FUNCTIONAL ANALYSIS

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Contents

1. Countable, uncountable sets

Definition 1.1. Let X, Y be sets, a function $f : X \to Y$ is said to be *injective* if for all $x, y \in X$, $f(x) = f(y)$, then $x = y$. The function f is said to be *surjective* if for all $y \in Y$, there is $x \in X$ such that $f(x) = y$. A function is said to be *bijective* if f is injective and surjective.

Remark 1.2. Note that f is bijective if and only if it has an inverse, meaning that there exists f^{-1} such that $f \circ f^{-1}$ is the identity in X and $f^{-1} \circ f$ is the identity in Y.

Definition 1.3. Let X be a set, X is said to be *finite* if there exists $N \in \mathbb{N}$ and $f : X \to \{0, \dots, N-1\}$ bijective. In this case we will say that N is the cardinal of X.

Definition 1.4. Let X be a set, X is said to be *infinite* if it is not finte. The set X is said to be *countable* if there $f : X \to \mathbb{N}$ bijective. If X is not finite or countable, is said to be uncountable.

Theorem 1.5. *Let X be countable and $A \subseteq X$ be infinite, then A is countable.*

Proof. Let's see first that every infinite set of $\mathbb{N}$ is countable. Let $A \subseteq \mathbb{N}$ be infinite. In particular, $A \neq \emptyset$. Since $A \subseteq \mathbb{N}$ is not empty, by the well ordering principle, there exist $a_0 \in A$ such that $a_0 = \min(A)$. Now, consider $A \setminus \{a_0\}$ and note that, since A is infinite, $A \setminus \{a_0\}$ is not empty, then, we can take $a_1 = \min(A \setminus \{a_0\})$. Note that since A is infinite, we can recursively get $a_0, a_1, a_2, \dots, a_n$, for all $n \in \mathbb{N}$. This procedure does not over, as A is inifnite. Now, note that $f : \mathbb{N} \to A, f(n) := a_n$ is a bijection.

Now, let X be countable and $A \subseteq X$ be infinite. Since X is countable, there is $f : X \to \mathbb{N}$ bijective. Consider the set $f(A) := \{f(a) : a \in A\}$ and note that since f is bijective, $f(A)$ is infinite. Then, $f(A)$ is countable, but since the function $g : A \to f(A), g(a) := f(a)$ is a bijection, A itself is countable. □

Theorem 1.6. *The set $\mathbb{N} \times \mathbb{N}$ is countable.*

Proof. Note that we can arrange $\mathbb{N} \times \mathbb{N}$ as follows:

$$\begin{array}{ccccc} (0,0) & (0,1) & (0,2) & (0,3) & \dots \\ (1,0) & (1,1) & (1,2) & (1,3) & \dots \\ (2,0) & (2,1) & (2,2) & (2,3) & \dots \end{array}$$

and note that we can arrange this as $(0,0),(0,1),(1,0),(2,0),(1,1),(0,2),(0,3),\dots$ which is a bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$. □

Lemma 1.7. The set $\mathbb{Z}$ is countable.

Proof. Consider the following function, $f:\mathbb{Z}\to\mathbb{N}$, $f(n)$ is going to be odd if n is positive, even if n is negative. This is a bijection and hence $\mathbb{Z}$ is countable. □

Theorem 1.8. *The set $\mathbb{Q}$ is countable.*

Proof. Note that $\mathbb{Q}$ can be seen as a subset of $\mathbb{Z}\times\mathbb{Z}$ as every rational number is the quotient of two integer numbers. Since is is an infinite set of a countable set, it is countable itself. □

Theorem 1.9. *The set $\mathbb{P}(\mathbb{N}) := \{A : A \subseteq \mathbb{N}\}$ of all subsets of natural numbers is not countable.*

Proof. By way of contradiction, assume that there is $f:\mathbb{N}\to\mathbb{P}(\mathbb{N})$ bijective, in particular is onto. Let $X := \{n\in\mathbb{N} : n\notin f(n)\}$. Clearly $X\in\mathbb{P}(\mathbb{N})$, since f is bijective, there exists $n\in\mathbb{N}$ such that $f(n)=X$, does $n\in X$? If $n\in X$, then $n\notin f(n)=X$, which is a contradiction. if $n\notin X$, then $n\in f(n)=X$, which is as well a contradiction. Then, $\mathbb{P}(\mathbb{N})$ is not countable. □

Theorem 1.10. *The set of all real numbers $\mathbb{R}$ is not countable.*

Proof. First, note that $(0,1)$ and $\mathbb{R}$ has the same cardinality as using for example $\tan^{-1}$ with proper scale. Now, by way of contradiction, assume that $(0,1)$ is countable, then we can arrange their elements as a sequence $x_0, x_1, x_2, \dots$. Note that, this means that we can write every element in $(0,1)$ as follows:

$$x_0 := 0, y_0^0 y_1^0 y_2^0 y_3^0 \dots,$$
$$x_1 := 0, y_0^1 y_1^1 y_2^1 y_3^1 \dots,$$
$$x_2 := 0, y_0^2 y_1^2 y_2^2 y_3^2 \dots$$

Now, define the following sequence,

$$\tilde{y}_n := \begin{cases} y_n^n + 1 & \text{if } y_n^n \neq 9; \\ 0 & \text{if } y_n^n = 9. \end{cases}$$

and consider the element $y := 0, \tilde{y}_0\tilde{y}_1\tilde{y}_2\tilde{y}_3\dots$, clearly $y\in(0,1)$, but does y is listed? Note that y disagrees in the diagonal from every element, so it cannot be in the list, meaning that $(0,1)$ was not countable. □

Theorem 1.11. *Let $(A_n)_n$ be a countable family of countable sets, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.*

Proof. Note that we can arrange each A_n as a sequence $(a_{ij})_{i,j}$, meaning that $\bigcup_n A_n$ can be embedded into $\mathbb{N}\times\mathbb{N}$. The union is an infinite subset of a countable set, hence is countable. □

Lemma 1.12. Let X be an uncountable set, if $A\subseteq X$ is countable, then A^c is uncountable.

Proof. By way of contradiction, assume that both A and A^c are countable. By the previous result, $A\cup A^c$ is countable, but $X = A\cup A^c$ is uncountable, a contradiction. □

Exercise 1.13. The empty set is a subset of every set.

