FIRST-ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS I: Introduction and Linear Systems

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

Contents

1. Introduction

1.1. Normal Forms and Solutions. For the remainder of the course we will study first-order systems of n ordinary differential equations for functions $x_j(t)$, $j = 1, 2, \cdots, n$ that can be put into the normal form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(t, x_1, x_2, \cdots, x_n)\,, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(t, x_1, x_2, \cdots, x_n)\,, \\ &\vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} &= f_n(t, x_1, x_2, \cdots, x_n)\,. \end{aligned} \tag{1.1}$$

We say that n is the *dimension* of this system.

System (1.1) can be expressed more compactly in *vector notation* as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x})\,, \tag{1.2}$$

where $\mathbf{x}$ and $\mathbf{f}(t, \mathbf{x})$ are given by the n-dimensional *column vectors*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, x_1, x_2, \cdots, x_n) \\ f_2(t, x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \cdots, x_n) \end{pmatrix}.$$

We thereby express the system of n equations (1.1) as the single vector equation (1.2). We will use boldface, lowercase letters to denote column vectors.

You should recall from Calculus III what it means for a vector-valued function $\mathbf{u}(t)$ to be either continuous or differentiable at a point.

- We say $\mathbf{u}(t)$ is *continuous at* time t if *every entry* of $\mathbf{u}(t)$ is continuous at t.
- We say $\mathbf{u}(t)$ is *differentiable at* time t if *every entry* of $\mathbf{u}(t)$ is differentiable at t.

Given these definitions, we define what it means for a vector-valued function $\mathbf{u}(t)$ to be either continuous, differentiable, or continuously differentiable over an interval.

- We say $\mathbf{u}(t)$ is *continuous over* a time interval (t_L, t_R) if it is continuous at every t in (t_L, t_R).
- We say $\mathbf{u}(t)$ is *differentiable over* a time interval (t_L, t_R) if it is differentiable at every t in (t_L, t_R).
- We say $\mathbf{u}(t)$ is *continuously differentiable over* a time interval (t_L, t_R) if it is differentiable over (t_L, t_R) and its derivative is continuous over (t_L, t_R).

Definition. We say that $\mathbf{x}(t)$ is a *solution* of system (1.2) over a time interval (t_L, t_R) when

1. $\mathbf{x}(t)$ is differentiable at every t in (t_L, t_R);
2. $\mathbf{f}(t, \mathbf{x}(t))$ is defined for every t in (t_L, t_R);
3. equation (1.2) holds at every t in (t_L, t_R).

1.2. **Reductions to First-Order Systems.** Many differential equation problems can be recast in terms of a first-order system in the normal form (1.2). For example, every n^{th}-order ordinary differential equation in the normal form

$$y^{(n)} = g\big(t, y, y', \cdots, y^{(n-1)}\big) ,$$

can be expressed as an n-dimensional first-order system in the form (1.2) with

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ g(t, x_1, x_2, \cdots, x_n) \end{pmatrix} , \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} .$$

Notice that the first-order system in expressed solely in terms of the entries of $\mathbf{x}$. The "dictionary" that relates $\mathbf{x}$ to y, y', $\cdots$, $y^{(n-1)}$ is given as a separate equation.

Example. Recast as a first-order system

$$y''' + yy' + e^t y^2 = \cos(3t) .$$

Solution. Because this single equation is third order, the first-order system will have dimension three. It will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \cos(3t) - x_1 x_2 - e^t x_1^2 \end{pmatrix} , \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} .$$

More generally, every d-dimensional m^{th}-order ordinary differential system in the normal form

$$\mathbf{y}^{(m)} = \mathbf{g}\big(t, \mathbf{y}, \mathbf{y}', \cdots, \mathbf{y}^{(n-1)}\big) ,$$

can be expressed as an md-dimensional first-order system in the form (1.2) with

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \\ \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m) \end{pmatrix} , \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}' \\ \vdots \\ \mathbf{y}^{(m-1)} \end{pmatrix} .$$

Here each $\mathbf{x}_k$ is a d-dimensional vector while $\mathbf{x}$ is the md-dimensional vector constructed by stacking the vectors $\mathbf{x}_1$ through $\mathbf{x}_m$ on top of each other.

Example. Recast as a first-order system

$$q_1'' + f_1(q_2 - q_1) = 0 , \qquad q_2'' + f_2(q_2 - q_1) = 0 .$$

Solution. Because this two dimensional system is second order, the first-order system will have dimension four. It will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ -f_1(x_2 - x_1) \\ -f_2(x_2 - x_1) \end{pmatrix} , \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_1' \\ q_2' \end{pmatrix} .$$

1.3. **Numerical Methods.** One advantage of expressing an initial-value problem in the form of a first-order system is that you can then apply all the numerical methods that we studied earlier in the setting of single first-order equations. Suppose you wish to construct a numerical approximation over the time interval $[t_I, t_F]$ to the solution $\mathbf{x}(t)$ of the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}) \,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I \,.$$

