Characterization Theorem 1: (wAC) and Convexity

Theorem (De Philippis, De Rosa, Ghiraldin, 2016): In codimension 1, an integrand $G$ satisfies the atomic condition if and only if the function $G$ is strictly convex.

Definition: An integrand $G$ satisfies the weak atomic condition (wAC) if

(AC1): $\limsup_{\nu \to 0} \mu(A_\nu) \leq 1$ for all $\mu \in P(G(n,d))$;

(AC2): if $\limsup_{\nu \to 0} \mu(A_\nu) = 1$, then $\mu$ is supported on $A_\nu$ for some $\nu \in S^{n-1}$ where $A_\nu = \{v \in S^{n-1} : \nabla G(v) = \nabla G(u)\}$.

Theorem: In codimension 1, an integrand $G$ satisfies (wAC) if and only if $G$ is convex.

Characterization Theorem 2: (USAC) and Strong Convexity

A 1-homogeneous, even integrand $G : \mathbb{R}^n \to \mathbb{R}$ is strongly convex if there exists a constant $c > 0$ such that

$$G(u) - \langle (\nabla G(u), v) \rangle \geq c \|u - v\|^2 \quad \forall u, v \in S^{n-1}.$$

Theorem: In dimension or codimension 1, a $C^2$ integrand $G$ satisfies (USAC) if and only if $G$ is strongly convex.

Lemma: In codimension 1, $G$ satisfying (USAC) is equivalent to there existing some $k > 0$ such that

$$G(\nu(x)) - \langle (\nabla G(u), v) \rangle \geq k \|u - v\|^2 \quad \forall u, v \in S^{n-1}.$$

Corollary: In dimension and codimension 1, for any $p > 2$, the $f_p$ norm on $\mathbb{R}^n$ defined by

$$G(v) = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}$$

does not satisfy (USAC). We speak about the $f_p$ norm as a function on the Grassmannian manifold in dimension and codimension 1 via $\Psi(f_p(v)) = G(v)$ and $\Psi(v) = \Psi(f_p(v)) = G(v)$, respectively.

Dimension 1 Results

Theorem: Every positive integrand on $G(n,1)$ satisfies (AC1). Hence, any varifold with locally bounded first variation has $H^1$-rectifiable support.

This means any curve in $\mathbb{R}^n$ (for which $\gamma(V)$ is a Radon measure) is $H^1$-rectifiable (union of countably many Lipschitz graphs). The same is not true for codimension 1, and counterexamples exist in $G(3,2)$.

Theorem: Let $G$ be a positive 1-homogeneous integrand on $\mathbb{R}^n$. Define $\Psi : G(n,n-1) \to \mathbb{R}$ as $\Psi(v) = G(v)$ (hence $|\Psi(v)| = G(v)$). Then:

1. $\Psi$ satisfying (AC) is equivalent to strict convexity of $G$.
2. $\Psi$ satisfying (AC) is equivalent to $\Psi$ satisfying (AC).
3. $\Psi$ satisfying (AC) is equivalent to convexity of $G$.

Future Work

As it stands, (wAC) is only defined in dimension and codimension 1. We have a generalized notion, but it needs to be explored further.

The relationship between ellipticity conditions and notions of convexity in higher dimension and codimension.

References

(1) Guido De Philippis, Antonio De Rosa, and Francesco Ghiraldin, Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies, 2016. The authors thank Professor Antonio De Rosa, Seyed Abdolhamid Banighashemi, and Vasantha Pidaparthi for providing guidance throughout our REU program.