Proof. We want to see that for every set X and every $x \in \emptyset, x \in X$. However, since the empty set does not have any element, the condition satisfies trivially. □

Exercise 1.14. A complex number $z \in \mathbb{C}$ is said to be *algebraic* if there are rational numbers $a_1, \ldots, a_n$ not all zero such that $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$. The set of algebraic numbers is countable.

Proof. Let $n \in \mathbb{N}$, consider the set

$$P_n := \{z \in \mathbb{C} : a_n z^n + \cdots + a_0 = 0\}.$$

Consider the mapping $f : P_n \to \mathbb{Q}^n, a_n z^n + \cdots + a_0 \mapsto (a_0, \ldots, a_n)$ and note that f is injective. Note also that $f(P_n)$ is infinite for all $n \in \mathbb{N}$, so it is countable, as is a infinite set of a countable set. Therefore, P_n is countable, as the restriction of f to the range of P_n is bijective. From this, we can conclude that the set of the algebraic numbers, which is the union of all P_n's is countable as is the countable union of countable sets. □

Exercise 1.15. There are real numbers that there are not algebraic.

Proof. We have shown that $\mathbb{R}$ is uncountable. Suppose that every real number is algebraic, then $\mathbb{R}$ has to be countable, which is a contradiction. □

Exercise 1.16. The set of irrational numbers is non countable.

Proof. Note that $\mathbb{R}$ is equal to the rational numbers disjoint union with the irrational numbers, if the irrational numbers where countable, then $\mathbb{R}$ would be countable, which is a contradiction. □

2. Metric spaces

Definition 2.1. Let $X \neq \emptyset$, a *metric* is a function $d : X \times X \to [0, \infty)$ such that

(1) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
(2) For all $x, y \in X$, $d(x, y) = d(y, x)$.
(3) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

The tuple (X, d) is called a *metric space.*

Example 2.2. Let's provide some examples.

(1) Let $X \neq \emptyset$, define the following metric

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

It can be shown that d is a metric on X, it is called the *discrete* metric.

(2) In $\mathbb{R}^n$ we define the euclidean metric as

$$d(x, y) := \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.$$

The function d is a metric over $\mathbb{R}^n$.

(3) More interesting, if (X, d) is a metric space, the following are metrics over d as well,

$$d_1(x, y) := \min(d(x, y), 1),$$
$$d_2(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

Definition 2.3. Let $(X, d_X), (Y, d_Y)$ be metric spaces, we can define a metric on the product as follows,

$$d_{X\times Y}((x_1, x_2)), (y_1, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Other possible metrics over $X \times Y$ are,

$$d_\infty((x_1, x_2)), (y_1, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2),$$
$$d_p((x_1, x_2)), (y_1, y_2)) := (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{1/p}, p \geq 1.$$

Definition 2.4. Let (X, d) be a metric space, for $r > 0$ and $x \in X$ we define *the ball of radius* $r > 0$ *centered at* x as follows,

$$B_r(x) := \{y \in X : d(x, y) < r\}.$$

The closed ball can be defined using $\leq$ instead of $<$.

Definition 2.5. Let (X, d) be a metric space, $A \subseteq X$ is said to be

(1) *open* if for every $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$.
(2) *closed* if is the complement of an open set.

A point $a \in A$ is said to be *a limit point* if for every $r > 0$, $A \setminus \{a\} \cap B_r(a) \neq \emptyset$.

Lemma 2.6. Let (X, d) be a metric space, $(C_\lambda)_\lambda \subseteq X$ be a family of closed sets, $(U_\lambda)_\lambda$ be a family of open sets, then $\bigcup_\lambda U_\lambda$ is open and $\bigcap_\lambda C_\lambda$ is closed.

Proof. Let $(U_\lambda)_\lambda$ be a family of open sets and call $U := \bigcup_\lambda U_\lambda$, we want to see U is open. Let $x \in U$, then there exist $\lambda \in \Lambda$ such that $x \in U_\lambda$, since U_λ is open, there is $r > 0$ such that $B_r(x) \subseteq U_\lambda$, hence $B_r(x) \subseteq U$ and U is open.

For the other part, call $C := \bigcap_\lambda C_\lambda$ and note that $C^c = \bigcup_\lambda C^c\lambda$ is open by the previous paragraph, so C is closed. □

Now we are going to introduce a key concept in analysis which is going to be very useful.

Definition 2.7. Let (X, d) be a metric space, $(x_n)_n$ be a sequence, i.e, a countable set in X. Let $x \in X$, we will say that x_n *converges* to x, $x_n \to x$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B_\varepsilon(x)$.

Lemma 2.8. Limits of sequences are unique.

Proof. Let $(x_n)_n$ be a sequence and suppose that converges to x, y such that $x \neq y$. Let $\varepsilon := d(x, y)/2$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \in B_\varepsilon(x)$ and also there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2, x_n \in B_\varepsilon(y)$. Take $N := \max\{N_1, N_2\}$ and note that therefore for all $n \geq N, x_n \in B_\varepsilon(x) \cap B_\varepsilon(y)$, but this is impossible, as this intersection is empty. □

Lemma 2.9. Let (X, d) be a metric space, $(x_n)_n \subseteq X$ be a sequence, $x \in X$. Then $x_n \to x$ if and only if for all $\varepsilon > 0$, $B_\varepsilon(x)$ all but except finitely many points.

Proof. Suppose that $x_n \to x$ and let $\varepsilon > 0$, then, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B_\varepsilon(x)$, but this just means that all except $x_1, \ldots, x_N$ are in the complement of $B_\varepsilon(x)$. For the other direction, let $\varepsilon > 0$, by hypothesis, $B_\varepsilon(x)$ contains all but finitely many elements in the sequence. Call them $x_1, \ldots, x_N$ and without loss of generality assume that the indices are arranged (otherwise, just take a permutation). Then, for all $n \geq N$, $x_n \in B_\varepsilon(x)$ and we are done. □

Lemma 2.10. Let (X, d) be a metric space, $A \subseteq X$. Then $a \in X$ is a limit point of A if and only if there exist a sequence $(x_n)_n \subseteq A$ such that $x_n \to a$.

