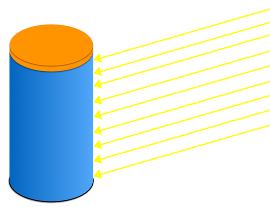
Take-home Final exam. Due May 18, 10 AM

1. Suppose an empty cylindric tin covered with an insulating top is standing on ice. The sun located low above the horizon is shining on one side of the tin as shown in the figure.



Formulate a Boundary Value Problem describing this situation. Note that you need to pick adequate boundary conditions and design a right-hand side accurately describing the intensity of heating. Solve the BVP numerically and find the stationary temperature distribution. Plot a figure showing the distribution.

Submit the .m file with your code. The formulation of the BVP and its justification can be written either in the comments to the code or submitted in a separate PDF file.

2. Consider the Greenberg traffic model

$$\rho_t + [-\rho \log(\rho)]_x = 0, \quad \rho(x,0) = \rho_0(x). \tag{1}$$

Here, ρ is the density of cars, and the velocity v depends on the density according to $v(\rho) = v_{\max} \log\left(\frac{\rho_{\max}}{\rho}\right)$, where $v_{\max}$ and $\rho_{\max}$ are set to be 1 for convenience.

(a) Find the formula for the characteristic $x(t)$ of Eq. (1) starting at the point $(x = x_0, t = 0)$ (the curve $x(t)$ passing through $(x = x_0, t = 0)$ along which ρ is constant, i.e., $\frac{d}{dt}\rho(x(t), t) = 0$).

(b) Plot the characteristics and the shock line on the xt-plane for the Riemann problem

$$\rho_0(x) = \begin{cases} 0.1, & x < 0, \\ 0.9, & x > 0. \end{cases}$$

(c) Suppose

$$\rho_0(x) = 0.5 + \frac{0.9}{\pi}\arctan(x). \tag{2}$$

Find the time when the shock appears. Then find the eventual shock speed.

(d) Solve Eq. (1) numerically with the initial condition (2) at the time interval $t \in [0, 5]$. If you choose the Godunov method, make sure to solve the Riemann problems at each time step correctly. The Godunov numerical flux is given by $F(u_j, u_{j+1}) = f(u^*(u_j, u_{j+1}))$, where $u^*(u_l, u_r)$ is defined as follows. The sonic point η is defined from $f'(\eta) = 0$. The solution of the local Riemann problem is a shock wave if $u_l < u_r$. Then

$$u^*(u_l, u_r) = \begin{cases} u_l, & \text{if } s > 0, \\ u_r, & \text{if } s \le 0, \end{cases}$$

where $s = (f(u_l) - f(u_r))/(u_l - u_r)$ is the shock speed. If $u_l \ge u_r$, the exact solution of the local Riemann problem is rarefaction fan. In this case, we set

$$u^*(u_l, u_r) = \begin{cases} u_l, & \text{if } u_l < \eta, \\ \eta, & \text{if } u_l \ge \eta \ge u_r, \\ u_r, & \text{if } u_r > \eta. \end{cases}$$

(e) Compare theoretical estimates of the time of shock appearance and the shock speed obtained in (c) with the ones inferred from your numerical solution.

Submit the .m file with your code. Your code should plot the shock diagram on the (x-t)-plane, compute the numerical solution, and infer the eventual shock speed and the time of the shock appearance. The answers to the theoretical questions should be submitted in a separate PDF file.

3. Consider the equation

$$u_t + u_x + u_{xxx} = u_{xx} \tag{3}$$

on the interval $[0, 4\pi]$ with periodic boundary conditions and the initial condition

$$u(x, 0) = \sin(x)\sin(5x)\sin(25x). \tag{4}$$

Solve this equation numerically using the discrete Fourier transform and exact time integration on the interval $t \in [0, 0.005]$. Plot the numerical solution at times $0 : 0.001 : 0.005$. Also, find the formula for exact analytical solution and plot it at the same moments of time.

Hint: to do the last task, you might want to use the Euler formula

$$\sin(ax) = \frac{e^{iax} - e^{-iax}}{2i}.$$

Submit the .m file with your code. The formula for the exact solution should be visible in the code. The derivation of the exact solution should be submitted in a separate PDF file.