminus \Theta = \{\alpha_1, ..., \alpha_m\}$. For every $j \in \mathbf{Z}$, define

$$\Pi_j = \{\Sigma_{i=1}^r n_i\alpha_i \mid n_i \in \mathbf{Z}^{\leq 0} \ \forall\ i \text{ or } n_i \in \mathbf{Z}^{\geq 0} \ \forall\ i \text{ and } \Sigma_{i=1}^m n_i = j\}$$

Then put $\mathfrak{g}_j = \Sigma_{\alpha\in\Pi_j}\mathfrak{g}_\alpha$, and $\mathfrak{g}^j = \Sigma_{i\geq j}\mathfrak{g}_i$.

Let k be the greatest integer such that $\Pi_k \neq \emptyset$. Associated to the parabolic subalgebra $\mathfrak{p}$ are a grading

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

as well as a filtration

$$\mathfrak{g}^k \subset \mathfrak{g}^{k-1} \subset ... \subset \mathfrak{g}^{-k} = \mathfrak{g}$$

with $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. Notice that $\mathfrak{p} = \mathfrak{g}^0$.

6.2 Parabolic geometries

Let G be as above, and $P < G$ a parabolic subgroup with Lie algebra $\mathfrak{p}$. Let $\mathfrak{g}_i$, for $i = -k, \ldots, k$, be the subspaces of the associated grading, as above. A *parabolic geometry* is a Cartan geometry (M, B, ω) modeled on $\mathbf{X} = G/P$. As illustrated in section 1.3 of the introduction, many interesting examples of geometric structures have an interpretation as a parabolic geometry.

The Cartan connection ω of a parabolic geometry modeled on $\mathbf{X} = G/P$ is *regular* if the corresponding curvature function satisfies for any $i, j < 0$

$$\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$$

The regularity assumption on a Cartan connection is natural. Indeed, in every case in which the equivalence problem is solved for parabolic geometries, regularity is one of the normalization conditions on the Cartan connection to make it unique.

6.3 Flatness and completeness: proof of 1.5

Let G be a semisimple Lie group and P a parabolic subgroup as above. Throughout this section, (M, B, ω) is a Cartan geometry modeled on $\mathbf{X} = G/P$ with ω regular.

Inside $\mathfrak{a}$, define the positive Weyl chamber

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0 \ \forall \ \alpha \in \Pi^+\}$$

Let (a_k) be a sequence of A tending to infinity—that is, leaving every compact subset of A. Fix any norm on $\mathfrak{g}$, and write

$$a_k = \exp(X_k) = \exp(||X_k|| u_k)$$

where $||u_k|| = 1$. An *asymptotic direction of* (a_k) is any cluster point $u \in \mathfrak{a}$ of the sequence (u_k).

The notion of *holonomy sequences* will be important in the proof below. Let $H < \mathrm{Aut}\ M$, for (M, B, ω) a Cartan geometry modeled on G/P, let $b \in B$, and $x = \pi(b)$. For $h_k \in H$ such that $h_k.x \to y_\infty \in M$, a *holonomy sequence associated to* (b, h_k), is a sequence (p_k) of P such that

$$h_k.b.p_k^{-1} \to b_\infty \in \pi^{-1}(y_\infty)$$

If (p_k) is a holonomy sequence associated to (b, h_k), then all other holonomy sequences associated to (b, h_k) are $(l_k p_k)$ with (l_k) a convergent sequence of P. Also, if $b' = b.p'$, then any holonomy sequence associated to (b', h_k) is of the form $(p_k p')$, where (p_k) is a holonomy sequence associated to (b, h_k). These observations lead naturally to the following notion of equivalence.

Definition 6.1. *Two sequences (p_k) and (p'_k) are* equivalent *if there are a convergent sequence (l_k) in P and $q \in P$ such that $p'_k = l_k p_k q$, for all $k \in \mathbf{N}$.*

Note that a sequence of P and its conjugate by some $p \in P$ are equivalent. Also, two converging sequences of P are equivalent.

Definition 6.2. *A* holonomy sequence at $b \in B$ *is a holonomy sequence associated to (b, h_k) for some (h_k) in H. For $x \in M$, a* holonomy sequence at x *is a holonomy sequence at some $b \in B$ for which $\pi(b) = x$.*

Remark that if (p_k) is a holonomy sequence at x, then any sequence equivalent to (p_k) is also a holonomy sequence at x.

