Homework 1. Due Thursday, Sept. 17

1. Show that the matrix
$$\begin{bmatrix} G & A_{\mathcal{W}}^\top \\ A_{\mathcal{W}} & 0 \end{bmatrix} \begin{bmatrix} -\mathbf{p}_k \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \nabla f(\mathbf{x}_k) \\ 0 \end{bmatrix}. \quad (1)$$
with G $d \times d$ being symmetric positive definite and A $m \times d$ having linearly independent rows, is of *saddle-point type*, i.e., it has d positive eigenvalues and m negative ones. *Hint: Omit the subscript $\mathcal{W}$ for brevity. Find matrices X and S (S is called the **Schur compliment**) such that*
$$\begin{bmatrix} G & A^\top \\ A & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & X^\top \\ 0 & I \end{bmatrix}.$$
Then use Sylvester's Law of Inertia (look it up!) to finish the proof.

2. Consider an equality-constrained QP (G is symmetric)
$$\frac{1}{2}\mathbf{x}^\top G\mathbf{x} + \mathbf{c}^\top \mathbf{x} \ \to \ \min \quad \text{subject to} \quad (2)$$
$$A\mathbf{x} = \mathbf{b}. \quad (3)$$
$$(4)$$
Assume that A is full rank (i.e., its rows are linearly independent) and $Z^\top G Z$ is positive definite where Z is a basis for the null-space of A, i.e., $AZ = 0$.

   (a) Write the KKT system for this case in the matrix form.

   (b) Show that the matrix of this system K is invertible. *Hint: assume that there is a vector $\mathbf{z} := (\mathbf{x}, \mathbf{y})^\top$ such that $K\mathbf{z} = 0$. Consider the form $\mathbf{z}^\top K\mathbf{z}$, and so on ... . You should arrive at the conclusion that then $\mathbf{z} = 0$.*

   (c) Conclude that there exists a unique vector $(\mathbf{x}^*, \boldsymbol{\lambda}^*)^\top$ that solves the KKT system. Note that since we have only equality constraints, positivity of $\boldsymbol{\lambda}$ is irrelevant.