Proof. Note that if $x_n \to a$, for all $\varepsilon > 0$, $B_\varepsilon(a)$ has infinitely many points, so clearly is not empty. On the other hand, suppose that a is a limit point of A and let $\varepsilon > 0$. Since a is a limit point of A, there is $x_1 \in A \setminus \{a\} \cap B_\varepsilon(a)$. Now, without loss of generality, we may assume that $x_1 \notin B_{\varepsilon/2}(a)$ (otherwise, just re-scale), then, we may find $x_2 \in A \setminus \{a, x_1\} \cap B_{\varepsilon/2}(a)$. This recursive procedure constructs a sequence that eventually converges to a. Note that there is no actually loss of generality, as the re-scaling only depends on re-sizing up to finitely many elements. ☐

Lemma 2.11. Let (X, d) be a metric space, $A \subseteq X$, define A' to be the set of all limit points of A. Then A is closed if and only if $A = A \cup A'$.

Proof. Suppose A is closed, then A^c is open. By way of contradiction, assume $A' \nsubseteq A$, then there exists $a \in A' \cap A^c$. By the previous lemma, we may find a sequence $(x_n)_n \subseteq A$ such that $x_n \to a$, and being A^c open, we may find $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A^c$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B_\varepsilon(a)$, which is impossible as A and A^c are disjoint.

On the other hand, by contrapositive, assume that $A \neq A \cup A'$, we want to show that A is not closed. By way of contradiction, assume that A is closed, then A^c is open. Since $A \neq A \cup A'$, then $A^c \cap A \cup A' \neq \emptyset$, but since A and A^c are disjoint, this implies that $A^c \cap A' \neq \emptyset$. Let $x \in A^c \cap A'$ and take $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A^c$. Now, since $x \in A'$ is a limit point, we may find a sequence $(x_n)_n \subseteq A$ such that $x_n \to x$, this means that there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B_\varepsilon(x)$, but this is a contradiction, as $B_\varepsilon(x) \subseteq A^c$. ☐

Definition 2.12. Let (X, d) be a metric space, $A \subseteq X$, we define the *closure of A* as follows,

$$\overline{A} := \bigcap_{C \supseteq A, C \text{ closed}} C.$$

Note that $\overline{A}$ is the smallest closed set containing A, as is the intersection of closed sets containing A. Let's see now that $\overline{A} = A \cup A'$.

Lemma 2.13. Let (X, d) be a metric space, $A \subseteq X$, then $\overline{A} = A \cup A'$.

Now, the following result will be very important when we were dealing with normed linear spaces.

Theorem 2.14. *Let (X, d) be a metric space and $A \subseteq X$ be a set. Then A is closed if and only if every convergent sequence in A converges to an element in A.*

Proof. Suppose A is closed and let $(x_n)_n$ be a convergent sequence in A. By way of contradiction, assume that the limit is not in A. Note that A being closed implies by definition that A^c is open. Since $x \in A^c$ is open, there is $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A$. Now, by definition of convergence, there is $N \in \mathbb{N}$ such that for all $n \geq N, x_n \in B_\varepsilon(x)$, but this is impossible, as $B_\varepsilon(x) \cap A = \emptyset$.

Now, for the other implication, assume that A is not closed, then A is not equal to its closure and since $\overline{A} = A \cup A'$, this means that $A' \cap A^c \neq \emptyset$, so let $x \in A' \cap A^c$. Since x is a limit point of A, there exists a sequence $(x_n)_n \subseteq A$ converging to x, but $x \in A^c$, which is a contradiction. ☐

Definition 2.15. Let (X, d) be a metric space, $A \subseteq X$ is said to be *bounded* if

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\} < \infty.$$

Now, let's explore more important properties of metric spaces.

2.1. **Compactness.** Compactness (in my personal opinion) is one of the most important topological properties and it can be extrapolated to many settings. For example, a form of compactness allow us to prove the following statement: "let $(p_\lambda)_\lambda$ be an infinite family of polynomials (not necessarily countable), such that every finite subfamily has a common root. Then, the entire $(p_\lambda)_\lambda$ has a common root"!

Definition 2.16. Let (X, d) be a metric space, $K \subseteq X$ is said to be *compact* if every covering of K by open sets can be reduced to a finite covering.

Lemma 2.17. Let (X, d) be a metric space, $K \subseteq X$ compact, then K is closed.

Proof. Let's see that K^c is open, so let $x \in K^c$, for every $k \in K$, consider $\varepsilon_k := d(x, k)/2$ and note that $B_{\varepsilon_k}(x) \cap B_{\varepsilon_k}(k) = \emptyset$. Note that the collection $(B_{e_k}(k))_k$ is a covering of K, by compactness, there exists a finite subcovering, meaning that $K \subseteq B_{\varepsilon_{k_1}}(k_1) \cup \cdots \cup B_{e_{k_l}}(k_l)$. Now, note that the set $\bigcap_{j=1}^{l} B_{e_{k_j}}(x)$ is open and does not intersect K as is disjoint from the union of the balls. So, K^c is open and K is closed. □

Lemma 2.18. Let (X, d) be a metric space, $K \subseteq X$ be compact and $C \subseteq K$ be closed, then C is compact.

Proof. Let $(U_\alpha)_\alpha$ be an open covering of C, since C is closed, C^c is open and $(U_\alpha)_\alpha \cup C^c$ is an open covering of K, by compactness, there is a finite subcovering of K, but since $C \subseteq K$, this subcovering also works for C. □

Lemma 2.19. Let (X, d) be a metric space, $K \subseteq X$ be compact, then K is bounded.

Proof. By way of contradiction, assume that K is unbounded, then $diam(K) = \infty$. Let $x \in K$, choose $x_1 \in K$ such that $d(x, x_1) = 1$, note that keeping this procedure, since K is infinite, otherwise couldn't be unbounded, there is $x_n \in K \setminus \{x_1, \ldots, x_{n-1}\}$ such that $d(x, x_n) = n$. Note that the family $(B_n(x_n))_n$ is an open covering of K that does not admit a finite subcovering, hence K is not compact, contradiction. □

Remark 2.20. In a metric space (X, d) compact sets are closed and bounded. However, the reverse does not hold in general. However, in $\mathbb{R}^n$ this property holds, K is compact if and only if is closed and bounded, this is called the *Heine-Borel Property.*

Theorem 2.21. *Let (X, d) be a metric space, $K \subseteq X$ be a set, then the following statements are equivalent:*

(1) K is compact.
(2) Every infinite set in K has a limit point in K
(3) Every sequence in K has a convergent subsequence in K.