If $b \in B$ is such that $\kappa_b = 0$, then $\kappa_{b'} = 0$ for every b' in the same fiber. In the following, by a slight abuse of language, we will say that the curvature function vanishes at $x \in M$, if $\kappa_b = 0$ for all b in the fiber of x. We can now prove

Proposition 6.3. *Let $H < Aut\ M$ be a Lie subgroup, and $S < H$.*

1. *If there is $b \in B$ such that $Ad\ A \subset \check{S}^b$, then $\kappa_b = 0$.*
2. *Suppose there is $x \in M$ such that for any $u \in \mathfrak{a}^+$, there is a holonomy sequence $(p_k) \subset A$ at x with asymptotic direction the same as the direction of u. Then there is an open neighborhood U of x on which the curvature vanishes.*

Proof: We begin with the proof of point (2). Let $i \leq j < 0$, and $l > i + j$. Let $\alpha \in \Pi_i$, $\beta \in \Pi_j$, and $\nu \in \Pi_l$ be three roots of $\mathfrak{g}$. Choose nonzero vectors

$v \in \mathfrak{g}_\alpha$ and $w \in \mathfrak{g}_\beta$. Choose $u \in \mathfrak{a}^+$, such that $\alpha(u) + \beta(u) < \nu(u)$ (Such u always exits, since $\alpha + \beta$ is in Π_{i+j}, hence cannot equal ν). Finally, choose (p_k) a holonomy sequence at x with u for asymptotic direction. Passing to a subsequence if necessary, write $p_k = \exp(X_k)$ with

$$\lim_{k\to\infty} \frac{X_k}{||X_k||} = u$$

By definition of (p_k), there is a sequence (h_k) of H and $b \in B$ over x such that $h_k.x \to x_\infty \in M$, and

$$b_k = h_k.b.p_k^{-1} \to b_\infty \in \pi^{-1}(x_\infty)$$

For all $k \in \mathbf{N}$

$$(\mathrm{Ad}\ p_k).\kappa_b(v,w) = \kappa_{b_k}((\mathrm{Ad}\ p_k).v, (\mathrm{Ad}\ p_k).w) = e^{\alpha(X_k)+\beta(X_k)}\kappa_{b_k}(v,w)$$

Since $\alpha(u) + \beta(u) - \nu(u) < 0$,

$$e^{-\nu(X_k)}(\mathrm{Ad}\ p_k).\kappa_b(v,w) \to 0$$

Then the component of $\kappa_b(v,w)$ on $\mathfrak{g}_\nu$ is zero. Since this is true for any $\nu \in \Pi_l$ with $l > i+j$, the component of $\kappa_b(v,w)$ on $\mathfrak{g}^{i+j+1}$ is zero, which implies $\kappa_b(v,w) = 0$ by the regularity condition. Then at any point $x \in M$ where the assumptions of point (2) hold, the curvature vanishes.

Let $b \in B$ and $\xi \in T_bB$. Let X_ξ be the unique vector field on B having the property that $\omega_{b'}(X_\xi(b')) = \omega_b(\xi)$ for all $b' \in B$. If ϕ^t is the flow generated by X_ξ, and if ϕ^1 is defined, then put $\exp_b(\xi) = \phi^1.b$. The map $\exp_b$ is defined on a neighborhood of 0_b in T_bB, and yields a diffeomorphism of a sufficiently small neighborhood of 0_b onto its image.

Let u be in $\mathfrak{a}^+$ and (p_k) a holonomy sequence at x with asymptotic direction u. Let $h_k \in H$ and $b \in \pi^{-1}(x)$ such that

$$b_k = h_k.b.p_k^{-1} \to b_\infty \in B$$

Choose an open $U \subset \mathfrak{n}^- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ containing $0_{\mathfrak{n}^-}$ such that $\exp_b$ is a diffeomorphism from $\omega_b^{-1}(U)$ onto its image; note that $\pi \circ \exp_b(\omega_b^{-1}(U))$ is open in M. For every $k \in \mathbf{N}$,

$$h_k.\exp_b(\omega_b^{-1}(U)).p_k^{-1} = \exp_{b_k}(\omega_{b_k}^{-1}((\mathrm{Ad}\ p_k).U))$$