A numerical method selects times $\{t_n\}_{n=0}^N$ such that

$$t_I = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = t_F \,,$$

and computes vectors $\{\mathbf{x}_n\}_{n=0}^N$ such that $\mathbf{x}_0 = \mathbf{x}^I$ and $\mathbf{x}_n$ approximates $\mathbf{x}(t_n)$ for $n = 1, 2, \cdots, N$. If we do this by using N uniform time steps (as we did earlier) then we set

$$h = \frac{t_F - t_I}{N} \,, \qquad \text{and} \qquad t_n = t_I + nh \quad \text{for } n = 0, 1, \cdots, N \,,$$

where h is called the *time step.* Below we show that the vectors $\{\mathbf{x}_n\}_{n=0}^N$ can be computed easily by the four explicit methods that we studied earlier — namely, the forward Euler, Heun-Tradezoidal, Huen-Midpoint, and Runge-Kutta methods. Implicit methods such as the backward Euler method are often extremely useful for computing approximate solutions for first-order systems, but are more complicated to implement because they require the numerical solution of algebraic systems, which is beyond the scope of this course.

Forward Euler Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\mathbf{f}_n = \mathbf{f}(t_n, \mathbf{x}_n) \,, \qquad \mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n \,,$$

where $t_n = t_I + nh$.

Heun-Trapezoidal Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n) \,, & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_n \,, \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1}) \,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{2}h[\mathbf{f}_n + \tilde{\mathbf{f}}_{n+1}] \,, \end{aligned}$$

where $t_n = t_I + nh$.

Heun-Midpoint Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n) \,, & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2}h\mathbf{f}_n \,, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}}) \,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_{n+\frac{1}{2}} \,, \end{aligned}$$

where $t_n = t_I + nh$ and $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$.

Runge-Kutta Method. Set $\mathbf{x}_0 = \mathbf{x}^I$ and then for $n = 0, \cdots, N-1$ cycle through

$$
\begin{aligned}
\mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n)\,, & \tilde{\mathbf{x}}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2} h \mathbf{f}_n\,,\\
\tilde{\mathbf{f}}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \tilde{\mathbf{x}}_{n+\frac{1}{2}})\,, & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \tfrac{1}{2} h \tilde{\mathbf{f}}_{n+\frac{1}{2}}\,,\\
\mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}})\,, & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h \mathbf{f}_{n+\frac{1}{2}}\,,\\
\tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1})\,, & \mathbf{x}_{n+1} &= \mathbf{x}_n + \tfrac{1}{6} h \Big[\mathbf{f}_n + 2\tilde{\mathbf{f}}_{n+\frac{1}{2}} + 2\mathbf{f}_{n+\frac{1}{2}} + \tilde{\mathbf{f}}_{n+1}\Big]\,,
\end{aligned}
$$

where $t_n = t_I + nh$ and $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$.

2. Linear Systems: General Methods and Theory

The n-dimensional first-order system (1.1) is called *linear* when it has the form

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\,, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\,, \\
&\vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)\,.
\end{aligned} \tag{2.1}
$$

The functions $a_{jk}(t)$ are called *coefficients* while the functions $f_j(t)$ are called *forcings*. Linear system (2.1) can be written compactly using *matrix notation* as

$$
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \tag{2.2}
$$

where $\mathbf{x}$ and $\mathbf{f}(t)$ are the n-dimensional column vectors

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},
$$

while $\mathbf{A}(t)$ is the $n\times n$ matrix

$$
\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.
$$

System (2.2) is said to be *homogeneous* if $\mathbf{f}(t) = 0$ and *nonhomogeneous* otherwise. The product $\mathbf{A}(t)\mathbf{x}(t)$ appearing in system (2.2) denotes column vector that results from the *matrix multiplication* of the matrix $\mathbf{A}(t)$ with the column vector $\mathbf{x}$. The sum appearing in (2.2) denotes column vector that results from the *matrix addition* of the column vector $\mathbf{A}(t)\mathbf{x}$ with the column vector $\mathbf{f}(t)$. These matrix operations are presented in Appendix B.

2.1. **Initial-Value Problems.** We will consider linear initial-value problems in the form

$$
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,, \tag{2.3}
$$

where $\mathbf{A}(t)$ is called the *coefficient matrix*, $\mathbf{f}(t)$ is called the *forcing vector*, and $\mathbf{x}^I$ is called the *vector of initial values*, or simple *initial vector*.

A major theme of this section is that for every fact that we studied about higher-order linear equations there is an analogous fact about linear first-order systems. For example, the basic existence and uniqueness theorem is the following.

Theorem 2.1. If $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous over the time interval (t_L, t_R) then for every initial time t_I in (t_L, t_R) and every initial vector $\mathbf{x}^I$ the initial-value problem (2.3) has a unique solution $\mathbf{x}(t)$ that is continuously differentiable over (t_L, t_R). Moreover, if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are k-times continuously differentiable over the time interval (t_L, t_R) then $\mathbf{x}(t)$ will be is $(k+1)$-times continuously differentiable over (t_L, t_R).

