SIR Model on Random Graphs

Amy Xu

Walt Whitman High School

SIR Model

- Used to model the spread of certain diseases
- Population divided into three states: susceptible, infected, and recovered
- $s(t)$, $i(t)$, and $r(t)$ are the fractions of the population in each state

$$\frac{ds}{dt} = -\beta is \qquad \frac{di}{dt} = \beta is - \gamma i \qquad \frac{dr}{dt} = \gamma i$$

β = rate at which infected individuals contact other individuals.
γ = rate at which infected individuals recover.

Random Graphs

Erdös-Renyi Random Graphs: $G(n, p)$, where n = number of nodes, p = probability that a possible edge exists.

Watts-Strogatz Random Graphs: $G(n, k, p)$, where n = number of nodes, k = number of nearest neighbors to which each node starts connected, p = probability of rewiring an edge
• Have the small-world property

Barabási-Albert Model: $G(n, m)$, where n = number of nodes, m = number of edges attached from a new node to existing nodes
• Generates scale-free graphs using preferential attachment

Stochastic Block Model (SBM):
• Generates graphs that contain communities

• Degree of a node: number of nodes to which it is connected
• Degree distribution: probability distribution for the degrees across the whole network

Goal

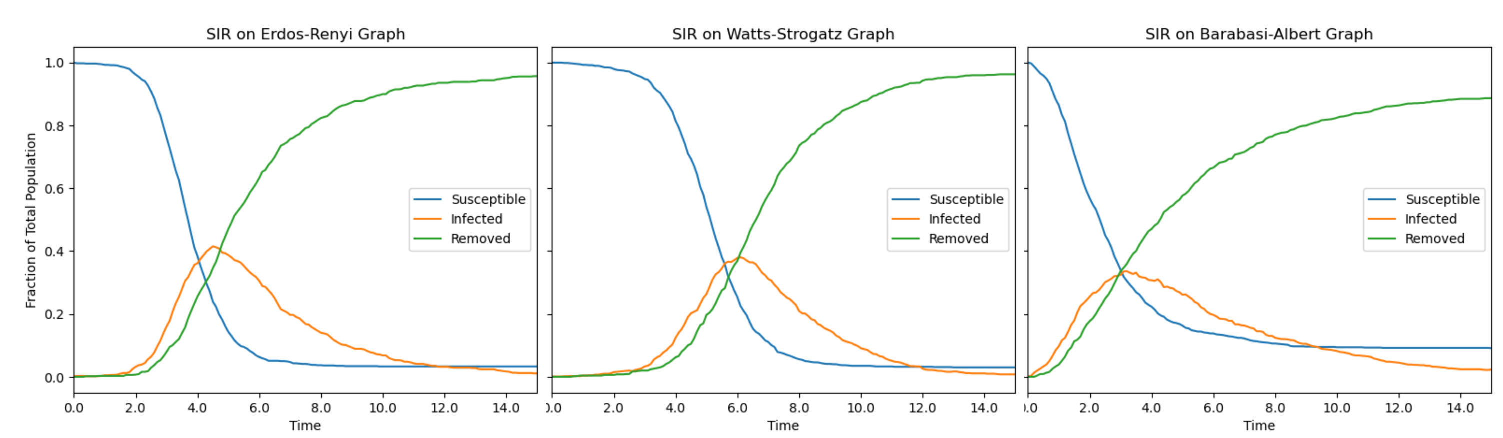
Analyze SIR on various random graphs and explore the outbreak sizes.

SIR on Random Graphs

- Each node represents an individual
- An edge indicates that two individuals are in contact
- The disease can only spread between nodes in contact

- For each time step dt, a susceptible individual in contact with an infected individual becomes infected with probability βdt, an infected individual recovers with probability γdt

- Probability a susceptible node in contact with an infected node becomes infected: $T = \dfrac{\beta}{\beta + \gamma}$



Simulated SIR for 1000 time steps on Erdös-Renyi, Watts-Strogatz, and Barabási-Albert random graphs with mean degree $z = 10$, $dt = .1, \beta = .3, \gamma = .05$

Generating Functions [1]