Taking a subsequence if necessary, we assume that u is the only asymptotic direction of (p_k). Since $u \in \mathfrak{a}^+$, the elements Ad p_k act by contraction on $\mathfrak{n}^-$, which implies that $(\mathrm{Ad}\ p_k).U \to 0_{\mathfrak{g}}$ for the Hausdorff topology. As a consequence,

$$\exp_{b_k}(\omega_{b_k}^{-1}((\mathrm{Ad}\ p_k).U)) = h_k.\exp_b(\omega_b^{-1}(U)).p_k^{-1} \to \{b_\infty\}$$

Let $V = \pi \circ \exp_b(\omega_b^{-1}(U))$. For any $x' \in V$, the sequence (p_k) is a holonomy sequence at x'. The hypotheses of point (2) of the proposition are thus satisfied at any $x' \in V$. By what was said above, the curvature vanishes at every point of the open set V.

We now prove point (1). Since $\check{S}^b$ preserves κ_b, any sequence (p_k) of A satisfies

$$(\mathrm{Ad}\ p_k).\kappa_b(v,w) = \kappa_b((\mathrm{Ad}\ p_k).v, (\mathrm{Ad}\ p_k).w)\ \forall\ v,w \in \mathfrak{g}$$

The proof is then basically the same as the first part of that of point (2). $\diamondsuit$

We can now prove the first point of theorem 1.5. By the hypothesis on $\mathrm{rk}^{\mathrm{alg}}(\mathrm{Ad}\ H)$, there is an abelian $\mathbf{R}$-split subgroup of Ad H with Zariski closure $\bar{S}$ of dimension rk G. Let $S = \mathrm{Ad}^{-1}(\bar{S}) \subset H$. Since the kernel of Ad on H is amenable by assumption (§1.1), S is amenable as well. Theorem 4.1 gives $b \in B$ such that ρ_b maps $\check{S}^b$ surjectively on $\bar{S} = \bar{S}_d$.

As in the proof of theorem 1.3 in section 5.1, there is an abelian $\mathbf{R}$-split subalgebra $\mathfrak{z}^s \subset \check{\mathfrak{s}}^b$ such that $d\rho_b(\mathfrak{z}^s) = \bar{\mathfrak{s}}$. Then $\dim \mathfrak{z}^s = \mathrm{rk}\ G$, and there exists a maximal $\mathbf{R}$-split subalgebra $\mathfrak{a}' \subset \mathfrak{p}$ such that ad $\mathfrak{a}' = \mathfrak{z}^s$. There exists $p_o \in P$ such that $(\mathrm{Ad}\ p_0)(\mathfrak{a}') = \mathfrak{a}$. Since

$$\mathrm{Ad}\ A < \check{S}^{b.p_0} = (\mathrm{Ad}\ p_0)\check{S}^b(\mathrm{Ad}\ p_0^{-1})$$

we can apply point (1) of proposition 6.3. Together with lemma 2.1, it yields that $-\iota_{b.p_0} : \mathfrak{h} \to \mathfrak{g}$ is an embedding of Lie algebras, proving the first point of the theorem.

We now come to the second part of theorem 1.5. By assumption, there is a connected Lie group $A_H < H$ with Lie algebra $\mathfrak{a}_H \subset \mathfrak{h}$, such that Ad A_H is $\mathbf{R}$-split, abelian, and has dimension rk G. Moreover, since the kernel of Ad is amenable by assumption, so is A_H. We first show

Lemma 6.4. *Every closed A_H-invariant subset of M contains an A_H-fixed point x with an open neighborhood on which the curvature vanishes.*

Proof: Let F be a closed A_H-invariant subset. Apply theorem 4.1 with $S = A_H$. It gives $b \in \pi^{-1}(F)$ such that $\check{S}^b$ is mapped surjectively by ρ_b on $(\bar{A}_H)_d$. Exactly as in the proof of the first point, changing b into $b.p_0$ if necessary, we obtain Ad $A < \check{S}^b$, and ρ_b maps Ad A onto the identity component of $(\bar{A}_H)_d$.