Proof. Let's see (1) ⇒ (2), so let $A \subseteq K$, we want to see that if A is infinite, then A has an accumulation point in K, let's prove this by contrapositive. Suppose that A does not have accumulation points in K, therefore $A' = \emptyset$ and A needs to be closed.

Note that, since A does not have limit points, for every $a \in A$, we can find $\varepsilon_a > 0$ such that $B_{\varepsilon_a}(a) \cap A = \{a\}$. Therefore, the covering $(B_{\varepsilon_a}(a))_a \cup A^c$ is an open covering of K and since it is compact, admits a finite subcovering. Hence, A must be finite.

To see (2) ⇒ (3), let $(x_n)_n \subseteq K$ be a sequence, note that it is an infinite set of K and therefore it has a limit point in K. But then we can find a sequence in the range of $(x_n)_n$ converging to x, as we wanted.

I am not proving (3) ⇒ (1), because this result is actually hard. It is only true for metric spaces, we are going to use the result and let you to check the proof! □

2.2. **Continuous functions.**

Definition 2.22. Let (X, d) be a metric space, a subset $D \subseteq X$ is said to be *dense* if for all $x \in X$, there exists $(d_n)_n \subseteq D$ such that $d_n \to x$.

Example 2.23. The set $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 2.24. Let $(X, d_X), (Y, d_Y)$ be metric spaces, a function $f : X \to Y$ is said to be continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Theorem 2.25. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a function. Then f is continuous if and only if the preimage of every open set in Y under f is open in X.*

Theorem 2.26. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a function. Then f is continuous if and only if for every sequence $(x_n)_n \subseteq X$ such that $x_n \to x$, then $f(x_n) \to f(x)$.*

Proof. Suppose that f is continuous and consider a convergent sequence $(x_n)_n$ in X. Let $\varepsilon > 0$, we need $N \in \mathbb{N}$ such that for all $n \geq N$, $f(x_n) \in B_\varepsilon(f(x))$. Since f is continuous, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. Since $x_n \to x$, there is $N \in \mathbb{N}$ such that for all $n \geq N, x_n \in B_\delta(x)$, but by continuity this entails that for all $n \geq N, f(x_n) \in B_\varepsilon(f(x))$, as we wanted.

Now, suppose that f exchanges limits, let's see that f is continuous. Let's prove it by contrapositive, suppose that if f is not continuous, then there is a convergent sequence in X such that its images does not converge. Since f is not continuous, there are $x, y \in X$ and $\varepsilon > 0$ such that for all $\delta > 0, d_X(x, y) < \delta$, but $d_Y(f(x), f(y)) \geq \varepsilon$. Note that we can choose $\delta = 1/n$ and in that way we can construct a sequence $(x_n)_n$ converging to x, but whose all images are a distance greater or equal than $\varepsilon > 0$ from $f(x)$, this proves the statement. □

Lemma 2.27. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be continuous and $K \subseteq X$ be compact, then $f(K)$ is compact.

Proof. Let $(U_\alpha)_\alpha$ be an open covering of $f(K)$ and note that since f is continuous, $(f^{-1}(U_\alpha))_\alpha$ is an open covering of K, being K compact, this means that admits a finite subcovering. Note that going back to Y, this provides a finite covering of $f(K)$. □

Theorem 2.28. *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be continuous, let $K \subseteq X$ be compact, then K attains its minimum and its maximum on K.*

Proof. Note that $f(K)$ is a compact subset of $\mathbb{R}$, hence is closed and bounded. Call $m := \inf(f(K)), M := \sup(f(K))$. By the definition of the supremum, we may find $(x_n)_n \subseteq K$ such that $f(x_n) \to M$. Since $(x_n)_n$ is a sequence in K and K is compact, we may find a convergent subsequence to $x \in K$, without loss of generality, assume that $(x_n)_n$ itself converges to x. By continuity of f, $f(x_n) \to f(x)$, but we already knew that $f(x_n) \to M$, by uniqueness of the limits, $f(x) = M$. Note that if we are working with a subsequence, we have to note that a sequence converges to x if and only if every subsequence converges to the same limit. The proof for the value whose image is m is analogue. □

Definition 2.29. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be a function. We will say that f is *uniformly continuous* if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$, if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Note that the difference between continuity and uniform continuity is that for continuity, the δ might depend on the points, while for uniform continuity, δ is uniform for every pair of elements in X.

2.3. **Complete metric spaces.**

Definition 2.30. Let (X, d) be a metric space, a sequence $(x_n)_n$ in X is said to be a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

Definition 2.31. Let (X, d) be a metric space, X is said to be *complete* if every Cauchy sequence in X converges to an element in X.

Lemma 2.32. Let (X, d) be a metric space and $(x_n)_n$ be a convergent sequence. Then $(x_n)_n$ is a Cauchy sequence.

Proof. Let $x \in X$ be the limit of $(x_n)_n$ and let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon/2$. Let $n, m \geq N$ and note that $d(x_n, x_m) \leq d(x_n, x)+d(x, x_m) < \varepsilon$, then is a Cauchy sequence. □

Example 2.33. The set of the real numbers is a complete metric space, also every $\mathbb{R}^n$.

Definition 2.34. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$, the function f is said to be an *isometry* if $d_X(x, y) = d_Y(f(x), f(y))$.

Note that clearly isometries are continuous and actually uniformly continuous. Isometries preserves the structuree of the metric space.

Theorem 2.35. *Let (X, d) be a metric space, then there exists a complete metric space X^* such that X can be isometrically embedded into X and X is dense in X^*, such space is called the* completion *of X.*

Proof. The idea of this proof is to construct a new metric space by taking all the Cauchy sequences on X and defining the following relation, $(x_n)_n \sim (y_n)_n$ if and only if $d(x_n, y_n) \to 0$. This relation is an equivalence relation, the set X^* is just $X/\sim$. □

Example 2.36. The set $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Definition 2.37. Let (X, d) be a metric space, X is said to be *separable* if it admits a countable dense set.