2.2. **Homogeneous Systems.** Just as with higher-order linear equations, the key to solving a first-order linear system (2.2) is understanding how to solve its associated homogeneous system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,. \tag{2.4}$$

We will assume throughout this section that the coefficient matrix $\mathbf{A}(t)$ is continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied. We will exploit the following property of homogeneous systems.

Theorem 2.2. (Linear Superposition). If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of system (2.4) then so is $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ for any values of the constants c_1 and c_2. More generally, if $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_m(t)$ are m solutions of system (2.4) then so is the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\,, \tag{2.5}$$

for any values of the constants c_1, c_2, $\cdots$, c_m.

Reason. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ $\cdots$, $\mathbf{x}_m(t)$ solve (2.4), a direct calculation starting from the linear combination (2.5) shows that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\big) \\
&= c_1\frac{\mathrm{d}\mathbf{x}_1}{\mathrm{d}t}(t) + c_2\frac{\mathrm{d}\mathbf{x}_2}{\mathrm{d}t}(t) + \cdots + c_m\frac{\mathrm{d}\mathbf{x}_m}{\mathrm{d}t}(t) \\
&= c_1\mathbf{A}(t)\mathbf{x}_1(t) + c_2\mathbf{A}(t)\mathbf{x}_2(t) + \cdots + c_m\mathbf{A}(t)\mathbf{x}_m(t) \\
&= \mathbf{A}(t)\big(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\big) \\
&= \mathbf{A}(t)\mathbf{x}(t)\,.
\end{aligned}$$

Therefore $\mathbf{x}(t)$ given by the linear combination (2.5) solves system (2.4). □

Remark. This theorem states that any linear combination of solutions of (2.4) is also a solution of (2.4). It thereby provides a way to construct a whole family of solutions from a finite number of them.

Now consider the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.6}$$

Suppose you know n "different" solutions of (2.4), $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$. It is natural to ask if you can construct the solution of the initial-value problem (2.6) as a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$. Set

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)\,. \tag{2.7}$$

By the superposition theorem this is a solution of (2.4). You only have to check that values of c_1, c_2, $\cdots$, c_n can be found so that $\mathbf{x}(t)$ will also satisfy the initial conditions in (2.6) — namely, so that

$$\mathbf{x}^I = \mathbf{x}(t_I) = c_1\mathbf{x}_1(t_I) + c_2\mathbf{x}_2(t_I) + \cdots + c_n\mathbf{x}_n(t_I) = \mathbf{\Psi}(t_I)\mathbf{c}\,,$$

where $\mathbf{c}$ is the $n\times 1$ column vector given by

$$\mathbf{c} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}^T\,,$$

while $\mathbf{\Psi}(t_I)$ is the $n\times n$ matrix given by

$$\mathbf{\Psi}(t_I) = \begin{pmatrix} \mathbf{x}_1(t_I) & \mathbf{x}_2(t_I) & \cdots & \mathbf{x}_n(t_I) \end{pmatrix}\,.$$

This notation indicates that the k^{th} column of $\mathbf{\Psi}(t_I)$ is the column vector $\mathbf{x}_k(t_I)$. In this notation the question becomes whether there is a vector $\mathbf{c}$ such that

$$\mathbf{\Psi}(t_I)\mathbf{c} = \mathbf{x}^I\,, \quad \text{for every } \mathbf{x}^I\,.$$

This linear algebraic system will have a solution for every $\mathbf{x}^I$ if and only if the matrix $\mathbf{\Psi}(t_I)$ is invertible, in which case the solution is unique and is given by

$$\mathbf{c} = \mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,.$$

Of course, the matrix $\mathbf{\Psi}(t_I)$ is invertible if and only if $\det\big(\mathbf{\Psi}(t_I)\big) \neq 0$.

2.3. **Wronskians and Fundamental Matrices.** Given any set of n solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ to the homogeneous equation (2.4), we define its *Wronskian* by

$$W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = \det\begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}\,. \tag{2.8}$$

The *Abel Theorem* for first-order systems is

$$\frac{\mathrm{d}}{\mathrm{d}t}W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = \operatorname{tr}\big(\mathbf{A}(t)\big)\, W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t)\,, \tag{2.9}$$

where $\operatorname{tr}\big(\mathbf{A}(t)\big)$ denotes the *trace* of $\mathbf{A}(t)$, which is defined by

$$\operatorname{tr}\big(\mathbf{A}(t)\big) = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)\,. \tag{2.10}$$

Upon integrating the first-order linear equation (2.9) we see that

$$W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t_I)\, \exp\bigg(\int_{t_I}^{t} \operatorname{tr}\big(\mathbf{A}(s)\big)\,\mathrm{d}s\bigg)\,. \tag{2.11}$$

As was the case for higher-order equations, this shows that if the Wronkian is nonzero somewhere then it is nonzero everywhere, and that if it is zero somewhere, it is zero everywhere!