Now $(\bar{A}_H)_d$ acts trivially on $\mathfrak{a}_H$, so Ad A acts trivially on $\mathfrak{a}_H^b = \iota_b(\mathfrak{a}_H)$. Thus $\mathfrak{a}_H^b \subset \mathfrak{a} \oplus \mathfrak{m}$, the centralizer of $\mathfrak{a}$; in particular, $\mathfrak{a}_H^b \subset \mathfrak{p}$, which proves that A_H fixes $x = \pi(b)$.

Since $\mathfrak{a}_H^b \subset \mathfrak{p}$, lemma 2.1 ensures that $-\iota_b$ is a Lie algebra isomorphism between $\mathfrak{a}_H$ and $\mathfrak{a}_H^b$, so $\mathfrak{a}_H^b$ is abelian. In fact, for $u \in \mathfrak{a}_H$ and $v \in \mathfrak{h}$

$$(\text{ad } \iota_b(u))(\iota_b(v)) = -\iota_b((\text{ad } u).v)$$

In particular, ad $\mathfrak{a}_H^b$ acts on $\iota_b(\mathfrak{h})$ as an $\mathbf{R}$-split subalgebra of dimension rk G. Decompose $u \in \mathfrak{a}_H^b$ into $u_\mathfrak{a} + u_\mathfrak{m}$. Because $\mathfrak{a}$ and $\mathfrak{m}$ are simultaneously diagonalizable on $\mathfrak{g}$ and u preserves $\iota_b(\mathfrak{h})$, both ad $u_\mathfrak{a}$ and ad $u_\mathfrak{m}$ preserve $\iota_b(\mathfrak{h})$; moreover, the restriction of ad $u_\mathfrak{m}$ is trivial. Then the projection of $\mathfrak{a}_H^b$ modulo $\mathfrak{m}$ is a surjection onto $\mathfrak{a}$.

Let A_H^b be the connected subgroup of P with Lie algebra $\mathfrak{a}_H^b$. Any sequence in A_H^b is a holonomy sequence at the fixed point $x = \pi(b)$. From above, $A_H^b < A \times M$, and projects onto the first factor. Thus any sequence of A is equivalent to a sequence in A_H^b, and it follows from the remark after definition 6.2 that any sequence of A is a holonomy sequence at x. We can now apply proposition 6.3 to obtain an open neighborhood V containing x on which the curvature vanishes. $\diamondsuit$

Corollary 6.5. *The Cartan geometry (M, B, ω) is flat.*

Proof: Let $N_0 \subset M$ be the subset on which the curvature of ω vanishes. By the previous lemma, N_0 has nonempty interior. If $x \in \partial N_0$, the topological boundary of N_0 in M, then $\overline{H.x}$, the closure of the H-orbit of x, is closed and H-invariant. By lemma 6.4, there are $y \in \overline{H.x}$ and an open neighborhood V of y such that the curvature vanishes on V. Then y could not have been in ∂N_0, so $\partial N_0 = \emptyset$, which means $N_0 = M$. $\diamondsuit$

Corollary 6.6. *The geometry* (M, B, ω) *is geometrically isomorphic to* $\Gamma\backslash\widetilde{\mathbf{X}}$.

Proof: By the corollary directly above, the geometry (M, B, ω) is flat—in other words, there is a $(G, \mathbf{X})$-structure on M. The proof of the completeness of this structure is a straightforward generalization of that in [FZ]. The $(G, \mathbf{X})$-structure on M defines a developing map $\delta : \widetilde{M} \to \mathbf{X}$, which is a geometric immersion, and a holonomy morphism $\rho : \mathrm{Aut}\ \widetilde{M} \to G$ satisfying the equivariance relation $\rho \circ \delta = \delta \circ \rho$ (see [Th]). Let $\pi_1(M)$ be the fundamental group of M, which acts by automorphisms on $\widetilde{M}$, and consider $\mathrm{Hol}(M) = \rho(\pi_1(M)) \subset G$, the *holonomy group* of M.

At the infinitesimal level, there is a morphism of Lie algebras

$$d\rho_e : \chi^{kill}(\widetilde{M}) \to \mathfrak{g}$$

where $\chi^{kill}(\widetilde{M})$ denotes the Lie algebra of Killing fields on $\widetilde{M}$. Since H acts by automorphisms on M, there is an infinitesimal action of $\mathfrak{h}$ by Killing fields on $\widetilde{M}$, so that $\mathfrak{h} \hookrightarrow \chi^{kill}(\widetilde{M})$. Observe that any Killing field of $\widetilde{M}$ lifted from M is centralized by $\pi_1(M)$.

