Descriptive set theory and uncountable model theory

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In the early days of the development of model theory it was considered natural and was certainly beneficial to assume that the theories under investigation were in a *countable* language. The primary advantage of this assumption was the presence of the Omitting Types Theorem of Grzegorczyk, Mostowski, and Ryll-Nardzewski [1], which generalized arguments of Henkin [3] and Orey [8]. Following this, Vaught [13] gave a very pleasing analysis of the class of countable models of such a theory. This led to Morley's categoricity theorem [7] for certain classes of uncountable models of theories in a countable language.

The landscape was completely altered by the subsequent work of Shelah (see e.g. [11]). He saw that the salient features of Morley's proof did not require the assumption of the language being countable. Indeed, many of notions that were central to Shelah's work, including unstability, the fcp, the independence property and the strict order property, are *local*. That is, a theory possesses such a property if and only if some formula has the property. Consequently, the total number of formulas in the language is not relevant. Still other notions, such as superstability, are not local but can be described in terms of *countable fragments* of the theory. That is, a theory of any cardinality is superstable if and only if all of its reducts to countable fragments of the theory are superstable. Using a vast collection of machinery, Shelah was able to answer literally hundreds of questions about

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the class of *uncountable* models of certain theories. Most[1] of his arguments do not depend on the cardinality of the underlying language. In particular, he gave a proof of Łoś' conjecture, that the analogue of Morley's theorem[2] holds for theories in languages of any size. Somewhat curiously, whereas Shelah's methods were very good in classifying uncountable models of a theory, they had considerably less to say about the countable models of a theory.

The 'party line' changed as a result of this – A general belief was that if one were concerned with uncountable models of a theory, then the countability of the language should not be relevant. In what follows we give two examples where this is not the case. In both examples the countability of the language allows us to apply classical methods of Descriptive Set Theory in order to prove theorems about the uncountable models of a theory. The first section is by now folklore. It is included to set notation and to indicate that rudimentary descriptive set theory plays a role in the basic underpinnings of model theory. The second section highlights the overall argument of [2] wherein new dividing lines are given to complete the classification of the uncountable spectra of complete theories in countable languages. The third section highlights the main result of [4], which proves a structure theorem for saturated models of a stable theory in a countable language. One consequence of the main theorem is that one dividing line in Shelah's attempt at describing the class of $\aleph_1$-saturated models of a countable theory is redundant (Corollary 3.6(2)).

This article is definitely intended to be a survey. Many definitions, although standard, are not given and proofs of theorems are merely sketched. The reader who is interested in more rigor is referred to [2] for the material in Section 2 and to [4] for Section 3. Throughout the article, classical descriptive set-theoretic results are referred to as 'Facts.' It is noteworthy that all of the facts needed here are relatively soft.

1 Stone spaces and T^{eq}

Fix a complete theory T in a language L. For a fixed choice of variables $\bar{x} = \langle x_1, \ldots, x_n \rangle$, the quotient of all L-formulas $\varphi(\bar{x})$ whose free variables are

[1] One place where countability of the theory is useful is his analysis of theories with NOTOP, the negation of the omitting types order property.

[2] Specifically, if a theory in a language of size κ is categorical in some cardinality greater than κ, then it is categorical in all cardinals greater than κ.

among $\bar{x}$ modulo T-equivalence is a Boolean algebra. Its associated Stone space $S_{\bar{x}}$ consists of the complete types in the variables $\bar{x}$. The Stone space is naturally topologized by declaring the set of subsets U_φ to be a basis, where each $U_\varphi = \{p \in S_{\bar{x}} : \varphi \in p\}$.

It is routine to verify that each of the spaces $S_{\bar{x}}$ is compact, Hausdorff, and totally disconnected. Many model theoretic concepts translate easily into this setting. As examples, for any model M of T, the set of types in $S_{\bar{x}}$ realized in M is dense, and a type p is isolated in $S_{\bar{x}}$ if and only if it contains a complete formula (i.e., is a principal type).