Theorem 2.38 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space and $(U_n)_n$ be a sequence of open dense sets in X, then the intersection is also dense in X.*

Proof. First note that D is dense in X if and only if every open set in X intersects D. This follows from the definition using sequences. So, let $U \subseteq X$ open and take $x_0 \in U$, $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq U$. Since U_1 is dense in X, $B_\varepsilon(x_0) \cap U_1 \neq \emptyset$, so, let $x_1 \in B_\varepsilon(x_0) \setminus \{x_0\} \cap U_1$ and now consider $B_{\varepsilon/2}(x_1) \cap U_2$, again since U_2 is dense, we may find x_2 in such intersection. Note that following in this way, we can find $(x_n)_n$ such that each $x_n \in U_n$. Note that $(x_n)_n$ is a Cauchy sequence, by the shrinking radius condition and since X is complete, $x_n \to x \in X$. Note also that such $x \in B_{\varepsilon/n}(x_n) \cap U_n \subseteq U$ for all n, hence $x \in \cap \bigcap_{n=1}^{\infty} U_n \cap U$. Since U was arbitrary, this shows that the intersection is dense in X. □

2.4. **Descriptive set theory.**

Definition 2.39. Let (X, d) be a metric space, a set $A \subseteq X$ is said to be an F_σ if it is the countable union of closed sets, it is said to be a G_δ if it is the countable intersections of open sets.

By taking countable intersections of G_δ sets and countable unions of F_σ sets we can move on the Hierarchy, this how the Borel σ-algebra is generated. In this way we can prove that the cardinality of the Borel σ-algebra is the cardinality of the continuum. Every family in the construction has $2^{\aleph_0}$ sets and by not assuming the continuum hypothesis, this procedure ends up in the first uncountable ordinal $\omega_1 < 2^{\aleph_0}$, then we have $2^{\aleph_0} \times \omega_1 = 2^{\aleph_0}$ Borel sets.

A set $A \subseteq X$ is said to be *analytic* if it is the image of a continuous function of a Borel set. Analytic sets are "big" in X in the following sense: they cannot be the countable union of closed nowhere dense sets.

Definition 2.40. Let (X, d) be a metric space, X is said to be a *Polish space* if its completely metrizable and has a countable dense set.

2.5. Basic topology.

Definition 2.41. Let $X \neq \emptyset$, a family of sets $\tau \subseteq \mathbb{P}(X)$ is said to be a *topology* if

(1) $\emptyset \in \tau$,
(2) τ is closed under arbitrary unions, meaning if $(U_\lambda)_\lambda \subseteq \tau$, then $\bigcup_\lambda U_\lambda \in \tau$,
(3) τ is closed under finite intersections, meaning that if $U_1, \dots, U_n \in \tau$, then $\bigcap_{i=1}^n U_i \in \tau$.

The pair (X, τ) is called a topological space. Sets in τ are called *open sets*, the complement of an open set is a *closed* set.

Definition 2.42. Let (X, τ) be a topological space, a set $\mathcal{B} \subseteq \tau$ is said to be a *basis for* τ if

(1) $\mathcal{B}$ covers X, i.e, for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
(2) For every $B_1, B_2 \in \mathcal{B}$ and for every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq \mathcal{B}$. A basis generates the topology by taking arbitrary unions of elements in the basis. If $\mathcal{B}$ is a basis, clearly every basic set is open.

Example 2.43. Let's see some examples of topological spaces.

(1) Every metric space is a topological space, note that for $r > 0$, the set $\mathcal{B} = (B_r(x))_{r>0,x})$ is a basis for the topology. Clearly it covers X and we have to note that if we have $B_r(x)$ and $B_q(y)$ such that they are not disjoint and we take $z \in B_r(x) \cap B_q(y)$, then we can find $\varepsilon > 0$, take for example $\operatorname{diam}(B_r(x) \cap B_q(y))/2$, such that $B_\varepsilon(z) \subseteq B_r(x) \cap B_q(y)$. The topology generated by the open balls is called the metric topology.
   This is consistent on what we have, that every open set can be expressed as the union of arbitrary balls centered at their points.
(2) Let $(L, \leq)$ be a totally ordered set, meaning $L \neq \emptyset$ and $\leq$ is a relation over L satisfying the following,
   (a) For all $x \in L, x \leq x$ (reflexivity).
   (b) For all $x, y \in L$, $x \leq y$ and $y \leq x$ implies $x = y$ (anti symmetry).
   (c) For all $x, y, z \in L$, $x \leq z$ and $z \leq y$, implies $x \leq y$ (transitivity).
   (d) For all $x, y \in L$, one of the following holds always: $x \leq y$ or $y \leq x$ or $x = y$ (totality).

   Over L, define the following sets in the standard way:

$$
\begin{aligned}
(x, y) &:= \{z : x < z < y\} \text{ open interval,} \\
[x, y) &:= \{z : x \leq z < y\} \text{ right half-closed interval,} \\
(x, y] &:= \{z : x < z \leq y\} \text{ left half-closed interval,} \\
[x, y] &:= \{z : x \leq z \leq y\} \text{ closed interval.}
\end{aligned}
$$

   Let $\mathcal{B}$ be the family of all open intervals in L, then $\mathcal{B}$ is a basis. Clearly covers L and note that for $(a, b), (c, d) \in \mathcal{B}$, if $x \in (a, b) \cap (c, d)$ then by taking $\alpha := \min(a, c), \beta := \min(b, d)$, the interval $(\alpha, \beta) \in \mathcal{B}$ contains x and is fully contained in the intersection. The topology generated by this basis is called the *order topology.*
(3) Consider the set $\mathbb{N}^\mathbb{N}$ of functions from the natural numbers to the natural numbers, i.e, $\mathbb{N}^\mathbb{N} := \{f : \mathbb{N} \to \mathbb{N}\}$. Define the following sets over $\mathbb{N}^\mathbb{N}$, for $n \in \mathbb{N}$, $f \in \mathbb{N}^\mathbb{N}$,

$$V_{f,n} := \{g \in \mathbb{N}^\mathbb{N} : f(i) = g(i), i = 1, \dots, n\}$$

   . Let's see that the $\mathcal{B} = (V_{f,n})_{f \in \mathbb{N}^\mathbb{N}, n \in \mathbb{N}}$ is a basis in $\mathbb{N}^\mathbb{N}$. Clearly it covers the space, now for $f, h \in \mathbb{N}^\mathbb{N}, n, m \in \mathbb{N}$, take $V_{f,n} \cap V_{h,m}$ and note that this set it is just

$$V_{f,n} \cap V_{h,m} = \{g \in \mathbb{N}^{\mathbb{N}} : g(i) = f(i), i = 1, \ldots, n \wedge g(i) = h(i), i = 1 \ldots, m\}.$$