Now the morphism $d\rho_e : \mathfrak{h} \to \mathfrak{g}$ maps $\mathfrak{h}$ (resp. $\mathfrak{a}_H$) injectively onto a Lie subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ (resp. $\mathfrak{a}'_H \subset \mathfrak{h}'$), which moreover is centralized by $\mathrm{Hol}(M)$. It was established in the proof of lemma 6.4 that $\mathfrak{a}'_H$ is included, up to conjugacy, in the sum $\mathfrak{a} \oplus \mathfrak{m}$, and projects onto $\mathfrak{a}$. Choose $u_1, \ldots, u_s$ spanning $\mathfrak{a}'_H$, with projections spanning $\mathfrak{a}$. Since those projections are just the $\mathbf{R}$-split parts of $u_1, \ldots, u_s$, they are centralized by $\mathrm{Hol}(M)$. Then $\mathrm{Hol}(M)$ centralizes $\mathfrak{a}$, so is contained in $A \times M_G$. Now $\mathrm{Hol}(M)$ centralizes $\mathfrak{h}'$, and A acts faithfully on $\mathfrak{h}'$ because its action is conjugated to that of $\mathrm{Ad}\ A_H$ on $\mathfrak{h}$. It follows that the projection of $\mathrm{Hol}(M)$ on A is trivial, namely $\mathrm{Hol}(M) \subset M_G$. The compact group M_G is included in K, a maximal compact subgroup of G, which acts transitively on $\mathbf{X}$.

In conclusion, the $(G, \mathbf{X})$-structure on M can be seen as a $(K, \mathbf{X})$-structure. These latter structures are known to be *complete* on closed manifolds (see proposition 3.4.10 of [Th]): the developing map δ is a covering map (The key point is that δ is not changed by viewing the Cartan geometry as a $(K, \mathbf{X})$-structure rather than a $(G, \mathbf{X})$-structure). Completeness means $\widetilde{M}$ is geometrically isomorphic to $\widetilde{\mathbf{X}}$, so $\mathrm{Aut}\ \widetilde{M} \cong \widetilde{G}$, and M is geometrically isomorphic to $\Gamma\backslash\widetilde{\mathbf{X}}$, with Γ a discrete subgroup of $\widetilde{G}$ acting properly discontinuously on $\widetilde{\mathbf{X}}$. $\diamondsuit$

7 Results about local freeness

7.1 Local freeness almost everywhere: proof of 1.6

Here (M, B, ω) is a connected Cartan geometry modeled on G/P, and $H <$ Aut M preserves a finite measure ν on M with full support. Theorem 4.1 yields a subset $\Omega \subset M$ of full measure such that for every $b \in B$ projecting on $x \in \Omega$, the image $\rho_b(\check{H}^b) = \bar{H}_d$. The stabilizer Lie algebra $\mathfrak{h}_x$ has image $\iota_b(\mathfrak{h}_x) = \iota_b(\mathfrak{h}) \cap \mathfrak{p}$. Since $\iota_b(\mathfrak{h}) \cap \mathfrak{p}$ is invariant by $\check{H}^b$, the Lie algebra $\mathfrak{h}_x$ is invariant by $\bar{H}_d$, as desired.

Assume now that $\bar{H}_d$ leaves stable only a finite family $\{0\} = F_0, \ldots, F_s$ of subalgebras in $\mathfrak{h}$. Write $\Omega = \Omega_0 \cup \cdots \cup \Omega_s$, where Ω_i contains all $x \in \Omega$ such that $\mathfrak{h}_x = F_i$. Let H_i be the connected Lie subgroup of H with Lie algebra F_i. The action of H_i is faithful, so that for $i \neq 0$, the interior of Ω_i is empty. This follows from the lemma

Lemma 7.1. *Let $\{h^t\}_{t \in \mathbf{R}} <$ Aut M be a 1-parameter subgroup, and $U \subset M$ an open subset such that $h^t\big|_U = Id$ for every $t \in \mathbf{R}$. Then $h^t = Id$ for every $t \in \mathbf{R}$.*