If, in addition, we assume that the language L is countable, then each of the spaces $S_{\bar{x}}$ can be endowed with a complete metric, hence it is a Polish space.[3]

Fact 1.1 *Every Polish space is either* countable *(in which case the isolated types are dense) or it contains a perfect subset.*

Theorem 1.2 *For T a complete theory in a countable language,*

1. *T has a countable saturated model if and only if $S_{\bar{x}}$ is countable for all $\bar{x}$; and*

2. *T has a prime (atomic) model if and only if the isolated types of $S_{\bar{x}}$ are dense for all $\bar{x}$.*

In order to 'beautify' these statements and to provide a natural setting for arguments such as Shelah's existence theorem for semiregular types, one commonly passes from the theory T to its expansion T^{eq}. Specifically, let $\mathcal{E}$ denote the set of all L-formulas $E(\bar{x}, \bar{y})$ that are equivalence relations in models of T. We form the language L^{eq} by adjoining to L a new unary predicate symbol U_E and a new function symbol f_E for each $E \in \mathcal{E}$. Then each model M of T has a canonical expansion and extension to an L^{eq}-structure M^{eq} which can be described as follows:

The universe of M^{eq} is the disjoint union of the interpretations of the U_E's, where for each $2n$-ary relation E, $U_E(M^{eq}) = \{\bar{a}/E : \bar{a} \in M^n\}$. We identify the original structure M with $U_=(M^{eq})$. For each $2n$-ary $E \in \mathcal{E}$, f_E is interpreted as the canonical mapping from $U_=(M^{eq})^n$ onto $U_E(M^{eq})$ given by $\bar{a} \mapsto \bar{a}/E$.

[3]Unlike some texts, we allow Polish spaces to have isolated points.

We let T^{eq} denote the theory of any M^{eq} and denote *the* Stone space of T^{eq} to consist of all complete L^{eq} 1-types that contain exactly one formula of the form $U_E(x)$. Since powers of M embed naturally into M^{eq} (for each n, M^n is identified with $U_n(M^{eq})$, where U_n corresponds to the $2n$-ary equivalence relation of componentwise equality) S can be thought of as being a 'disjoint union' of the spaces $S_{\bar{x}}$, together with Stone spaces of other, nontrivial quotients.

This space S is not compact, but it is locally compact and Hausdorff. Furthermore, if the original language L is countable, then so is L^{eq} and S is a Polish space. So, the 'beautification' of Theorem 1.2 above is immediate:

Theorem 1.3 *For T a complete theory in a countable language,*

1. *T has a countable saturated model if and only if S is countable; and*
2. *T has a prime (atomic) model if and only if the isolated types of S are dense.*

We note in passing that the classical theorem of Vaught asserting that for a complete theory T in a countable language, if T has a countable saturated model then T has an atomic model follows immediately from this and Fact 1.1.

Fact 1.4 (Baire Category Theorem) *Polish spaces are not meagre (i.e., they are not the countable union of nowhere dense sets).*

From this and the characterizations given above, one can easily obtain the following strengthenings of the omitting types theorem:

Theorem 1.5 *Let T be a complete theory in a countable language.*

1. *If $\Gamma \subseteq S$ is meagre and no $p \in \Gamma$ is isolated, then there is a model M of T omitting each $p \in \Gamma$;*
2. *If, in addition, T does not have a prime model, then there are $2^{\aleph_0}$ nonisomorphic countable models omitting every $p \in \Gamma$.*

The proof of (1) is merely a Henkin construction with conditions added to ensure that no type in Γ gets realized. For (2) one performs a Henkin construction on a tree of models indexed by $2^{<\omega}$. The continuum models are described by branches through the tree. At any stage of the construction, one has assigned 'formula-much' information to finitely many nodes in the tree with the understanding that any model that is coded by a branch passing through the node will satisfy the formula. Conditions are included to ensure that each branch will produce a model of the theory that omits every type in Γ. Additional conditions ensure that distinct branches give rise to nonisomorphic models. In spirit, the proof is like arguments employing Sacks forcing. Variations of this technique are used in the verification of Proposition 2.4 in the next section.

2 The uncountable spectrum of a theory

Given a complete theory T in a countable language, it is natural to ask how many nonisomorphic models of T are present in any cardinality κ. Whereas the case of $\kappa = \aleph_0$ is still problematic, a full answer is known when κ is uncountable. As notation, let $I(T, \kappa)$ denote the number of pairwise nonisomorphic models of T of size κ. For a theory T, its *uncountable spectrum* is the mapping $\kappa \mapsto I(T, \kappa)$ for $\kappa > \aleph_0$.

The following theorem appears in [2]:

Theorem 2.1 *1. Among all theories in countable languages, there are exactly twelve 'species' of uncountable spectra (some of which involve a parameter);*

2. Fix a countable language L. The equivalence relation on complete L-theories of 'having the same uncountable spectrum' is $\mathbf{\Pi}^1_1$ in the standard topology of complete L-theories.

