Final Exam. Solutions.

1. **(20 pts)** Consider the following iteration:

$$e_1^{k+1} = e_1^k + \lambda\left(-2e_1^k + e_2^k\right),$$
$$e_2^{k+1} = e_2^k + \lambda\left(e_1^k - 2e_2^k\right),$$

where λ is a positive parameter. For each level k, look at the two-norm $\|e^k\| = \sqrt{(e_1^k)^2 + (e_2^k)^2}$. Find analytically the exact range of positive values of λ such that, starting from any initial vector $[e_1^0, e_2^0]$, the norm $\|e^k\|$ remains bounded for all $k \in \mathbb{N}$ as a result of this iteration. *Hint: rewrite the iteration in the matrix form $e^{k+1} = Ae^k$. Use your knowledge of linear algebra.*

Submit a pdf file with your solution.

2. **(20 pts)** Let $f(x)$ be a function $(N+1)$-times continuously differentiable on $[x_0, x_1]$, $x_0 < x_1 < \ldots < x_N$ be a set of distinct nodes, and $p_N(x)$ be the unique interpolation polynomial for f of degree $\le N$ in these nodes.

   (a) Prove that $p_N'(x)$ interpolates $f'(x)$ at some nodes $y_0 < y_1 < \ldots < y_{N-1}$ where $y_j \in (x_j, x_{j+1})$. *Hint: use the identity*

$$g(x_{j+1}) = g(x_j) + \int_{x_j}^{x_{j+1}} g'(t)dt \quad \text{for any differentiable function } g.$$

   (b) Let $v_j := p_N(x_j) = f(x_j)$, and $w_j := p_n'(x_j)$, $j = 0, 1, \ldots, N$. Prove that then

$$w = Dv,$$

   where $v = [v_0, \ldots, v_N]^\top$, $w = [w_0, \ldots, w_N]^\top$, and D is the $(N+1)\times(N+1)$ differentiation matrix whose entries are given by:

$$D_{ij} = \frac{a_i}{a_j(x_i - x_j)}, \quad i \ne j, \quad D_{jj} = \sum_{\substack{k=0\\k\ne j}}^{N} \frac{1}{(x_j - x_k)},$$

$$a_j := \prod_{\substack{k=0\\k\ne j}}^{N} (x_j - x_k).$$

   Hint: write out the polynomial p_N in the Lagrange form and see Exercise 6.1 (p. 58) in L. N. Trefethen "Spectral Methods in Matlab".

   (c) What is the null space of the matrix D? Prove your answer.

Submit a pdf file with your solution.

3. **(20 pts)** Design a quadrature rule with minimal number of nodes that evaluates exactly the integral

$$I = \int_0^\infty e^{-x/2} \left(\frac{1}{x^{1/2}} - x^{3/2} + x^{5/2} \right) dx.$$

- Use this quadrature to evaluate this integral. Provide an answer with fourteen digits.
- Check your answer using a linear combination of some known values of Gamma function.
- Also estimate this integral using any appropriate general-purpose quadrature rule of your choice (e.g. adaptive Simpson's rule, composite trapezoid rule, etc.). For the latter, use a finite interval $[a, b]$ where a is very close to zero and b is reasonably large. It suffices if you get a few first digits correctly.

Hint: you will likely need Chapter 5 in Gil, Segura, Temme, "Numerical Methods for Special Functions" and this and this Wiki articles.

Submit a pdf file with your solution and your code pasted into it.