Homework 4. Due Thursday, Oct. 22

1. **(5 pts)** Let A be a symmetric matrix. Consider the Rayleigh quotient

$$\rho_A(\mathbf{x}) = \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top \mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Note that since $\rho_A(\mathbf{x})$ is invariant along every line passing through the origin, one can consider the function $\phi(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ restricted to the unit sphere $\|\mathbf{x}\|_2^2 = 1$ instead. Prove that

(a) $\mathbf{v}$ is a stationary point of ρ_A if and only if $\mathbf{v}$ is an eigenvector of A, and $\rho_A(\mathbf{v}) = \lambda_{\mathbf{v}}$, the corresponding eigenvalue.

(b) The only local minima and maxima of ρ_A are the global minimum and maximum, and all other stationary points are saddles.

2. **(5 pts)** Prove the Eckart-Young-Mirsky theorem for any Ky-Fan norm, i.e., if $A = U\Sigma V^\top$ is the SVD of A, and M is any matrix of the size of A such that $\mathsf{rank}(M) \le k$, then

$$\|A - M\| \ge \|A - U_k\Sigma_k V_k^\top\| \quad \text{for any Ky-Fan norm} \quad \|\cdot\|.$$