In other words, the determination of the uncountable spectrum of T can be made by analyzing which *countable* configurations of elements embed into models of T. Thus, despite the fact that the uncountable spectrum involves classes of uncountable models, the determination of which species the map $\kappa \mapsto I(T, \kappa)$ belongs to reduces to questions about the class of *countable* models of T. It is because of this that methods of Descriptive Set Theory proved useful in the proof of Theorem 2.1. In addition to the facts mentioned in the previous section, two more classical results, respectively due to Mazurkiewicz (see e.g., [10]) and Silver [12] are relevant:

Fact 2.2 *1. A G_δ subspace of a Polish space is Polish.*

2. A Borel equivalence relation on a Borel subset of a Polish space has either countably many or $2^{\aleph_0}$ equivalence classes.

Much of the work in computing uncountable spectra was done by Shelah. His analysis proceeds in a 'top-down' fashion. First, he described the theories having the maximal spectrum (i.e., $I(T,\kappa) = 2^\kappa$ for all uncountable κ). It turns out that the uncountable spectrum is maximal unless T is superstable and, for any triple of models[4] M_0, M_1, M_2, if $\{M_1, M_2\}$ are independent over M_0, then there is a prime and minimal model over their union, which we denote by $M_1 \bigoplus_{M_0} M_2$. Using these algebraic facts Shelah provided a structure theorem for the class of uncountable models of theories whose spectrum is not maximal: Any uncountable model of such a theory is prime and minimal over a well-founded, independent tree of *countable* submodels. As there are at most $2^{\aleph_0}$ nonisomorphic countable models, an upper bound on the number of nonisomorphic models of size κ can be computed from a bound on the depth of the trees that can occur. If the theory admits trees of infinite depth, then Shelah is able to use coding tricks (using certain levels of the tree as 'markers') to obtain a matching lower bound. So we concentrate on countable theories in which every uncountable model is prime and minimal over an independent tree of countable models in which each branch has length at most some fixed finite number d. For such theories, the naive upper bound on the number of nonisomorphic models of size $\aleph_\alpha$ alluded to above is $\beth_{d-1}(|\omega + \alpha|^{2^{\aleph_0}})$.

In order to find lower bounds on $I(T, \aleph_\alpha)$ we consider the following scenario: Fix $1 \leq n < d$ and an increasing chain

$$\overline{\mathcal{M}} := M_0 \subset_{na} M_1 \subset_{na} \ldots \subset_{na} M_{n-1}$$

of countable models satisfying $wt(M_{i+1}/M_i) = 1$ and (when $i > 0$) M_{i+1}/M_i is orthogonal to M_{i-1}. We wish to describe the set of chains $\overline{\mathcal{N}}$ of length $n+1$ extending $\overline{\mathcal{M}}$ and count the number models of a fixed uncountable cardinality that are prime and minimal over a tree of countable models, every branch of which is isomorphic to some $\overline{\mathcal{N}}$.

Accordingly, we call a countable model N a *leaf* of $\overline{\mathcal{M}}$ if $wt(N/M_{n-1}) = 1$ and N/M_{n-1} is orthogonal to M_{n-2}. Leaves of $\overline{\mathcal{M}}$ are connected with the

[4]It is equivalent to require that M_0, M_1, M_2 be *countable* models.

space of types

$$R(\overline{\mathcal{M}}) : \{p \in S(M_{n-1}) : p \text{ regular and } p \perp M_{n-2}\}$$

(when $n = 1$ we delete the final condition). Indeed, if a realizes a type in $R(\overline{\mathcal{M}})$ and N is dominated by $M_{n-1} \cup \{a\}$, then N is a leaf of $\overline{\mathcal{M}}$. Conversely, every leaf of $\overline{\mathcal{M}}$ realizes at least one type in $R(\overline{\mathcal{M}})$. Unfortunately, the correspondence between leaves of $\overline{\mathcal{M}}$ and $R(\overline{\mathcal{M}})$ is not tight. It is possible for a single leaf to contain realizations of several types in $R(\overline{\mathcal{M}})$, and it is possible that the same type in $R(\overline{\mathcal{M}})$ can be realized in several nonisomorphic leaves.

For a subset Y of leaves, a *Y-tree* is an independent tree in which every branch is isomorphic to $\overline{\mathcal{M}} \frown \langle N \rangle$ for some $N \in Y$, and a model of T is a *Y-model* if it is prime and minimal over a Y-tree. In order to get a lower bound on the number of Y-trees of a certain cardinality we examine subsets of leaves with certain 'separation properties.' Specifically, we call a family Y of leaves of $\overline{\mathcal{M}}$ *diffuse* if

$$N \oplus_{M_{n-1}} V \not\cong_V N' \oplus_{M_{n-1}} V$$

for all distinct $N, N' \in Y$ and all Y-models V. Similarly, a set Y of leaves is *diverse* if $N \bigoplus_{M_{n-2}} V \not\cong_V N' \bigoplus_{M_{n-2}} V$, again for all distinct $N, N' \in Y$ and all Y-models V. (It is not hard to show that a diffuse family is diverse.) The following is the content of Propositions 5.6 and 5.8 of [2].

