# SEISMIC VELOCITY ESTIMATION FROM TIME MIGRATION

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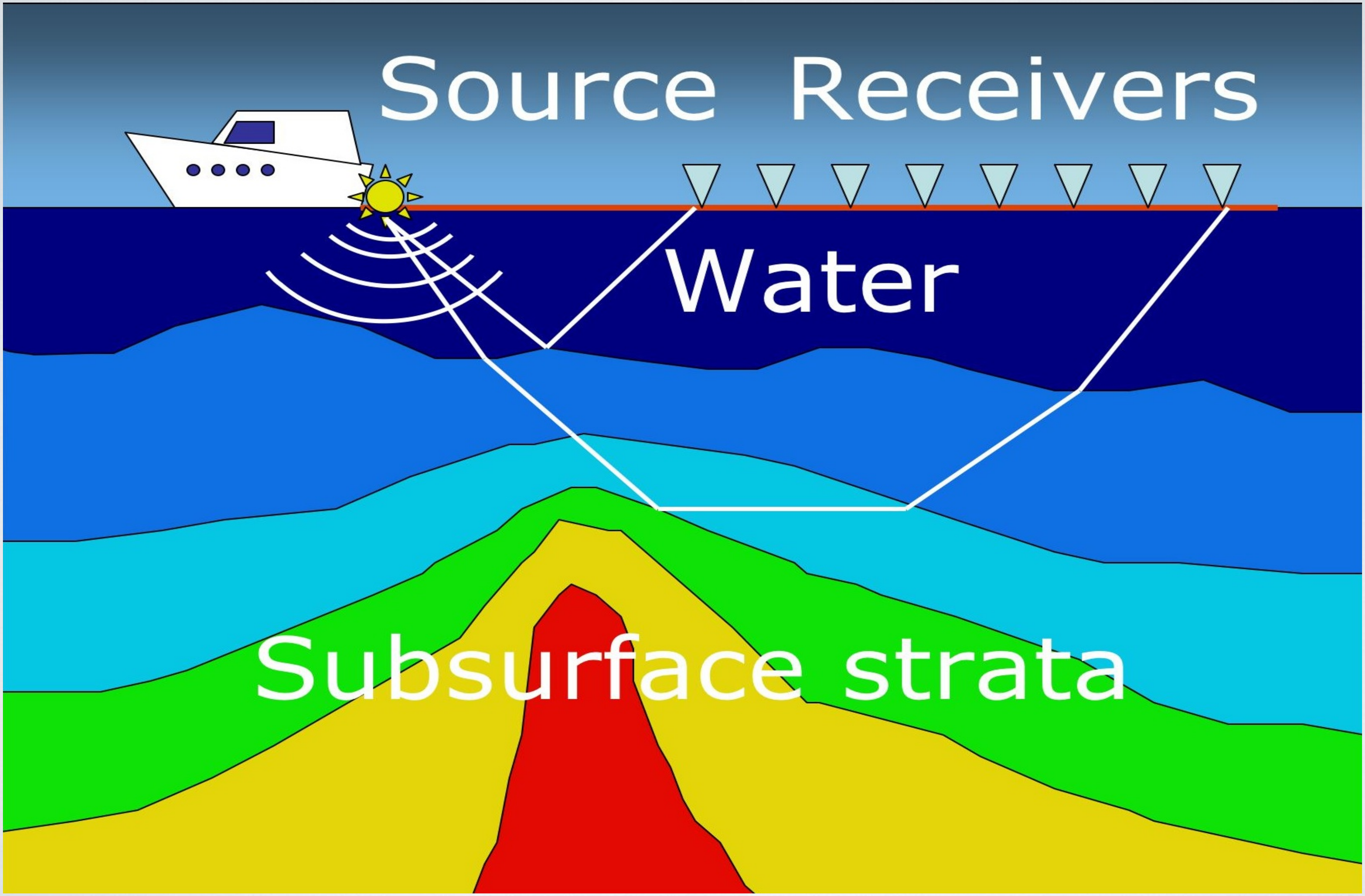
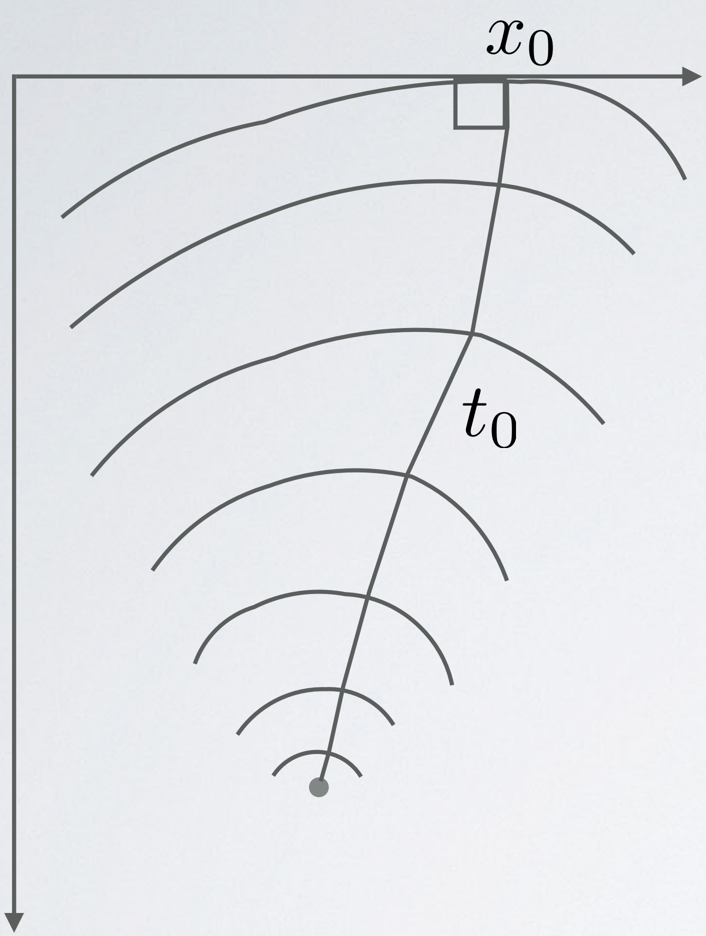
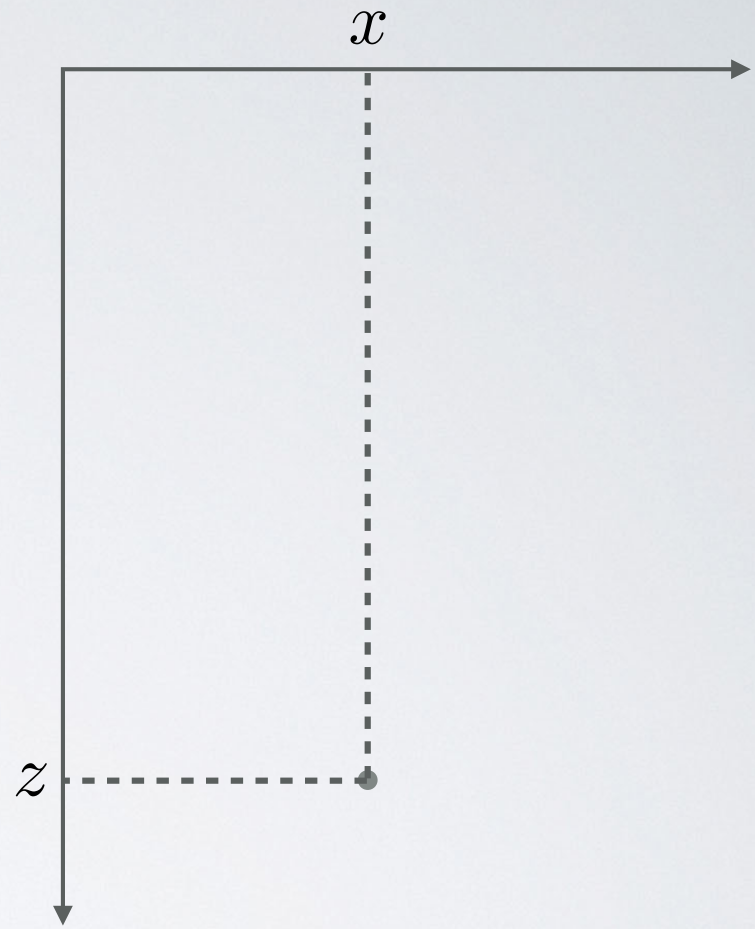
# Seismic data