Without loss of generality, assume $n \geq m$ and let w be in that intersection. Then $w(i) = f(i)$ for all $i = 1, \ldots, n$ and $w(i) = h(i)$ for all $i = 1 \ldots, m$. Since $n \geq m$, this later statement just means that $w(i) = f(i)$ for all $i = 1, \ldots, n$, as for $i = 1, \ldots, m$, f and h agrees and after that w only needs to agree with f. Consider the set $V_{w,n}$ and note that this set is the set of all functions g such that $g(i) = w(i)$ for $i = 1, \ldots, n$. Then $g \in V_{f,n} \cap V_{h,m}$.

The topology generated by this topology is the same as the product topology in $\mathbb{N}^{\mathbb{N}}$. It can be shown that this topology is metrizable under the metric

$$d(f, g) := \sum_{n=1}^{\infty} \frac{\min(|f(n) - g(n)|, 1)}{2^n}.$$

Under this metric, the space is complete, it is called the Baire space. The Baire space is a Polish space. Actually it can be proved that is homeomorphic to the irrational numbers.

Remark 2.44. For a topological space, notions as compacntess, limit point, continuous function, are defined as in metric spaces as well.

Example 2.45 (Sequentially compact space that is not compact)**.** Consider ω_1, the first uncountable ordinal and give $[0, \omega_1]$ the order topology. Note that with the order induced by the $\in$ relation one can show that the ordinal numbers are totally ordered. Under that order, $(\omega_1, \in)$ is a topological space and it has the least upper bound property as ordinals are well-ordered.

It can be shown that K is compact in the order topology if and only if K closed and bounded provided that the order satisfies the least upper bound property. Therefore $[0, \omega_1)$ is not compact. Take any sequence $(\alpha_n)_n \subseteq [0, \omega_1)$, note that this is just a countable family of countable sets. Also, note that the limit of such sequence is just $\alpha := \bigcup_{n=1}^{\infty} \alpha_n$. Being a countable union of countable sets, α itself is countable and therefore is an element in $[0, \omega_1)$, so every sequence has a convergent subsequence (even stronger), but $[0, \omega_1)$ is not compact.

3. Normed linear spaces

Remark 3.1. Here all vector spaces are going to be consider over the fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Definition 3.2. Let $(X, +, \cdot)$ be a vector space. A *norm* is a function $\|\cdot\| : X \to [0, \infty)$ such that

(1) For all $x \in X, \|x\| = 0$ if and only if $x = 0$.
(2) For all $x \in X, \alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$.
(3) For all $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a *normed* space.

Remark 3.3. Every normed space is a metric space, consider the metric $d(x, y) := \|x - y\|$, however, the converse is not true.

Definition 3.4. A normed space which is complete with respect to the metric induced by the norm is a *Banach space*.

Definition 3.5. Let X be a normed space and $(x_n)_n \subseteq X$, we are going to say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n := \sum_{j=1}^{n} x_j$ converges with respect to the norm in X. The series is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

Theorem 3.6. *Let X be a normed linear space, then X is a Banach space if and only if every absolutely convergent series in X converges in X*

Proof. Suppose X is a Banach space and let $\sum_{n=1}^{\infty}$ an absolutely convergent series. Let $m, n \in \mathbb{N}$ and without loss of generality, assume that $n > m$. Consider the difference

$$\|s_n - s_m\| = \left\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \right\| = \left\| \sum_{j=m}^{n} x_j \right\| \leq \sum_{j=m}^{n} \|x_j\|$$

and note that the later term goes to zero as $n, m \to \infty$ as since is a convergent series in $\mathbb{R}$ is a Cauchy sequence. So, $(s_n)_n$ is a Cauchy sequence in X and since X is a Banach space, it converges.

Suppose now that every convergent series in X is convergent and let $(x_n)_n$ be a Cauchy sequence in X, therefore for all $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that for all $n, m \geq N_k$, $\|s_n - s_m\| < 2^{-k}$. Without loss of generality, assume that the $(N_k)_k$ is an increasing function.

Now, consider the sequence $\sum_{k=1}^{\infty}(x_{N_{k+1}} - x_{N_k})$ and note that the later condition entails that this sequence is absolutely convergent and then it converges to $x \in X$. Now, note that the sequence

$$x_{N_k} = x_{N_1} + (x_{N_2} - x_{N_1}) + \cdots + (x_{N_k} - x_{N_{k-1}})$$

converges to such $x \in X$. Therefore,

$$\|x_n - x\| \leq \|x_n - x_{N_k}\| + \|x_{N_k} - x\|$$

and note that the first term goes to zero since our sequence was a Cauchy sequence and the second term goes to zero by the convergence of (x_{N_k}). □

Definition 3.7. Let X be a vector space and $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X, they are said to be *equivalent* if there are $\alpha, \beta > 0$ such that for all $x \in X$,

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1.$$

Theorem 3.8. *All the norms on $\mathbb{R}^n$ are equivalent.*

Proof. We are going to prove that any norm is equivalent to the maximum norm, $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$. Let $\|\|$ be any norm over $\mathbb{R}^n$ and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and note that $x = \sum_{i=1}^{n} x_i e_i$. Therefore,

$$\|x\| = \left\| \sum_{i=1}^{n} x_i e_i \right\| \leq \sum_{i=1}^{n} |x_i| \|e_i\| \leq n \max\{\|e_i\|\} \|x\|_\infty.$$

□

Call $\beta = n \max\{\|e_i\|\}$. Consider now the following function, $f : (\mathbb{R}^n, \|\cdot\|_\infty) \to \mathbb{R}, f(x) = \|x\|$. Note that the previous inequality implies that

$$|f(x) - f(y)| = |\|x\| - \|y\|| \leq \|x - y\| \leq \beta\|x - y\|_\infty$$

and therefore f is a continuous function. Consider the unit closed ball in $\mathbb{R}^n$ and remember that by the Heine-Borel property, $K = \overline{B_1(0)}$ is compact as is closed and bounded. Therefore, the restriction of f to K has a minimum and a maximum in K. Let $k \in K$ be the minimum of f over K and note that, for all $x \in K$, $f(k) \leq f(x)$.