Proof: As already noticed, the action of h^t on B is free because it preserves a framing. So if suffices to show that h^t fixes a point on B for every $t \in \mathbf{R}$. Pick $x \in U$ and $b \in B$ projecting on x. Thanks to the Cartan connection, it is possible to define a developing map $\mathcal{D}_x^b$ from the space $C^1([-1,1], U)$ of C^1 curves γ in U with $\gamma(0) = x$ to the space $C^1([-1,1], \mathbf{X})$ of C^1 curves β in $\mathbf{X}$ with $\beta(0) = o$, the class of P in $\mathbf{X} = G/P$. We refer to [Shar] for the construction of the developing map. Now there is a 1-parameter subgroup $\{p^t\} < P$ such that $h^t.b.p^{-t} = b$. It is not difficult to show that $\mathcal{D}_x^b(h^t.\gamma) = p^t.\mathcal{D}_x^b(\gamma)$. Because $h^t\big|_U = Id$, all developments of curves in U are pointwise fixed by Ad p^t. The union of all these developments is a subset of $\mathbf{X}$ with nonempty interior, pointwise fixed by p^t. As a consequence, p^t acts trivially on $\mathbf{X}$ for every $t \in \mathbf{R}$. Since the kernel of the action of P on $\mathbf{X}$ was assumed to be finite, $p^t = e$ for every $t \in \mathbf{R}$. $\diamondsuit$

Now because Ω is conull for a measure of full support, it is dense in M, and $\Omega_1 \cup \cdots \cup \Omega_s$ has empty interior. Thus Ω_0, on which H is locally free, is dense in M. Since the subset of M where the action is locally free is clearly open, theorem 1.6 follows. $\diamondsuit$

7.2 Open orbit implies homogeneity: proof of 1.7

In this section, we assume that H is a Lie subgroup of Aut M. The Cartan geometry (M, B, ω) is connected and n-dimensional, equipped with an H-invariant volume form ν of finite total volume.

Recall that σ is the $(\mathrm{Ad}_{\mathfrak{g}}P)$-equivariant projection from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{p}$. For every $b \in B$ projecting on $x \in M$, there is a natural isomorphism $j_b : T_xM \to \mathfrak{g}/\mathfrak{p}$ with $j_b(u) = \sigma \circ \omega_b(\check{u})$, where $\check{u} \in T_bB$ projects on u. It is easy to check that this isomorphism is well-defined, and $j_{b.p} = (\mathrm{Ad}\ p^{-1}) \circ j_b$ for every $p \in P$ and $b \in B$.

By this remark, we see that a vector field on M is nothing else than a map $\phi : B \to \mathfrak{g}/\mathfrak{p}$ satisfying $\phi(b.p) = (\mathrm{Ad}\ p^{-1}).\phi(b)$. In the same way, a continuous n-form ν on the manifold M can be seen as a continuous map $\nu : B \to \wedge^n(\mathfrak{g}/\mathfrak{p})^*$, such that for any n-tuple $(u_1, \ldots, u_n)$ of $(\mathfrak{g}/\mathfrak{p})^n$,

$$\nu(b.p)(u_1, \ldots, u_n) = \nu(b)((\mathrm{Ad}\ p).u_1, \ldots, (\mathrm{Ad}\ p).u_n)$$

Our hypothesis is that $\nu(h.b) = \nu(b)$ for all $h \in H$.

Let U be the real algebraic variety $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g}) \times \wedge^n(\mathfrak{g}/\mathfrak{p})^*$. The map $\phi : b \mapsto (\iota_b, \nu(b))$ is continuous and $H \times P$-equivariant. Here, the action of H on $\wedge^n(\mathfrak{g}/\mathfrak{p})^*$ is trivial, and for $\alpha \in \wedge^n(\mathfrak{g}/\mathfrak{p})^*$ and $\check{p} \in \mathrm{Ad}_{\mathfrak{g}}P$,

$$(\check{p}.\alpha)(u_1, \ldots, u_n) = \alpha(\check{p}.u_1, \ldots, \check{p}.u_n).$$

