Homework Assignment 10. Due Thursday April 30.

1. (5 pts) Derive the one-sided backward difference second-order estimator for the first derivative

\[ D^1_{2-}[f](x, h) = \frac{3f(x) - 4f(x - h) + f(x - 2h)}{2h}. \]  

and an error estimate for it using the method of undetermined coefficients. I.e., write

\[ D^1_{2-}[f](x, h) = \frac{1}{h} \left[ af(x) + bf(x - h) + cf(h - 2h) \right] = f'(x) + Ch^2 + \ldots, \]  

Taylor-expand the terms \( f(x - h) \) and \( f(x - 2h) \) around \( x \), and set up and solve an appropriate linear system for the coefficients \( a, b, \) and \( c \). As a result, you will be also find the constant \( C \) in the main error term.

2. (a) (5 pts) Use one step of Richardson extrapolation to derive (1). Proceed as follows. Start with the first-order backward difference estimator:

\[ F(h) := \frac{f(x) - f(x - h)}{h}. \]

Taylor-expand \( f(x - h) \) around \( x \) and obtain a series:

\[ F(h) = f'(x) + a_1h + a_2h^2 + a_3h^3 + \ldots. \]

Then write out this estimator for \( h \) and \( 2h \). Use a linear combination of \( F(h) \) and \( F(2h) \) to knock out the error term proportional to \( h \). Obtain a second-order estimator for \( f' \) and an error formula for it. This estimator must coincide with (1).

(b) (5 pts) Denote the second-order estimator obtained in the previous item by \( F(h, 2h) \). Use \( F \) with \( 2h \) and \( 3h \) to obtain another second-order estimator \( F(2h, 3h) \). Use an appropriate linear combination of \( F(h, 2h) \) and \( F(2h, 3h) \) to get a third-order estimator for \( f' \). Write out the resulting estimator in terms of \( f(x), f(x - h), f(x - 2h), \) and \( f(x - 3h) \). Check your result using the table “Backward finite difference” in Wiki: finite difference coefficient.

3. (5 pts) Write out the Lagrange polynomial \( p(x) \) interpolating a smooth function \( f(x) \) at \( x - h, \) \( x, \) and \( x + h \). Take the derivative of \( p(x) \) with respect to \( x \) and evaluate it at \( x - h, x, \) and \( x + h \). Compare your results with second-order backward-, central-, and forward-difference estimators for the first derivative of \( f \). Comment on your results.