Now, for all $x \neq 0$, take the element $x/\|x\|_\infty$, this element is clearly an element in K, therefore, $f(k) \leq f\left(\frac{x}{\|x\|}\right)$. But this just means the following,

$$\|k\| \leq \left\| \frac{x}{\|x\|_\infty} \right\| \Rightarrow \|k\| \|x\|_\infty \leq \|x\|.$$

Since x is arbitrary, the two norms are indeed equivalent.

Lemma 3.9. Let X be a normed linear space, then

(1) Every finite dimensional subspace of X is closed.
(2) If X is a Banach space and $Y \subseteq X$ is a closed subspace, then Y is a Banach space.

(1) Let V be finite dimensional, let $n := \dim(V)$. Now, pick a basis $v_1, \ldots, v_n$ for V and define the following mapping, $T : V \to \mathbb{R}^n$, $T(v_i) := e_i$. Note that T can be linearly extended and therefore V can be seen as $\mathbb{R}^n$. But by our previous result, all norms in $\mathbb{R}^n$ are equivalent, then V itself must be complete and then is closed.
(2) If Y is closed, contains all its limit points, then every convergent sequence in Y must have its limit in Y and Y is a Banach space.

Clearly the real numbers are a Banach space, the complex numbers as well. Any $\mathbb{R}^n$ is a Banach space. However, let's try to deal with more interesting examples.

Example 3.10. Let X be a compact metric space and consider

$$C(X) := \{f : X \to \mathbb{R} : f \text{ is continuous}\}.$$

Define the following function over X,

$$\|f\|_\infty := \sup_{x \in X} |f(x)|$$

it is easy to see that $\| \cdot \|_\infty$ is a norm over $C(X)$. With the point-wise addition and multiplication, $C(X)$ can be seen as a vector space. Let's see that $C(X)$ is a Banach space. Let $(f_n)_n$ be a Cauchy sequence with respect to $\| \cdot \|_\infty$. Now, for all $x \in X$, consider $(f_n(x))_n$ and note that $(f_n(x))_n$ is a sequence in $\mathbb{R}$. For $m, n \in \mathbb{N}$, note that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \to 0$$

as the sequence is a Cauchy sequence. Since $\mathbb{R}$ is complete, $f_n(x)$ converges to an element $f(x) \in \mathbb{R}$, define,

$$f(x) := \lim_{n \to \infty} f_n(x)$$

we just shown that f is well defined, now we need to see that it is continuous. Let $\varepsilon > 0$, we want $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $\|f - f_n\|_\infty < \varepsilon$. Since $(f_n)_n$ is a Cauchy sequence, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon/2$. Now, note that by continuity of the norm function

$$\lim_{n \to \infty} \|f_n - f_N\| = \|f - f_N\| < \varepsilon/2$$

and hence, for all $n \geq N$,

$$\|f - f_n\|_\infty \leq \|f - f_N\| + \|f_N - f_n\| < \varepsilon.$$

So, f_n converges uniformly to f. Since the uniform limit of continuous functions is continuous, $f \in C(X)$ and $C(X)$ is a Banach space.

Example 3.11 (Sequence spaces)**.** Now, let's consider some spaces of sequences as example.

(1) $\ell_\infty(\mathbb{N})$: Consider the set of all bounded real sequences

$$\ell_\infty(\mathbb{N}) := \left\{ (x_n)_n : \sup_{n\in\mathbb{N}} < \infty \right\}.$$

Endowed with the norm $\|x_n\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$ it becomes in a Banach space. It is easy to see that the function is a norm and that $\ell_\infty(\mathbb{N})$ is a vector space, let's see that it is complete. Let $(x_n^j)_j$ be a Cauchy sequence in $\ell_\infty(\mathbb{N})$, and note that we can arrange its elements as follows:

$$\begin{matrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 & \dots \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & \dots \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & \dots \end{matrix}$$

now, note that for n fixed, $(x_n^j)_j$ is a Cauchy sequence in $\mathbb{R}$ as for $i, j \in \mathbb{N}, |x_n^i - x_n^j| \leq \|x_n^i - x_n^j\|_\infty$ and the later term goes to zero when $i, j \to \infty$, because it is a Cauchy sequence. Since $\mathbb{R}$ is complete, the coordinate sequence converges, define,

$$x_n := \lim_{j\to\infty} x_n^j.$$

Let's see that this sequence is bounded, note that for $k \in \mathbb{N}$ fixed,

$$\|(x_n)_n\|_\infty \leq \|(x_n)_n - (x_n^k)_k\| + \|(x_n^k)_k\| = \lim_{j\to\infty} \|(x_n^j)_n - (x_n^k)_k\|_\infty + \|(x_n^k)_n\|_\infty$$

and note that the limit on the right goes to zero when $j, k \to \infty$, as is a Cauchy sequence and the term on the left remains bounded, as the sequence is in $\ell_\infty(\mathbb{N})$. Therefore, the limit is bounded.

Finally, by using again the continuity of the norm, we can show that the convergence to $(x_n)_n$ is uniform. Note that, for $k \in \mathbb{N}$

$$\|(x_n)_n - (x_n^j)_n\|_\infty = \lim_{i\to\infty} \|(x_n^i)_n - (x_n^j)_n\|_\infty$$

and the later term goes to zero when $i, j \to \infty$, as the sequence is a Cauchy sequence. Therefore, $(x_n)_n$ is actually the uniform limit of the sequence and $\ell_\infty(\mathbb{N})$ is a Banach space.