Again analogous to the case for higher-order equations, we have the following definitions.

Definition. A set of n solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ to the n-dimensional homogeneous linear system (2.4) called *fundamental* if its Wronskian is nonzero. Then the family

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{2.12}$$

is called a *general solution* of system (2.4).

However, now we introduce a new concept for first-order systems.

Definition. If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ is a fundamental set of solutions to system (2.4) then the $n\times n$ matrix-valued function

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t)\end{pmatrix} \tag{2.13}$$

is called a *fundamental matrix* for system (2.4).

Some basic fact about fundamental matrices are as follows.

Fact. Let $\mathbf{\Psi}(t)$ be a fundamental matrix for system (2.4). Then

- $\mathbf{\Psi}(t)$ satisfies

$$\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}\,, \qquad \det\big(\mathbf{\Psi}(t)\big) \neq 0\,; \tag{2.14}$$

- A general solution of system (2.4) is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}\,;$$

Reason. By (2.13) you see that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t)\end{pmatrix} \\
&= \begin{pmatrix}\frac{\mathrm{d}\mathbf{x}_1}{\mathrm{d}t}(t) & \frac{\mathrm{d}\mathbf{x}_2}{\mathrm{d}t}(t) & \cdots & \frac{\mathrm{d}\mathbf{x}_n}{\mathrm{d}t}(t)\end{pmatrix} \\
&= \begin{pmatrix}\mathbf{A}(t)\mathbf{x}_1(t) & \mathbf{A}(t)\mathbf{x}_2(t) & \cdots & \mathbf{A}(t)\mathbf{x}_n(t)\end{pmatrix} \\
&= \mathbf{A}(t)\begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t)\end{pmatrix} \\
&= \mathbf{A}(t)\mathbf{\Psi}(t)\,.
\end{aligned}$$

Also by (2.13) you see that

$$\begin{aligned}
\det\big(\mathbf{\Psi}(t)\big) &= \det\begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t)\end{pmatrix} \\
&= W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) \neq 0\,.
\end{aligned}$$

It should be clear from (2.13) that the general solution given by (2.12) can be expressed as $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}$.

Example. The vector-valued functions

$$\mathbf{x}_1(t) = e^{5t}\begin{pmatrix}1\\1\end{pmatrix}\,, \qquad \mathbf{x}_2(t) = e^{t}\begin{pmatrix}1\\-1\end{pmatrix}\,,$$

are solutions of the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\,.$$

Construct a general solution and a fundamental matrix for this system.

Solution. It is easy to check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are each solutions to the differential system. Because

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} = -2e^{6t} \,,$$

we see that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \,,$$

while a general solution is given by

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c} = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{pmatrix} \,.$$

Alternatively, you can construct a general solution as

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{pmatrix} \,.$$

Example. The vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix} \,, \qquad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix} \,,$$

are solutions of the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,.$$

Construct a general solution and a fundamental matrix for this system.

Solution. It is easy to check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are each solutions to the differential system. Because

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} = 1 \,,$$

we see that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \,,$$

while a general solution is given by

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t + c_2 \end{pmatrix} \,.$$

Alternatively, you can construct a general solution as

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t + c_2 \end{pmatrix} \,.$$

Remark. The solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ were given to you in the problems above. Sections 3 and 4 will present methods by which you can construct a fundamental set of solutions (and therefore a fundamental matrix) for any homogeneous system with a constant coefficient matrix. For systems with a variable coefficient matrix you will always be given solutions.

Remark. Any matrix-valued function $\mathbf{\Psi}(t)$ such that $\det\big(\mathbf{\Psi}(t)\big) \neq 0$ over some time interval (t_L, t_R) is a fundamental matrix for the first-order differential system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \text{where} \quad \mathbf{A}(t) = \frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t)\mathbf{\Psi}(t)^{-1}\,.$$

This can be seen by multiplying (2.14) on the left by $\mathbf{\Psi}(t)^{-1}$.

Example. Find a first-order differential system such that the vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1+t^3 \\ t \end{pmatrix}, \qquad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix},$$

comprise a fundamental set of solutions.

Solution. Set

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

Because $\det\big(\mathbf{\Psi}(t)\big) = 1$, you see that $\mathbf{\Psi}(t)$ is invertible. Set

$$\begin{aligned}\mathbf{A}(t) = \frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1+t^3 & t \\ t^2 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -t^2 & 1+t^3 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix}.\end{aligned}$$

Therefore $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are a fundamental set of solutions for the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2.4. **Natural Fundamental Matrices.** Consider the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.15}$$

Let $\mathbf{\Psi}(t)$ be any fundamental matrix for this system. Then a general solution of the system is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}\,.$$