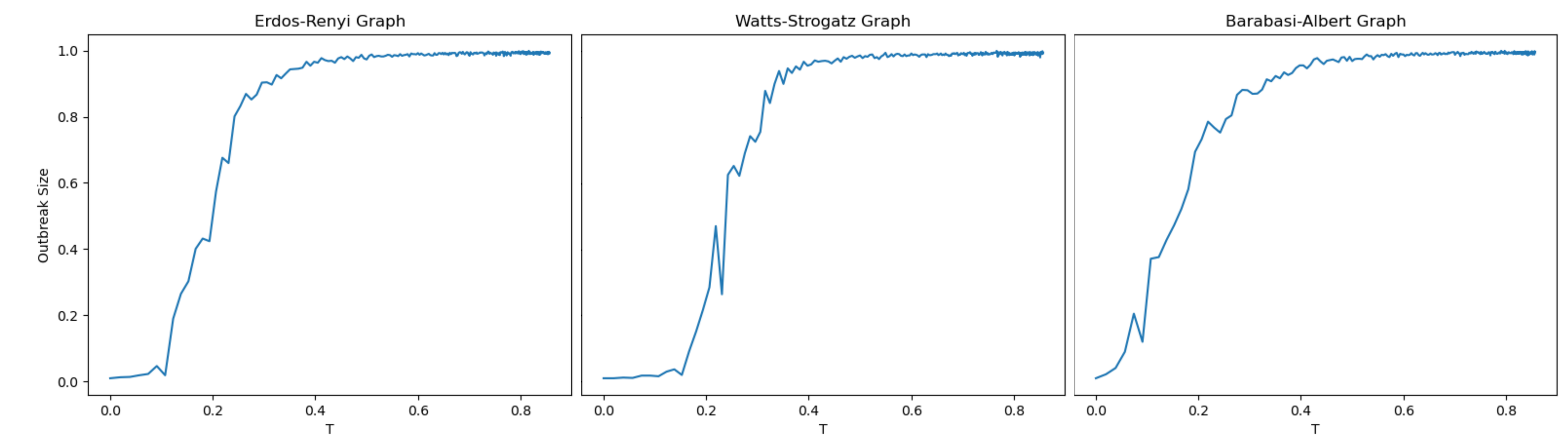
$G_0(x) = \sum_{k=0}^{\infty} p_k x^k$ generates the degree distribution for a randomly chosen vertex in the network.

$G_1(x) = \dfrac{G_0'(x)}{G_0'(1)}$ generates the excess degree distribution for a vertex reached by following a randomly chosen edge

$$G_0(x; T) = G_0(1 + (x-1)T)$$
$$G_1(x; T) = G_1(1 + (x-1)T)$$

Outbreak Size

$S(T) = 1 - G_0(u; T)$, where u is the solution to $u = G_1(u; T)$



SIR with 1000 time steps with $z = 10, dt = .1,$ final outbreak size as a function of T