(2) For $p \geq 1, \ell_p(\mathbb{N})$: Define the following space

$$\ell_p(\mathbb{N}) := \{(x_n)_n : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Define the function

$$\|(x_n)_n\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Here is not so straightforward to see that $\ell_p(\mathbb{N})$ is a vector space and that $\|\cdot\|_p$ is a norm. Let's start seeing that $\ell_p(\mathbb{N})$ is a vector space. Let $\alpha \in \mathbb{K}, (x_n)_n \in \ell_p$, then

$$\sum_{n=1}^{\infty} |\alpha x_n|^p = |\alpha|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$$

meaning if $(x_n)_n \in \ell_p$, then $\alpha(x_n)_n \in \ell_p$ as well. Now, let's see that ℓ_p is closed under sums. Note that

$$|x_n - y_n|^p \leq (2\max(|x_n|, |y_n|))^p \leq 2^p(|x_n|^p + |y_n|^p)$$

which means that

$$\sum_{n=1}^{\infty} |x_n - y_n|^p \leq 2^p \sum_{n=1}^{\infty}(|x_n|^p + |y_n|^p) \leq 2^p(\|(x_n)_n\|_p + \|(y_n)_n\|_p) < \infty.$$

So, ℓ_p is a vector space. The hard part of proving that $\|\cdot\|_p$ is a norm is the triangle inequality, which follows from Minkowski's inequality, which we are going to prove later.

Now, let $(x_n^j)_j$ be a Cauchy sequence in ℓ_p, as before, note that for fixed n and $j, i \in \mathbb{N}$, one has

$$|x_n^i - x_n^j| \leq \|(x_n^i)_n - (x_n^j)_n\|_p \xrightarrow{i,j\to\infty} 0$$

as the sequence was a Cauchy sequence in ℓ_p. Since $\mathbb{R}$ is complete, for fixed n the sequence converges, define,

$$x_n := \lim_{j\to\infty} x_n^j.$$

We need to see that $(x_n)_n \in \ell_p$ and that the sequence converges to this element. Let $\varepsilon > 0$, $M \in \mathbb{N}$ and choose $i, j \in \mathbb{N}$ such that $\|(x_n^i)_n - (x_n^j)_n\|_p < \varepsilon$. Note that

$$\sum_{n=1}^{M} |x_n^i - x_n^j|^p \leq \|(x_n^i)_n - (x_n^j)_n\|_p^p < \varepsilon^p$$

if we take the limit when $j \to \infty$, we get

$$\sum_{n=1}^{M} |x_n^i - x_n|^p < \varepsilon^p$$

and if we let $M \to \infty$, we actually get $\sum_{n=1}^{\infty} |x_n^i - x_n|^p < \varepsilon^p$. So, the element $(x_n^i)_n - (x_n)_n \in \ell_p$, but since ℓ_p is a vector space, the element $(x_n)_n = ((x_n)_n - (x_n^i)_n) + (x_n^i)_n \in \ell_p$. Convergence follows from the last inequality, as ε is arbitrary small.

(3) Subspaces of ℓ_∞: Consider the following subspaces of $\ell_\infty(\mathbb{N})$,

$$\begin{aligned}
d &:= \{x = (x_n)_n : x_n \neq 0 \text{ for all but finitely many } n\},\\
c_0 &:= \{x = (x_n)_n : x_n \to 0\},\\
c &:= \{x = (x_n)_n : x_n \text{ converges}\}.
\end{aligned}$$

Note that we have the following inclusions $d \subseteq c_0 \subseteq c \subseteq \ell_\infty$. c_0 and c are Banach spaces as subspaces of ℓ_∞, while d is dense in ℓ_∞, in particular, is not closed.

Let's see that c_0 and c are closed. Let $(x_n^j)_j$ be a sequence in c_0 converging to some $(x_n)_n \in \ell_\infty$, we want to see that $x_n \to 0$. Note that for all $j \in \mathbb{N}$

$$|x_n| \leq |x_n - x_n^j| + |x_n^j| \leq \|(x_n)_n - (x_n^j)_j\|_\infty + |x_n^j|.$$

Let $\varepsilon > 0$ be, since $(x_n^j)_j$ converges to $(x_n)_n$, there exists $N_1 \in \mathbb{N}$ such that for all $j \geq N_1$, $\|(x_n)_n - (x_n^j)_j\|_\infty < \varepsilon/2$. On the other hand, since for j fixed, $x_n^j \to 0$ for all $j \in \mathbb{N}$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $|x_n^j| < \varepsilon/2$. Take $N := \max(N_1, N_2)$, so for all $n, j \geq N$, $|x_n| < \varepsilon$.

The idea for c should be very similar, let $(x_n^j)_j$ be a sequence in c and let $(x_n)_n$ be its limit. For all $n \in \mathbb{N}$, $(x_n^j)_j$ is a convergent sequence in $\mathbb{R}$, call the limit y_n. Note that

$$|y_n - y_m| = \lim_{j\to\infty} |y_n - x_m^j| = \lim_{i\to\infty} \lim_{j\to\infty} |x_n^i - x_m^j| \leq \lim_{i\to\infty} \lim_{j\to\infty} \|(x_n^i)_i - (x_n^j)_j\|_\infty$$

and the later limit goes to zero, because the sequence was convergent and in particular a Cauchy sequence, so $(y_n)_n$ is convergent, call it y its limit.

We want to show that x_n converges to y. In order to do that, note that

$$\begin{aligned} |x_n - y| &\leq |x_n - x_n^j| + |x_n^j - y| \leq \|(x_n)_n - (x_n^j)_j\|_\infty + \lim_{k\to\infty} |x_n^j - y_k| \\ &= \|(x_n)_n - (x_n^j)_j\|_\infty + \lim_{k\to\infty} \lim_{i\to\infty} |x_n^j - x_k^i|. \end{aligned}$$

Note that the later term goes to zero when $i, j \to \infty$ as the sequence is a Cauchy sequence and the first goes to zero when $j \to \infty$ by convergence. Therefore, $x_n \to y$, as we wanted.

Finally, to see that d is dense in ℓ_∞, let $x = (x_n)_n$ be any element in ℓ_∞, we need a sequence $(x_n^j)_j$ in d such that $x_n^j \to x$. So, define the sequence

$$x_n^j := (x_1, x_2, \ldots, x_j, 0, 0, \ldots,)$$

note that for all $j \in \mathbb{N}$, $x_n^j \in d$ and the sequence $(x_n^j)_j - (x_n)_n$ converges to zero.