By imposing the initial condition from (2.15) we see that

$$\mathbf{x}^I = \mathbf{x}(t_I) = \mathbf{\Psi}(t_I)\mathbf{c}\,.$$

Because $\det\big(\mathbf{\Psi}(t_I)\big) \neq 0$, the matrix $\mathbf{\Psi}(t_I)$ is invertible and we can solve for $\mathbf{c}$ as

$$\mathbf{c} = \mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,.$$

Hence, the solution of the initial-value problem is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,. \tag{2.16}$$

Now let $\mathbf{\Phi}(t)$ be the matrix-valued function defined by

$$\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\,. \tag{2.17}$$

If we differentiate $\mathbf{\Phi}(t)$ and use the fact that $\mathbf{\Psi}(t)$ is a fundamental matrix for system (2.15) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\right) = \frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t_I)^{-1} = \mathbf{A}(t)\mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1} = \mathbf{A}(t)\mathbf{\Phi}(t)\,.$$

Moreover, from (2.17) we see that

$$\mathbf{\Phi}(t_I) = \mathbf{\Psi}(t_I)\mathbf{\Psi}(t_I)^{-1} = \mathbf{I}\,.$$

Therefore, $\mathbf{\Phi}(t)$ is the solution of the matrix-valued initial-value problem

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Phi}\,, \qquad \mathbf{\Phi}(t_I) = \mathbf{I}\,. \tag{2.18}$$

This shows three things.

1. $\mathbf{\Phi}(t)$ as a function of t is a fundamental matrix for system (2.15);
2. $\mathbf{\Phi}(t)$ is uniquely determined by the matrix-valued initial-value problem (2.18);
3. $\mathbf{\Phi}(t)$ is independent of our original choice of fundamental matrix $\mathbf{\Psi}(t)$ that was used to construct it in (2.17).

We call $\mathbf{\Phi}(t)$ the natural fundamental matrix associated with the initial time t_I.

Just like it was easy to express the solution of an initial-value problem for a higher-order equation in terms of its associated natural fundamental sets of solutions, you express the solution of the initial-value problem (2.15) in terms of its associated natural fundamental matrix as simply

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I\,. \tag{2.19}$$

Given any fundamental matrix $\mathbf{\Psi}(t)$, you construct the natural fundamental matrix associated with the initial time t_I by formula (2.17).

Example. Constuct the natural fundamental matrix associated with the initial time 0 for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions $x_1(0) = 4$ and $x_2(0) = -2$.

Solution. We have already seen that a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 0 is

$$\begin{aligned}\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}\frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^{t} & e^{5t}-e^{t} \\ e^{5t}-e^{t} & e^{5t}+e^{t} \end{pmatrix}.\end{aligned}$$

The solution of the initial-value problem is therefore

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I = \frac{1}{2}\begin{pmatrix} e^{5t}+e^{t} & e^{5t}-e^{t} \\ e^{5t}-e^{t} & e^{5t}+e^{t} \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{5t}+3e^{t} \\ e^{5t}-3e^{t} \end{pmatrix}.$$

Example. Constuct the natural fundamental matrix associated with the initial time 1 for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions $x_1(1) = 3$ and $x_2(1) = 0$.

Solution. We have already seen that a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 1 is

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1} &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 + t^3 - t^2 \\ t - 1 \end{pmatrix}.$$

2.5. **Nonhomogeneous Systems and Green Matrices.** We now consider the nonhomogeneous linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t). \tag{2.20}$$

If $\mathbf{x}_P(t)$ is a particular solution of this system and $\mathbf{\Psi}(t)$ is a fundamental matrix of the associated homogeneous system then a general solution of system (2.20) is

$$\mathbf{x}(t) = \mathbf{x}_H(t) + \mathbf{x}_P(t),$$

where $\mathbf{x}_H(t)$ is the general solution of the associated homogeneous problem given by

$$\mathbf{x}_H(t) = \mathbf{\Psi}(t)\mathbf{c}. \tag{2.21}$$

Recall that if you know a fundamental set of solutions to the associated homogeneous equation then you can use the methods of either Variations of Parameters or general Green Functions to construct a particular solution to a nonhomogeneous n^{th}-order linear equation in terms of n intergrals. Here we show that a similar thing is true for the first-order system (2.20). Specifically, if you know a fundamental matrix $\mathbf{\Psi}(t)$ for the associated homogeneous system then you can construct a particular solution to the n dimensional nonhomogeneous linear equation in terms of n intergrals.