Proposition 2.3 *If there is a diffuse set Y of leaves of cardinality $2^{\aleph_0}$, then for any $\alpha > 0$ there are $\min\{2^{\aleph_\alpha}, \beth_{n-1}(|\omega+\alpha|^{2^{\aleph_0}})\}$ nonisomorphic Y-models of size $\aleph_\alpha$. If such a Y is diverse then there are at least $\min\{2^{\aleph_\alpha}, \beth_{n+1}\}$ nonisomorphic Y-models of size $\aleph_\alpha$.*

Our method for computing the uncountable spectra will be to translate dichotomies occurring from descriptive set theory into dichotomies among theories, which will be expressed in terms of the existence or nonexistence of large diffuse or diverse sets of leaves. As an example, if $\{p_i : i \in \kappa\} \subseteq R(\overline{\mathcal{M}})$ are pairwise orthogonal and $Y = \{N_i : i \in \kappa\}$ is a set of leaves of $\overline{\mathcal{M}}$ such that each N_i realizes p_i, then Y is diffuse (see e.g., 3.6 of [2]). We will see below that $R(\overline{\mathcal{M}})$ is a Borel subset of $S(M_{n-1})$ and that nonorthogonality is a Borel equivalence relation. Hence, by Silver's theorem the number of nonorthogonality classes (i.e., the size of a maximal diffuse family obtained in this fashion) is either countable or has size $2^{\aleph_0}$.

In the case when the theory T is totally transcendental (equivalently $\aleph_0$-stable) choosing types from different nonorthogonality classes is the only way of obtaining diffuse sets of leaves. When T is t.t. any leaf N realizes a *strongly regular* type $p \in R(\overline{\mathcal{M}})$. In addition, this type p is realized in any leaf that realizes a type that is nonorthogonal to p. Moreover, for any a realizing such a type p, there is a prime model over $M_{n-1} \cup \{a\}$. Thus, there is a canonical choice of a leaf N among all possible leaves that realize a certain nonorthogonality class. Hence the size of a maximal diffuse family of leaves is precisely the number of nonorthogonality classes realized in $R(\overline{\mathcal{M}})$. As noted above this number is finite, countably infinite, or $2^{\aleph_0}$.

When T is superstable but not t.t. there are other mechanisms for producing diffuse or diverse sets of leaves. It might be that a leaf N does not realize any strongly regular types. Furthermore, even if N does realize a strongly regular type there need not be a prime model over M_{n-1} and a realization of the type. The effect of one of these 'failures' depends on whether the nonorthogonality class consists of *trivial* or *nontrivial* regular types. The following Proposition, which is the content of Propositions 3.21, 5.6 and 5.8 of [2], illustrates the effect of such failures. The constructions of the large families of leaves are similar to the construction of a large family of countable models in Theorem 1.5(2) in this paper.

Proposition 2.4 *Fix a type $q \in R(\overline{\mathcal{M}})$, a realization a of q, and an uncountable cardinal $\aleph_\alpha$. Suppose that* **either** *there is a perfect set of types in $R(\overline{\mathcal{M}})$ nonorthogonal to q* **or** *there is no prime model over $M_{n-1} \cup \{a\}$. Then:*

1. *If q is trivial then there is a diffuse set Y of leaves of size $2^{\aleph_0}$ (hence there are $\min\{2^{\aleph_\alpha}, \beth_{n-1}(|\omega+\alpha|^{2^{\aleph_0}})\}$ nonisomorphic Y-models of size $\aleph_\alpha$); and*

2. *If q is nontrivial then there is a diverse set Y of leaves of size $2^{\aleph_0}$ (hence there are at least $\min\{2^{\aleph_\alpha}, \beth_{n+1}\}$ nonisomorphic Y-models of size $\aleph_\alpha$).*

Sorting all of this out (i.e., computing $I(T, \aleph_\alpha)$ in all of the scenarios) is where Descriptive Set Theory comes into play. We begin with some rather crude computations.

