**SIR Model on Random Graphs**  
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### SIR Model
- Used to model the spread of certain diseases
- Population divided into three states: susceptible, infected, and recovered
- $s(t)$, $i(t)$, and $r(t)$ are the fractions of the population in each state

$$\frac{ds}{dt} = -\beta is \quad \frac{di}{dt} = \beta is - \gamma i \quad \frac{dr}{dt} = \gamma i$$

- $\beta$ = rate at which infected individuals contact other individuals.
- $\gamma$ = rate at which infected individuals recover.

### Goal
Analyze SIR on various random graphs and explore the outbreak sizes.

### SIR on Random Graphs
- Each node represents an individual
- An edge indicates that two individuals are in contact
- The disease can only spread between nodes in contact

- For each time step $dt$, a susceptible individual in contact with an infected individual becomes infected with probability $\beta dt$, an infected individual recovers with probability $\gamma dt$

- Probability a susceptible node in contact with an infected node becomes infected: $T = \frac{\beta}{\beta + \gamma}$

### Outbreak Size
\[ S(T) = 1 - G_0(u; T), \text{ where } u \text{ is the solution to } u = G_1(u; T) \]

- For an Erdös-Renyi graph: $G_0(x) = e^{-x(1-x)} \quad S(T) = 1 - u$
- For a SBM with community sizes $N_1, N_2, \ldots, N_n$ and edge probability matrix $P$:  
  \[ G_0(x) = e^{-P_0 N_1(1-x)} e^{-P_0 N_2(1-x)} \cdots e^{-P_0 N_n(1-x)} \]
  generates the degree distribution by community for a random vertex in community $i$ for large communities.

$$G_i^{\gamma} = \frac{\partial G_i^{\gamma}(x)}{\partial x_i} \quad G_i^{\gamma}(x) \text{ generates the excess degree distribution by community for a vertex reached following an edge from community } i \text{ to community } j.$$  
$u_i = G_0^{\gamma}(u; T), \quad S(T) = 1 - u_i$ is the outbreak size of community $i$

$S(T)$ is the weighted average of each $S_i(T)$

### Generating Functions [1]
\[ G_0(x) = \sum_{k=0}^{\infty} p_k x^k \]
\[ G_1(x) = \frac{G_0^\prime(x)}{G_0^\prime(1)} \]

- $G_0(x)$ generates the degree distribution for a randomly chosen vertex in the network.
- $G_1(x)$ generates the excess degree distribution for a vertex reached by following a randomly chosen edge

\[ G_0(x; T) = G_0(1 + (x - 1)T) \]
\[ G_1(x; T) = G_1(1 + (x - 1)T) \]

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### References