We begin with the analog of the method of Variation of Parameters. Because $\mathbf{x}_H(t)$ has the form (2.21), we *seek* a particular solution in the form

$$\mathbf{x}_P(t) = \mathbf{\Psi}(t)\mathbf{u}(t). \tag{2.22}$$

By differentiation you see that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_P(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Psi}(t)\mathbf{u}(t)\big) = \frac{\mathrm{d}\boldsymbol{\Psi}(t)}{\mathrm{d}t}\mathbf{u}(t) + \boldsymbol{\Psi}(t)\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} \\ &= \mathbf{A}(t)\boldsymbol{\Psi}(t)\mathbf{u}(t) + \boldsymbol{\Psi}(t)\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} \\ &= \mathbf{A}(t)\mathbf{x}_P(t) + \boldsymbol{\Psi}(t)\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t}\,.\end{aligned}$$

By comparing the right-hand side of this equation with the right-hand side of equation (2.20), you see that $\mathbf{x}_P(t)$ will solve (2.20) if

$$\boldsymbol{\Psi}(t)\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = \mathbf{f}(t)\,.$$

Because $\boldsymbol{\Psi}(t)$ is invertible, you find that

$$\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = \boldsymbol{\Psi}(t)^{-1}\mathbf{f}(t)\,. \tag{2.23}$$

If $\mathbf{u}_P(t)$ is a primitive of the right-hand side above then a general solution of this system has the form

$$\mathbf{u}(t) = \mathbf{c} + \mathbf{u}_P(t)\,.$$

When this solution is placed into the form (2.22), we find that a particular solution is given by

$$\mathbf{x}_P(t) = \boldsymbol{\Psi}(t)\mathbf{u}_P(t)\,. \tag{2.24}$$

If we take the particular solution of (2.23) given by

$$\mathbf{u}_P(t) = \int_{t_I}^{t} \boldsymbol{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,,$$

then (2.24) becomes

$$\mathbf{x}_P(t) = \boldsymbol{\Psi}(t)\int_{t_I}^{t} \boldsymbol{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.25}$$

The solution of the initial-value problem is then

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_I)^{-1}\mathbf{x}^I + \boldsymbol{\Psi}(t)\int_{t_I}^{t} \boldsymbol{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.26}$$

We define the Green matrix $\mathbf{G}(t,s)$ by

$$\mathbf{G}(t,s) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\,. \tag{2.27}$$

Then

$$\mathbf{x}(t) = \mathbf{G}(t,t_I)\mathbf{x}^I + \int_{t_I}^{t} \mathbf{G}(t,s)\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.28}$$

B. Appendix: Vectors and Matrices

An $m\times n$ matrix $\mathbf{A}$ consists of a retangular array of entries arranged in m rows and n columns

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \big(a_{jk}\big) . \tag{2.29}$$

We call a_{jk} the jk-entry of $\mathbf{A}$, m the row-dimension of $\mathbf{A}$, n the column-dimension of $\mathbf{A}$, and $m\times n$ the dimensions of $\mathbf{A}$. The entries of a matrix can be drawn from any set, but in this course they will be numbers. Special kinds of matrices include:

- $1\times m$ matrices are called *row vectors*;
- $n\times 1$ matrices are called *column vectors*;
- $n\times n$ matrices are called *square matrices*.

Two $m\times n$ matrices $\mathbf{A} = (a_{jk})$ and $\mathbf{B} = (b_{jk})$ are said to be *equal* if $a_{jk} = b_{jk}$ for every $j = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, n$, in which case we write $\mathbf{A} = \mathbf{B}$. We will use $\mathbf{0}$ to denote any matrix that has every entry equal to zero.

B.1. Vector and Matrix Operations.

Matrix Addition. Two $m\times n$ matrices $\mathbf{A} = (a_{jk})$ and $\mathbf{B} = (b_{jk})$ can be added to create a new $m\times n$ matrix sum $\mathbf{A}+\mathbf{B}$, called the *sum* of $\mathbf{A}$ and $\mathbf{B}$, defined by

$$\mathbf{A}+\mathbf{B} = \big(a_{jk} + b_{jk}\big) .$$

Matrix addition has the following properties.

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} && \text{—commutativity} , \\ (\mathbf{A}+\mathbf{B})+\mathbf{C} &= \mathbf{A}+(\mathbf{B}+\mathbf{C}) && \text{—associativity} , \\ \mathbf{A}+\mathbf{0} = \mathbf{0}+\mathbf{A} &= \mathbf{A} && \text{—additive identity} , \\ \mathbf{A}+(-\mathbf{A}) = (-\mathbf{A})+\mathbf{A} &= \mathbf{0} && \text{—additive inverse} . \end{aligned}$$

Here all matrices have the same dimensions and the matrix $-\mathbf{A}$ is defined by $-\mathbf{A} = \big(-a_{jk}\big)$ when $\mathbf{A} = \big(a_{jk}\big)$. We define $\mathbf{A}-\mathbf{B}$ to mean $\mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B})$.

Scalar Multiplication. A number α and an $m\times n$ matrix $\mathbf{A} = (a_{jk})$ can be multiplied to create a new $m\times n$ matrix $\alpha\mathbf{A}$, called the *multiple* of $\mathbf{A}$ by α, defined by

$$\alpha\mathbf{A} = \big(\alpha a_{jk}\big) .$$

Scalar Multiplication of Matrices has the following properties.

$$\begin{aligned} \alpha(\beta\mathbf{A}) &= (\alpha\beta)\mathbf{A} && \text{—associativity} , \\ \alpha(\mathbf{A}+\mathbf{B}) &= \alpha\mathbf{A}+\alpha\mathbf{B} && \text{—distributivity over matrix addition} , \\ (\alpha+\beta)\mathbf{A} &= \alpha\mathbf{A}+\beta\mathbf{A} && \text{—distributivity over scalar addition} , \\ 1\mathbf{A} = \mathbf{A} , \quad -1\mathbf{A} &= -\mathbf{A} && \text{—scalar identity} , \\ 0\mathbf{A} = \mathbf{0} , \quad \alpha\mathbf{0} &= \mathbf{0} && \text{—scalar multiplicative nullity} . \end{aligned}$$

Here all matrices have the same dimensions.