Lemma 2.5 *Let T be a stable theory in a countable language and let $M_0 \subseteq M$ be countable models of T. Then:*

1. *$\{p \in S(M) : p$ is trivial, weight 1$\}$ is a G_δ (i.e., $\mathbf{\Pi_2}$);*

2. *$\{p \in S(M) : p$ is almost orthogonal to M_0 over $M\}$ is a G_δ;*

3. *$\{p \in S(M) : p$ is almost orthogonal to $q\}$ is a G_δ for any fixed $q \in S(M)$;*

4. *$\{p \in S(M) : p$ is regular and orthogonal to $M_0\}$ is $\mathbf{\Pi_4}$;*

5. *$\{p \in S(M) : p$ is not orthogonal to $q\}$ is $\mathbf{\Sigma_4}$ for any fixed $q \in S(M)$.*

The verifications of all of these are routine. One should note the disparity in the complexity of determining 'almost orthogonality' versus 'orthogonality' in the statements above. The good news is that these bounds suffice to show that $R(\overline{\mathcal{M}})$ is a Borel subset of $S(M_{n-1})$ and that nonorthogonality is a Borel equivalence relation on $R(\overline{\mathcal{M}})$. The bad news is that $\mathbf{\Pi_4}$ and $\mathbf{\Sigma_4}$ sets of types do not have very good closure properties. Thus, we attempt to improve these bounds by showing that these sets have simpler descriptions in some restricted settings.

Our analysis begins as in the t.t. case. If $R(\overline{\mathcal{M}})$ has $2^{\aleph_0}$ nonorthogonality classes then there is a diffuse family Y of leaves of size $2^{\aleph_0}$, so the number of Y-models equals the naive upper bound mentioned above. Consequently, we can ignore this case and henceforth *assume that $R(\overline{\mathcal{M}})$ has only countably many nonorthogonality classes.*

Next, we concentrate on the *trivial* regular types in $R(\overline{\mathcal{M}})$. Note that any trivial, weight 1 type is necessarily regular. Also, a trivial, regular type is orthogonal to M_0 if and only if it is almost orthogonal to M_0. Thus, by Lemma 2.5(1) and (2),

$$R_{tr}(\overline{\mathcal{M}}) = \{p \in R(\overline{\mathcal{M}}) : p \text{ is trivial, regular, } p \perp M_{n-2}\}$$

is a G_δ subset of $S(M_{n-1})$ hence is a Polish space itself by Fact 2.2(1).

Let $\{r_i : i < j \leq \omega\}$ be a set of representatives of the nonorthogonality classes of $R_{tr}(\overline{\mathcal{M}})$. For each $i < j$, let

$$X_i = \{p \in R_{tr}(\overline{\mathcal{M}}) : p \text{ is not orthogonal to } r_i\}$$

Whereas this is typically a complicated set (*cf.* Lemma 2.5(5)), note that a type $p \in R_{tr}(\overline{\mathcal{M}})$ is an element of X_i if and only if it is orthogonal to some (equivalently to every) type in X_k for all $k < j$, $k \neq i$. But, since orthogonality is equivalent to almost orthogonality for trivial types, it follows from Lemma 2.5(3) and the countability of j that each X_i is a G_δ subset of $R_{tr}(\overline{\mathcal{M}})$, hence is a Polish space. But now, by Fact 1.1, either each X_i is countable, or for some $i < j$, X_i contains a perfect subset. In the latter case Proposition 2.4 yields a diffuse set of leaves of size $2^{\aleph_0}$. As we have dispensed that case, we may now *assume that each X_i is a countable Polish space.* Hence by Fact 1.1, each X_i has an isolated point. In this context, an isolated point must be strongly regular (see e.g., D.15 of [6]). So, to finish the analysis of the trivial types, we ask whether there is any trivial, strongly regular type p such that there is no prime model over M_{n-1} and a realization of p. If a prime model fails to exist, then by Proposition 2.4 there is a diffuse family of size $2^{\aleph_0}$. If the requisite prime models do exist, then the set of trivial types in $R(\overline{\mathcal{M}})$ acts as in the t.t. case.