Notice that the $H \times P$ action on U is the restriction of an algebraic action of $\bar{H}^{\mathrm{Ad}} \times \mathrm{Ad}_{\mathfrak{g}}P$. Following the same proof as for theorem 4.1, we obtain a P-invariant subset $\Lambda \subset B$ projecting on a set of full ν-measure, such that for every $b \in \Lambda$, $\phi(b)$ is a fixed point for $\bar{H}_d$. Now, $\bar{H}_d$ is cocompact in $\bar{H}^{\mathrm{Ad}}$ ([Shal, 1.4]). Let K be a compact subset such that $\bar{H}^{\mathrm{Ad}} = K\bar{H}_d$. Then we get that for $b \in \Lambda$, the $\bar{H}^{\mathrm{Ad}}$-orbit of $\phi(b)$ is compact and in fact equals $K.\phi(b)$. This means that for every $b \in \Lambda$, and $h \in H$, there exist $\check{p}_h \in \mathrm{Ad}_{\mathfrak{g}}P$ and $l_h \in K$ satisfying $\iota_{h.b}(u) = \check{p}_h.\iota_b(l_h.u)$ for every $u \in \mathfrak{h}$, and $\check{p}_h.\nu(b) = \nu(b)$.

Observe that if M has an open H-orbit, then it has an open orbit for H^o, the identity component of H. Indeed, if $H^o.x$ is not open for some $x \in M$, then it has volume zero. But H is a Lie group, so it has countably-many connected components. Thus $H.x$ is the union of countably-many sets of volume zero, so that it also has volume zero, contradicting the assumption that it was open.

Let $\mathcal{O}$ be an open orbit of H^o on M. Since any set of full measure for ν intersects $\mathcal{O}$, there is $b \in \Lambda$ projecting on $x \in \mathcal{O}$. Denote by $\overline{\mathcal{O}}$ the topological closure of $\mathcal{O}$ in M. Pick $x_\infty \in \overline{\mathcal{O}}$ and $b_\infty \in B$ projecting on x_∞. There is (h_k) a sequence of H^o such that $h_k.x \to x_\infty$, and thus a sequence (p_k) of P such that $h_k.b.p_k^{-1} \to b_\infty$. Let $X_1, ..., X_n \in \mathfrak{h}$.

The equivariance property implies that for every $p \in P$,

$$\nu(b.p)(\iota_{b.p}(X_1), \dots, \iota_{b.p}(X_n)) = \nu(b)(X_1, \dots, X_n)$$

The above equality together with H-invariance of ν implies that:

$$\begin{aligned}&\nu(h_k.b.p_k^{-1})(\sigma \circ \iota_{h_k.b.p_k^{-1}}(X_1), \dots, \sigma \circ \iota_{h_k.b.p_k^{-1}}(X_n))\\&= \nu(b)(\check{p}_{h_k}.\sigma \circ \iota_b(l_{h_k}.X_1), \dots, \check{p}_{h_k}.\sigma \circ \iota_b(l_{h_k}.X_n))\end{aligned}$$

Looking at a subsequence if necessary, we can assume that l_{h_k} tends to $l_\infty \in K$. Since $\check{p}_{h_k}.\nu(b) = \nu(b)$, the right-hand side in the previous equality tends to $\nu(b)(\sigma \circ \iota_b(l_\infty.X_1), \dots, \sigma \circ \iota_b(l_\infty.X_n))$, and the left one tends to $\nu(b_\infty)(\sigma \circ \iota_{b_\infty}(X_1), \dots, \sigma \circ \iota_{b_\infty}(X_n))$. Now, the tangent space $T_x(H^o.x) = T_xM$, so there are $Y_1, \dots, Y_n \in \mathfrak{h}$ such that $(\sigma \circ \iota_b(Y_1), \dots, \sigma \circ \iota_b(Y_n))$ is a basis of $\mathfrak{g}/\mathfrak{p}$, and thus $\nu(b)(\sigma \circ \iota_b(Y_1), \dots, \sigma \circ \iota_b(Y_n)) \neq 0$. Choosing $X_1 = l_\infty^{-1}.Y_1, ..., X_n = l_\infty^{-1}.Y_n$, we get that $\nu(b_\infty)(\sigma \circ \iota_{b_\infty}(X_1), \dots, \sigma \circ \iota_{b_\infty}(X_n)) \neq 0$, and $\sigma \circ \iota_{b_\infty}(\mathfrak{h}) = \mathfrak{g}/\mathfrak{p}$. It follows that the H^o-orbit of x_∞ is open, and that $x_\infty \in \mathcal{O}$. Because M is connected, $M = \mathcal{O}$, which proves that H^o acts transitively on M.