Matrix Multiplication. An $l \times m$ matrix $\mathbf{A}$ and an $m \times n$ matrix $\mathbf{B}$ can be multiplied to create a new $l \times n$ matrix $\mathbf{AB}$, called the *product* of $\mathbf{A}$ and $\mathbf{B}$, defined by

$$\mathbf{AB} = \big(c_{ik}\big)\,, \qquad \text{where} \qquad c_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}\,.$$

Remark. Notice that for some matrices $\mathbf{A}$ and $\mathbf{B}$, depending only on their dimensions, neither $\mathbf{AB}$ nor $\mathbf{BA}$ exist; for others excatly one of $\mathbf{AB}$ or $\mathbf{BA}$ exists; while for others both $\mathbf{AB}$ and $\mathbf{BA}$ exist.

Example. For the matrices

$$\mathbf{A} = \begin{pmatrix} 2-i5 & -1 \\ -4 & 6+i \end{pmatrix}\,, \qquad \mathbf{B} = \begin{pmatrix} 3 & 2 & 2+i3 \\ 5+i6 & i3 & 2-i \end{pmatrix}\,,$$

the product $\mathbf{AB}$ exists with

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} (2-i5)\cdot 3 + (-1)\cdot(5+i6) & (2-i5)\cdot 2 + (-1)\cdot(i3) & (2-i5)\cdot(2+i3) + (-1)\cdot(2-i) \\ (-4)\cdot 3 + (6+i)\cdot(5+i6) & (-4)\cdot 2 + (6+i)\cdot(i3) & (-4)\cdot(2+i3) + (6+i)\cdot(2-i) \end{pmatrix} \\ &= \begin{pmatrix} 12-i20 & 7-i10 & 18-i6 \\ 12+i41 & -11+i18 & 5-i16 \end{pmatrix}\,, \end{aligned}$$

while the product $\mathbf{BA}$ does not exist.

Remark. Notice that if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ then $\mathbf{AB}$ and $\mathbf{BA}$ both exist and are $n \times n$, but in general

$$\mathbf{AB} \neq \mathbf{BA}\,!$$

Example. For the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\,, \qquad \mathbf{B} = \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix}\,,$$

you see that

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\,, \\ \mathbf{BA} &= \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 45 \\ -5 & -15 \end{pmatrix}\,, \end{aligned}$$

whereby $\mathbf{AB} \neq \mathbf{BA}$.

Remark. The above example also shows that

$$\mathbf{AB} = \mathbf{0} \quad \text{does not imply that either } \mathbf{A} = \mathbf{0} \text{ or } \mathbf{B} = \mathbf{0}\,!$$

Matrix Multiplication does have the following properties.

$\mathbf{A}$	$\mathbf{B}$	$\mathbf{C}$		
$l \times m$	$m \times n$	$n \times p$	$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	—associativity,
$l \times m$	$m \times n$	$m \times n$	$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	—left distributivity,
$l \times m$	$l \times m$	$m \times n$	$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$	—right distributivity.

Here $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any matrices with dimensions related as indicated on the left.

An *identity matrix* is an $n\times n$ matrix $\mathbf{I}$ in the form

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} .$$

We will use $\mathbf{I}$ to denote any identity matrix; the dimensions of $\mathbf{I}$ will always be clear from the context. Identity matrices have the property for any $m\times n$ matrix $\mathbf{A}$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A} \qquad \text{—multiplicative identity} .$$

Notice that the first $\mathbf{I}$ above is $m\times m$ while the second is $n\times n$.

Transpose, Conjugate, and Hermitian Transpose. The transpose of the $m\times n$ matrix $\mathbf{A}$ given by (2.29) is the $n\times m$ matrix A^T given by

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} .$$

Matrix transpose has the following properties.

$$\begin{array}{ccrl} \mathbf{A} & \mathbf{B} & & \\ m\times n & m\times n & (\mathbf{A}+\mathbf{B})^T = & \mathbf{A}^T + \mathbf{B}^T , \\ m\times n & & (\alpha\mathbf{A})^T = & \alpha\mathbf{A}^T , \\ l\times m & m\times n & (\mathbf{AB})^T = & \mathbf{B}^T\mathbf{A}^T \quad \text{(note flip)} , \\ m\times n & & \left(\mathbf{A}^T\right)^T = & \mathbf{A} . \end{array}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are any matrices with dimensions related as indicated on the left.