Now, if we are not done already (i.e., produced a diffuse family of leaves of size continuum) then we have argued that only countably many nonorthogonality classes are represented in $R(\overline{\mathcal{M}})$ and the set $R_{tr}(\overline{\mathcal{M}})$ is countable. Let Q be a complete set of representatives of regular types represented in $R(\overline{\mathcal{M}})$. *Since we are free to work in T^{eq}* it follows from VIII 2.20 of [9] that for each $q \in Q$ there is a regular type q' nonorthogonal to q and a formula $\theta \in q'$ such that θ is q'-simple, has q'-weight 1 and moreover q'-weight is definable inside θ. Without loss, we may assume that $q' = q$. Then the set

$$Z_q = \{p \in S(M_{n-1}) : \theta \in p, w_q(p) = 1\}$$

is a closed subset of $S(M_{n-1})$. Note that if $p \in Z_q$, then $p \not\perp q$, hence $p \perp M_{n-2}$, so $p \in R(\overline{\mathcal{M}})$. Conversely, if $p \in R(\overline{\mathcal{M}})$, $p \not\perp q$, and $\theta \in p$, then $p \in Z_q$. The analysis is now similar to the above. If Z_q is uncountable for some q then by Lemma 1.1 it contains a perfect subset, so there is a diverse subset of size continuum by Proposition 2.4. On the other hand, if each Z_q is countable, then it contains an isolated point. That is, every nonorthogonality class represented in $R(\overline{\mathcal{M}})$ has a strongly regular representative. Next, if there is no prime model over M_{n-1} and a realization of some strongly regular type in $R(\overline{\mathcal{M}})$, then there is a diverse family of leaves of size $2^{\aleph_0}$. Finally, if there is always a prime model over such sets, then as in the t.t. case, we have a canonical choice of a leaf corresponding to each nonorthogonality class of $R(\overline{\mathcal{M}})$.

These considerations allow us to compute the spectrum of a theory in almost all cases.

3 Stable theories and the death of DIDIP

In this section we consider strictly stable theories (i.e., stable but not superstable) in a countable language. The important distinction is that in a superstable theory, any type over a model is based on a *finite* subset of the model, hence by a single element if we pass to T^{eq}. It is because of this fact that almost all of the common stability-theoretic adjectives (see e.g., Lemma 2.4) give rise to Borel subsets of the Stone space. However, when T is countable and strictly stable then types over models are based on *countable* subsets of the model. Thus, from the point of view of Descriptive Set Theory, we are forced upward into the projective hierarchy. Whereas the application of DST in this section only requires consideration of Σ^1_1 sets, it is my belief that further theorems about strictly stable theories in countable languages may require one to consider more complicated projective sets. The key to the proof of Theorem 3.3, which is the main theorem of this section, is the following classical theorem of Lusin and Sierpiński [5].

Fact 3.1 *Any $\mathbf{\Sigma^1_1}$-subset of a Polish space has the property of Baire (i.e., if A is $\mathbf{\Sigma^1_1}$ then there is an open set U such that the symmetric difference $A \triangle U$ is meagre).*

Fix a strictly stable theory T in a countable language. Since T is not superstable, results of Shelah (see e.g., [11]) demonstrate that one cannot reasonably classify the class of all uncountable models of T, so we pass to the subclass of $\aleph_1$-saturated models of T.[5] In this context, call a model M $\aleph_1$-*prime over* A if $A \subseteq M$, M is $\aleph_1$-saturated, and M embeds elementarily over A into any $\aleph_1$- saturated model containing A. In [11] Shelah proves that $\aleph_1$-prime models exist over any sets A.

Definition 3.2 Let T be a countable theory. T has NDOP if for all triples $\{M, M_0, M_1\}$ of $\aleph_1$-saturated models with $M = M_0 \cap M_1$ and $\{M_0, M_1\}$ independent over M, if N is $\aleph_1$-prime over $M_0 \cup M_1$ and $p \in S(N)$ is nonalgebraic, then p is not orthogonal to some M_i.

More generally, for any infinite cardinal μ, T has μ-NDOP if for all $\alpha < \mu$, all sets $\{M, N\} \cup \{M_i : i < \alpha\}$ of $\aleph_1$-saturated models such that $M \subseteq M_i$ for all $i < \alpha$, $\{M_i : i < \alpha\}$ are independent over M and N is $\aleph_1$-prime over

[5] In this setting a model is $\aleph_1$-saturated if and only if it is $\mathbf{F}^a_{\aleph_1}$-saturated in the terminology of Shelah [11] if and only if it is an 'a-model' in the terminology of Pillay [9].

$\bigcup\{M_i : i < \alpha\}$, every nonalgebraic type over N is not orthogonal to some M_i.

Theorem 1.3 of [4] shows that it is equivalent to require that N be $\aleph_1$-minimal over $\bigcup\{M_i : i < \alpha\}$, so the definition of NDOP given here coincides with the definition given in [11]. In [11] Shelah shows that if T does not have NDOP then T has 2^κ nonisomorphic $\aleph_1$-saturated models of size κ for all $\kappa \geq 2^{\aleph_0}$.