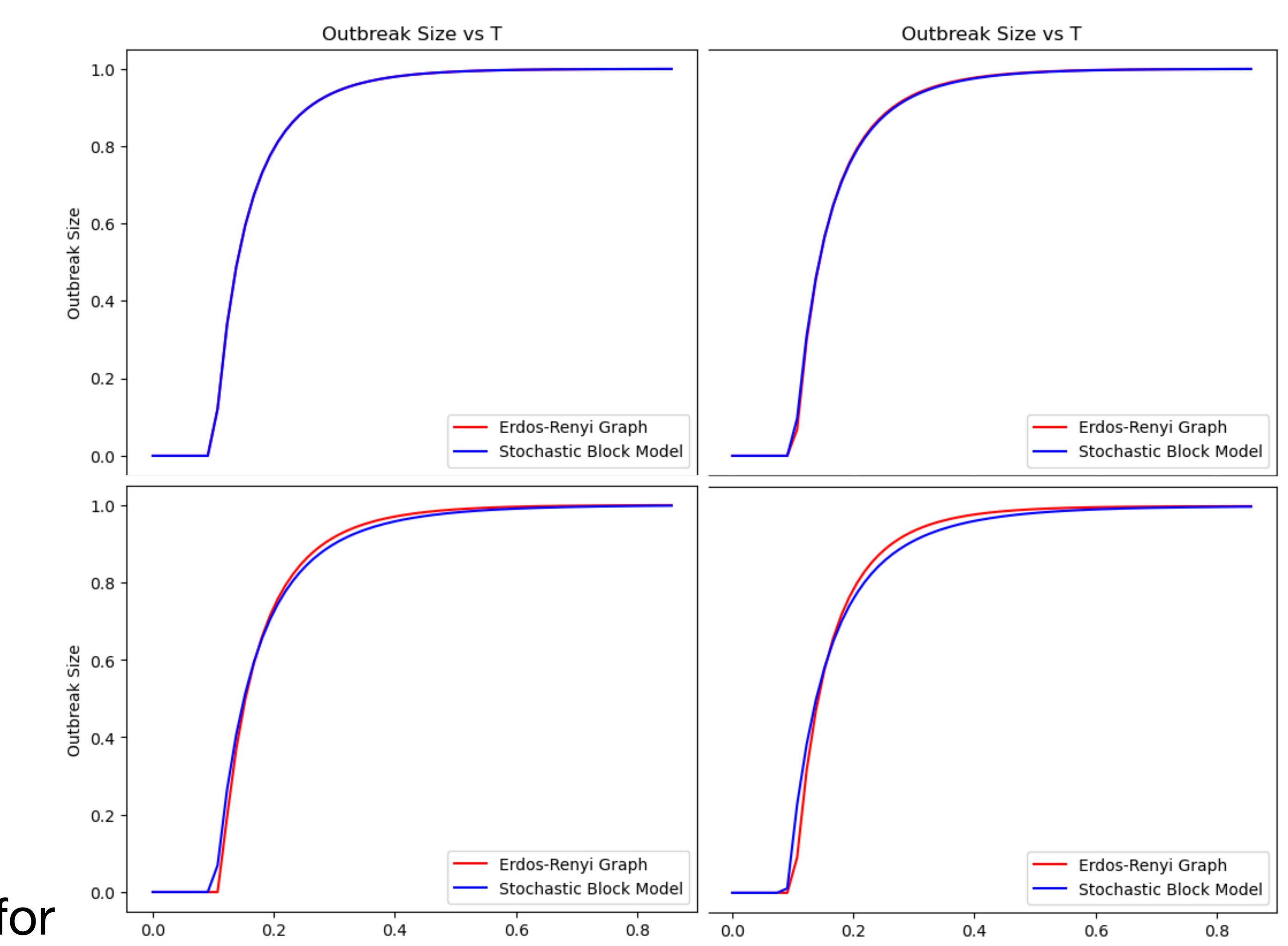
- For an Erdös-Renyi graph: $G_0(x) = G_1(x) = e^{-z(1-x)}$ $\qquad S(T) = 1 - u$

- For a SBM with community sizes $N_1, N_2, \ldots N_n$ and edge probability matrix P: $G_0^i(\mathbf{x}) = e^{-P_{i1}N_1(1-x_1)}e^{-P_{i2}N_2(1-x_2)} \ldots e^{-P_{in}N_n(1-x_n)}$ generates the degree distribution by community for a random vertex in community i for large communities.

$G_1^{i,j} = \dfrac{\frac{\partial G_0^j}{\partial x_i}(\mathbf{x})}{\frac{\partial G_0^j}{\partial x_i}(1,1,\ldots 1)} = G_0^j(\mathbf{x})$ generates the excess degree distribution by community for a vertex reached following an edge from community i to community j.

$u_i = G_0^i(\mathbf{u}; T)$ $\qquad S_i(T) = 1 - u_i$ is the outbreak size of community i

$S(T)$ is the weighted average of each $S_i(T)$



$S(T)$ vs T for an Erdös-Renyi graph and a SBM with the same mean degree and number of nodes

TOP LEFT: SBM with equal community sizes, $P_{ii} = .05$, $P_{ij} = .01$, $i \neq j$
TOP RIGHT: Equal community sizes, $P_{ii} = .05$, P_{ij} $i \neq j$ is a uniformly distributed random variable in $[0, .02)$
BOTTOM LEFT: Equal community sizes, P_{ii} is a uniformly distributed random variable in $[0, .1)$, $P_{ij} = .01$, $i \neq j$
BOTTOM RIGHT: Unequal community sizes, $P_{ii} = .05$, $P_{ij} = .01$, $i \neq j$

References

[1] Newman, M. E. (2002). Spread of epidemic disease on networks. *Physical review E*, *66*(1), 016128.

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