Homework 1. Due Thursday, Sept. 17

1. Show that the matrix
\[
\begin{bmatrix}
G & A^\top W \\
A W & 0
\end{bmatrix}
\begin{bmatrix}
-p_k \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
\nabla f(x_k) \\
0
\end{bmatrix}.
\] (1)

with $G$ being symmetric positive definite and $A$ having linearly independent rows, is of saddle-point type, i.e., it has $d$ positive eigenvalues and $m$ negative ones. *Hint: Omit the subscript $W$ for brevity. Find matrices $X$ and $S$ (S is called the Schur compliment) such that
\[
\begin{bmatrix}
G & A^\top \\
A & 0
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
X & I
\end{bmatrix}
\begin{bmatrix}
G & 0 \\
0 & S
\end{bmatrix}
\begin{bmatrix}
I & X^\top \\
0 & I
\end{bmatrix}.
\]

Then use Sylvester’s Law of Inertia (look it up!) to finish the proof.

2. Consider an equality-constrained QP ($G$ is symmetric)
\[
\frac{1}{2}x^\top G x + c^\top x \rightarrow \min \quad \text{subject to} \quad Ax = b.
\] (2) (3) (4)

Assume that $A$ is full rank (i.e., its rows are linearly independent) and $Z^\top G Z$ is positive definite where $Z$ is a basis for the null-space of $A$, i.e., $AZ = 0$.

(a) Write the KKT system for this case in the matrix form.

(b) Show that the matrix of this system $K$ is invertible. *Hint: assume that there is a vector $z := (x, y)^\top$ such that $Kz = 0$. Consider the form $z^\top K z$, and so on ... . You should arrive at the conclusion that then $z = 0$.

(c) Conclude that there exists a unique vector $(x^*, \lambda^*)^\top$ that solves the KKT system. Note that since we have only equality constraints, positivity of $\lambda$ is irrelevant.