It is easily proved by induction on $\alpha < \omega$ that if T has NDOP then T has ω-NDOP. Furthermore, if T happened to be superstable and had ω-NDOP, then as every type over N is based and stationary over a finite subset of N (hence over an $\aleph_1$-prime model over $\bigcup\{M_i : i \in X\}$ for some finite set $X \subseteq \alpha$) T would have μ-NDOP for all infinite cardinals μ. Arguing similarly, since our T is strictly stable in a countable language, all types over N are based and stationary over a countable subset. Thus, $\aleph_1$-NDOP implies μ-NDOP for all cardinals μ. However, at least at first glance there seems to be a gap between the notions of NDOP and $\aleph_1$-NDOP for such theories. Somewhat surprisingly (as demonstrated by Corollary 3.6(2)) this gap does not exist.

Theorem 3.3 *For stable theories in a countable language, if T has NDOP then T has $\aleph_1$-NDOP (hence μ-NDOP for all μ).*

The full proof of this theorem is given in Section 5 of [4], but it is instructive to see how it follows from the Lusin-Sierpiński theorem mentioned above. Fundamentally the proof of Theorem 3.3 is very much like the argument that there is no $\mathbf{\Sigma^1_1}$-definable nonprincipal ultrafilter on $\mathcal{P}(\omega)$, so we first review that argument. First of all, we endow $\mathcal{P}(\omega)$ with its standard topology by declaring the set $\mathcal{B} = \{U_{A,B} : A, B \text{ are finite subsets of } \omega\}$, where

$$U_{A,B} = \{X \in \mathcal{P}(\omega) : A \subseteq X, B \cap X = \emptyset\}$$

to be a basis of open sets. It is easily checked that $\mathcal{P}(\omega)$ is a Polish space. Note that a subset $Y \subseteq \mathcal{P}(\omega)$ is a nowhere dense if and only if for all pairs of disjoint finite sets (E, F), there is a pair of disjoint finite sets (E', F') with $E \subseteq E'$, $F \subseteq F'$ and $U_{E',F'} \cap Y = \emptyset$.

Now suppose that V is a $\mathbf{\Sigma^1_1}$-definable ultrafilter on $\mathcal{P}(\omega)$. We will show that it is principal. By Fact 3.1 there is an open subset U such that $V \bigtriangleup U$ is meager. That is, **either** V is meager **or** there is a disjoint pair (A, B) of finite subsets of ω such that $U_{A,B} \setminus V$ is meagre. However, if V is meagre, say $V = \bigcup\{Y_n : n \in \omega\}$ where each Y_n is nowhere dense, then by iteratively employing the characterization of nowhere denseness mentioned above, we could find a sequence $\langle (E_n, F_n) : n \in \omega \rangle$ of pairs of disjoint finite subsets of ω such that $E_0 = F_0 = \emptyset$, $E_n \subseteq E_{n+1}$, $F_n \subseteq F_{n+1}$, $U_{E_n,F_n} \cap Y_n = \emptyset$ and $U_{F_n,E_n} \cap Y_n = \emptyset$ for all $n \in \omega$. But then, if we let $X = \bigcup\{E_n : n \in \omega\}$, neither X nor $(\omega \setminus X)$ would be elements of any Y_n. In particular, neither X nor $(\omega \setminus X)$ would be elements of V, contradicting the assumption that V is an ultrafilter.

We conclude that $U_{A,B} \setminus V = \{Y_n : n \in \omega\}$ for some finite, disjoint sets $A, B \subseteq \omega$ and some nowhere dense sets Y_n. For each n let

$$Y_n^* = \{X : (X \cup A \setminus B) \in Y_n\}$$

Since any two nonempty basic open subsets of $\mathcal{P}(\omega)$ are homeomorphic, each Y_n^* is nowhere dense. Now construct a sequence $\langle (E_n, F_n) : n \in \omega \rangle$ of pairs of disjoint finite subsets of ω such that $E_0 = A$, $F_0 = B$, $E_n \subseteq E_{n+1}$, $F_n \subseteq F_{n+1}$, $U_{E_n,F_n} \cap Y_n = \emptyset$ and $U_{F_n,E_n} \cap Y_n^* = \emptyset$ for all $n \in \omega$. Let $X = \bigcup\{E_n : n \in \omega\}$ and $Z = (\omega \setminus X) \cup A \setminus B$. Clearly both X and Z are in $U_{A,B}$. Since $X \in U_{E_n,F_n}$ for each n, $X \notin Y_n$ for every n, hence $X \in V$. Similarly, $(\omega \setminus X) \notin Y_n^*$ for all n, so $Z \in V$. Since V is a filter, $X \cap Z = A \in V$. Since A is finite, V is principal.