The conjugate of the $m\times n$ matrix $\mathbf{A}$ given by (2.29) is the $m\times n$ matrix $\overline{A}$ given by

$$\overline{A} = \left(\overline{a}_{jk}\right) .$$

Matrix conjugate has the following properties.

$$\begin{array}{ccrl} \mathbf{A} & \mathbf{B} & & \\ m\times n & m\times n & \overline{(\mathbf{A}+\mathbf{B})} = & \overline{\mathbf{A}} + \overline{\mathbf{B}} , \\ m\times n & & \overline{(\alpha\mathbf{A})} = & \overline{\alpha}\overline{\mathbf{A}} , \\ l\times m & m\times n & \overline{\mathbf{AB}} = & \overline{\mathbf{A}}\overline{\mathbf{B}} \quad \text{(note no flip)} , \\ m\times n & & \overline{(\overline{\mathbf{A}})} = & \mathbf{A} . \end{array}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are any matrices with dimensions related as indicated on the left.

The Hermitian transpose of the $m\times n$ matrix $\mathbf{A}$ given by (2.29) is the $n\times m$ matrix $\mathbf{A}^* = \overline{\mathbf{A}}^T = \overline{A^T}$. Hermitian transpose has the following properties.

$\mathbf{A}$	$\mathbf{B}$	
$m\times n$	$m\times n$	$(\mathbf{A}+\mathbf{B})^* = \mathbf{A}^*+\mathbf{B}^*\,,$
$m\times n$		$(\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*\,,$
$l\times m$	$m\times n$	$(\mathbf{A}\mathbf{B})^* = \mathbf{B}^*\mathbf{A}^*$ (note flip),
$m\times n$		$\big(\mathbf{A}^*\big)^* = \mathbf{A}\,.$

Here $\mathbf{A}$ and $\mathbf{B}$ are any matrices with dimensions related as indicated on the left.

B.2. Invertibility and Inverses. An $n\times n$ matrix $\mathbf{A}$ is said to be *invertible* if there exists another $n\times n$ matrix $\mathbf{B}$ such that

$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{I}\,,$$

in which case $\mathbf{B}$ is said to be an *inverse* of $\mathbf{A}$.

Fact. A matrix can have at most one inverse.

Reason. Suppose that $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$. Then

$$\mathbf{B} = \mathbf{B}\mathbf{I} = \mathbf{B}(\mathbf{A}\mathbf{C}) = (\mathbf{B}\mathbf{A})\mathbf{C} = \mathbf{I}\mathbf{C} = \mathbf{C}\,.$$

If $\mathbf{A}$ is invertible then its unique inverse is denoted $\mathbf{A}^{-1}$.

Fact. If $\mathbf{A}$ is invertible and $\mathbf{A}\mathbf{B} = \mathbf{0}$ then $\mathbf{B} = \mathbf{0}$.

Reason.

$$\mathbf{B} = \mathbf{I}\mathbf{B} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}(\mathbf{A}\mathbf{B}) = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}\,.$$

Fact. Not all nonzero square matrices are invertible.

Reason. Earlier we gave an example of two nonzero matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}\mathbf{B} = \mathbf{0}$. The previous fact then implies that $\mathbf{A}$ is not invertible.

Fact. A matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

Example. For 2×2 matrices the inverse is easy to compute when it exists. If

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,,$$

then $\det(\mathbf{A}) = ad - bc$. If $\det(\mathbf{A}) = ad - bc \neq 0$ then $\mathbf{A}$ is invertible with

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\,.$$

This result follows from the calculation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ab-cd & -ab+ba \\ -cd+dc & -cb+da \end{pmatrix} = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(\mathbf{A})\mathbf{I}\,.$$

The following is a very important fact about determinants.

Fact. If $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ matrices then

$$\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})\,.$$

It follows that if $\mathbf{A}$ and $\mathbf{B}$ are invertible, whereby $\det(\mathbf{A}) \neq 0$ and $\det(\mathbf{B}) \neq 0$, then so is $\mathbf{AB}$ because $\det(\mathbf{AB}) \neq 0$.

Fact. If $\mathbf{A}$ and $\mathbf{B}$ are invertible and $\alpha \neq 0$ then

- $\alpha\mathbf{A}$ is invertible with $(\alpha\mathbf{A})^{-1} = \dfrac{1}{\alpha}\,\mathbf{A}^{-1}$;
- $\mathbf{AB}$ is invertible with $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (notice the flip);
- $\mathbf{A}^{-1}$ is invertible with $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$;
- $\mathbf{A}^T$ is invertible with $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$;
- $\overline{\mathbf{A}}$ is invertible with $\left(\overline{\mathbf{A}}\right)^{-1} = \overline{\mathbf{A}^{-1}}$;
- $\mathbf{A}^*$ is invertible with $\left(\mathbf{A}^*\right)^{-1} = \left(\mathbf{A}^{-1}\right)^*$;