AMSC 466. Spring 2020

Homework Assignment 7. Due Thursday April 9.

1. Let A be a real $n \times n$ matrix. A matrix H is called a principal submatrix of A if H is formed by taking an intersection of columns $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ of A and rows $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$ (with the same indices) of A, where $1 \le k \le n$ and the indices $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ are all distinct.

   (a) **(2 pts)** Let P be a permutation matrix, i.e., its rows are $e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}$ where σ is a permutation of $(1, 2, \ldots, n)$, and $e_{\sigma(j)}$'s are the unit vectors with entry 1 at position $\sigma(j)$ and zeros everywhere else. Prove that $P^{-1} = P^\top$, i.e, P is an orthogonal matrix.

   (b) **(2 pts)** Let A be $n \times n$ with rows $r_1, \ldots, r_n$ and columns $c_1, c_2, \ldots, c_n$ i.e.,

   $$A = \begin{bmatrix} r_1 & \rightarrow \\ r_2 & \rightarrow \\ \vdots & \\ r_n & \rightarrow \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

   Calculate PA and write the answer in terms of rows of the matrix A. Calculate $AP^\top$ and write the answer in terms of columns of A.

   (c) **(2 pts)** Let A be symmetric positive definite (SPD) matrix (i.e., $A = A^\top$, and for any $x \in \mathbb{R}^n$, $x \neq 0$, $x^\top A x > 0$). Prove that any its principal submatrix H formed by the intersection of the first k rows and the first k columns of A is SPD. (In the Matlab language, `H = A(1:k,1:k)`.)

   (d) **(2 pts)** Let H be a principal submatrix of A formed by the intersections of rows and columns of A with indices $i_1, \ldots, i_k$. Find a permutation matrix P such that $PAP^\top$ has H at its top left corner.

   (e) **(2 pts)** Prove that A is symmetric positive definite $\Longleftrightarrow$ any its principal submatrix is symmetric positive definite. *Hint: The* $\Longleftarrow$ *part is straightforward. Use the fact that* A *is SPD if and only if* $XAX^\top$ *is SPD for any invertible matrix* X *(Theorem in* `LectureMarch31.pdf` *available on* `ELMS/AMSC466/ZoomLectures`*), and the previous items to prove* $\Longrightarrow$.

2. **(5 pts)** Program the PLU algorithm for solving linear systems of algebraic equations from `LectureMarch31.pdf` available on `ELMS/AMSC466/ZoomLectures`. Apply it to 10 random matrices `A = rand(10)` and 10 random right-hand sides `b = rand(10,1)`. Compare your results with the ones obtained using Matlab's backslash operator `x = A \ b`. Comment on what you observe.

3. **(5 pts)** Write the polynomial interpolating the function $f(x) = \exp(-x)$ at $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$ (a) in Lagrange's form, (b) in Newton's form. Check that these polynomials coincide.