Towards the proof of Theorem 3.3, assume that T has NDOP and choose $\aleph_1$-saturated models M, N and $\{M_i : i \in \omega\}$ such that $M \subseteq M_i$ for all $i \in \omega$, $\{M_i : i \in \omega\}$ are independent over M, and N is $\aleph_1$-prime over their union. As well, fix a nonalgebraic type $p \in S(N)$. We will show that p is nonorthogonal to some M_i. To accomplish this let J denote the finite subsets of ω. We inductively build a 'stable system' $\{M_A : A \in J\}$ of submodels of N such that each M_A is $\aleph_1$-prime over $\bigcup\{M_B : B \subsetneq A\}$ and N is $\aleph_1$-prime over $\bigcup\{M_A : A \in J\}$. As notation, for any $X \subseteq \omega$ let $M_X = \bigcup\{M_A : A \in J \cap \mathcal{P}(X)\}$. For any finite $\Delta \subseteq L$, let

> $W_\Delta = \{X \subseteq \omega : p \not\perp N'$ for some $N' \preceq N$ such that N' is $\aleph_1$-prime over M_X, N is $\aleph_1$-prime over $N' \cup M_\omega$, and the nonorthogonality is witnessed by some $\varphi(x, y) \in \Delta\}$.

Much of the argument is devoted to showing that each W_Δ is a $\mathbf{\Sigma_1^1}$ subset of $\mathcal{P}(\omega)$ (see Claim 5.18 of [4]). So the set $W = \bigcup\{W_\Delta : \Delta \text{ finite}\}$ is a $\mathbf{\Sigma_1^1}$-subset of $\mathcal{P}(\omega)$ as well. The set W is not precisely an ultrafilter, but the following lemma shows that it is close to being one:

Lemma 3.4 *1. For all Δ and all $X \subseteq Y \subseteq \omega$, if $X \in W_\Delta$ then $Y \in W_\Delta$;*

2. *For all $X \subseteq \omega$, either $X \in W$ or $(\omega \setminus X) \in W$;*

3. *(Weak intersection) If, for some Δ and $A \subseteq \omega$, there are sets $\{X_j : j \in \omega\} \subseteq W_\Delta$ such that $X_j \cap X_k = A$ for all $j < k < \omega$, then $A \in W_\Delta$.*

Note that Condition (1) is trivial and (2) follows from NDOP. The verification of (3) follows easily from Proposition 2.16 of [4].

The argument now splits into cases as in the ultrafilter argument given above. On one hand, if every W_Δ is meagre then W is meagre. As in the ultrafilter argument we construct a set $X \in \mathcal{P}(\omega)$ such that $X \notin W$ and $(\omega \setminus X) \notin W$, contradicting Condition (2). On the other hand, if some W_Δ is nonmeagre then there is a pair (A, B) of disjoint finite sets such that $U_{A,B} \setminus W_\Delta$ is meagre. Now, by diagonalizing across the nowhere dense sets spanning this difference it is easy to produce an infinite family $\{X_j : j \in \omega\} \subseteq U_{A,B} \cap W_\Delta$ as in Condition (3). But this implies that $A \in W_\Delta$, hence p is nonorthogonal to M_A with A finite. Since NDOP implies ω-NDOP, Theorem 1.3 of [4] implies that p is nonorthogonal to M_i for some $i \in A$ and we finish.

In [4] we obtain the following corollaries to this theorem. The first is reminiscent of Shelah's 'Main Gap' for the class of models of a classifiable theory and the second is rather unexpected.

Definition 3.5 Let T be a stable theory in a countable language.

1. T is *deep* if there is an infinite elementary chain $\langle M_n : n \in \omega \rangle$ of models of T such that M_{n+1}/M_n is orthogonal to M_{n-1} for all $n \geq 1$.

2. T is *shallow* if it is not deep.

3. T has *DIDIP* if there is an elementary chain $\langle M_n : n \in \omega \rangle$ of $\aleph_1$-saturated models of T and a type $p \in S(N)$, where N is $\aleph_1$ prime over $\bigcup_{n \in \omega} M_n$, that is orthogonal to every M_n.

Proofs of the following Corollaries appear in [4].

Corollary 3.6 *Let T be a stable theory in a countable language.*

1. *If T has NDOP and is shallow then every saturated model of size at least $2^{\aleph_0}$ is $\aleph_1$-prime and minimal over an independent tree of $\aleph_1$-saturated models of size $2^{\aleph_0}$.*

2. *If T has NDOP and is shallow then T does not have DIDIP.*

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