AMSC 466. Spring 2020

Homework Assignment 12. Due Tuesday May 12.

1. **(5 pts)** Let $w(x)$ be a weight function and $\{p_n\}$ be a family of monic (i.e., their leading coefficients are 1) polynomials orthogonal w.r.t. the corresponding inner product. We have shown that p_n's satisfy the Three-Term Recurrence Relationship (TTRR) of the form

$$xp_0 = p_1 + B_0 p_0,$$

$$xp_k = p_{k+1} + B_k p_k + A_k p_{k-1}, \quad k = 1, 2, \ldots, \quad \text{where}$$
$$A_k = \frac{\|p_k\|^2}{\|p_{k-1}\|^2}, \ k \geq 1, \quad B_k = \frac{\langle xp_k, p_k \rangle}{\|p_k\|^2}, \ k \geq 0.$$

However, we often need the TTRR coefficients for the corresponding orthonormal set of polynomials $\{\tilde{p}_n\}$. Show that $\{\tilde{p}_n\}$ also satisfy a TTRR of the form

$$x\tilde{p}_0 = \alpha_1 \tilde{p}_1 + \beta_0 \tilde{p}_0,$$

$$x\tilde{p}_k = \alpha_{k+1}\tilde{p}_{k+1} + \beta_k \tilde{p}_k + \alpha_k \tilde{p}_{k-1}, \quad k = 1, 2, \ldots,$$

where
$$\alpha_k = \sqrt{A_k} = \|p_k\|/\|p_{k-1}\| \quad \text{and} \quad \beta_k = B_k.$$

Hint: Observe that $p_k = \lambda_k \tilde{p}_k$, where $\lambda_k = \|p_k\|$.

2. **(5 pts)** Design a quadrature rule (i.e., find nodes and weights) for evaluating integrals of the form
$$I(f) = \int_{-1}^{1} f(x)dx$$
with minimal number of nodes that is exact for all polynomials of degree 5 or less. This will be a so-called *Gauss-Legendre* quadrature rule.