Assume now that $\bar{H}^{\mathrm{Ad}} = \bar{H}_d$. Theorem 1.6 implies that for almost every $x \in M$, the stabilizer subalgebra $\mathfrak{h}_x$ is $\bar{H}^{\mathrm{Ad}}$-invariant. Because H^0 acts transitively, the stabilizer subalgebra is the same at every point of M. Since H acts faithfully, $\mathfrak{h}_x = \{0\}$ for every $x \in M$. Then $M = H^o/\Gamma$, where Γ is a discrete subgroup of H^o. The n-form ν is left-invariant, so it induces Haar measure on H^o/Γ. Since M has finite volume, Γ must be a lattice in H^o.

Deduction of Corollary 1.8

Let $\mathrm{Aut}_{loc}(M, \omega, \nu)$ (resp. $\mathrm{Aut}_{loc}(M, \mathcal{S}, \nu)$) be the pseudo-group of local automorphisms of the Cartan geometry (resp. of the structure $\mathcal{S}$) that also preserve ν. The hypotheses of corollary 1.8 imply $\mathrm{Aut}_{loc}(M, \omega, \nu) =$

$\mathrm{Aut}_{loc}(M, \mathcal{S}, \nu)$. The fact that H has a dense orbit ensures that $\mathrm{Aut}_{loc}(M, \omega, \nu)$, hence also $\mathrm{Aut}_{loc}(M, \mathcal{S}, \nu)$, has a dense orbit.

Adding an analytic volume form ν to the analytic rigid geometric structure of algebraic type $\mathcal{S}$ still yields an analytic rigid geometric structure of algebraic type $\mathcal{S}' = (\mathcal{S}, \nu)$. Gromov's open-dense theorem ([Gr] 3.3.A), implies that $\mathrm{Aut}_{loc}(M, \mathcal{S}, \nu)$ has an open dense orbit U. Note that the local Killing fields for this structure also act transitively on U.

Because the structure on M is analytic and M is simply connected, local Killing fields extend to all of M ([Am], [N], [Gr]). Because M is compact the extended Killing fields are complete. Then Aut $(M, \mathcal{S}, \nu)$ has an open orbit on M, hence so does Aut (M, ω, ν). The corollary now follows from theorem 1.7. $\diamondsuit$

7.3 Homogeneous reductive Cartan geometries

Finally, we prove theorem 1.9. The assumptions here are that the geometry is reductive, and the H-action is transitive and faithful, and preserves a finite volume. By theorem 1.7, the action is everywhere locally free.

Now use Theorem 4.1 to obtain $b \in B$ such that ρ_b is a surjection from $\check{H}^b$ to $\bar{H}^{\mathrm{Ad}}$. Let $\check{H} = \mathrm{Ad}_{\mathfrak{g}}^{-1}(\check{H}^b) < P$. Because H acts locally freely, $\iota_b(\mathfrak{h})$ is transverse to $\mathfrak{p}$ for all $b \in B$. Let σ be the $\mathrm{Ad}_{\mathfrak{g}}P$-equivariant projection from $\mathfrak{g}$ to $\mathfrak{n}$ associated to the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$. If Ad p is trivial on $\iota_b(\mathfrak{h})$, then it is trivial on $\mathfrak{n}$. Because $\mathrm{Ad}_{\mathfrak{g}}P$ is faithful on $\mathfrak{n}$, the kernel of $\rho = \rho_b \circ \mathrm{Ad}_{\mathfrak{g}} : \check{H} \to \bar{H}^{\mathrm{Ad}}$ is trivial, so ρ is an isomorphism. The map $\iota = \sigma \circ \iota_b$ is an isomorphism from $\mathfrak{h}$ to $\mathfrak{n}$, and by theorem 4.1, it intertwines the representations of $\bar{H}^{\mathrm{Ad}}$ on $\mathfrak{h}$ and Ad $\check{H}$ on $\mathfrak{n}$. Set $H' = \rho^{-1}(\mathrm{Ad}\ H)$. When H is connected, then H is a central extension of H' as claimed in the second point of theorem 1.9. $\diamondsuit$

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