# Time coordinates vs depth coordinates




$(x_0, t_0)$ = the location of
the first arrival at the surface
and and the travel time

$(x, z)$ = the cartesian
coordinates of
the subsurface location

# Time migration vs depth migration

| | Time migration | Depth migration |
|---|---|---|
| Adequate for | Mild lateral velocity variation | Arbitrary lateral velocity variation |
| Input | Seismic data $P(s,r,t)$ | Seismic data $P(s,r,t)$ and seismic velocity $v(x,z)$ |
| Output | Image in the time coordinates | Image in the depth coordinates |

# Idea

Time migration

↓

Time migration velocity

$$v_m(x_0, t_0)$$

↓

Seismic velocity and transition matrices

$$v(x,z), \quad x_0(x,z), \quad t_0(x,z)$$

↓

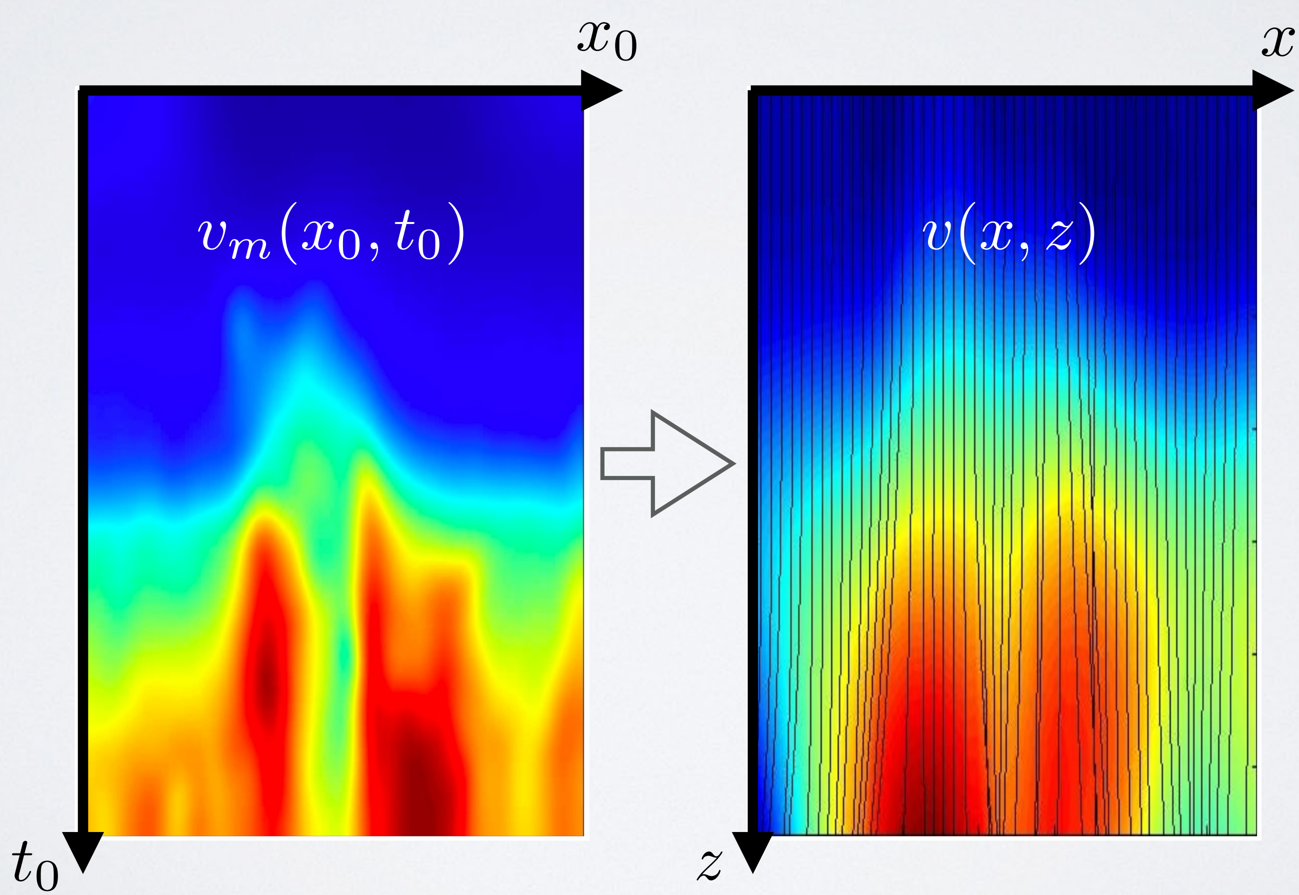
Depth migration using the seismic velocity

$$v(x,z)$$

# The Goal:

Develop a fast algorithm to convert:

- Time migration velocities to seismic velocities
- Time coordinates to depth coordinates



# The eikonal approximation

The wave equation:

$$\frac{\partial^2 P}{\partial t^2} = v^2(x,y,z)\Delta P$$

$v(x,y,z)$ = unknown seismic velocity
= the of propagation of P-waves

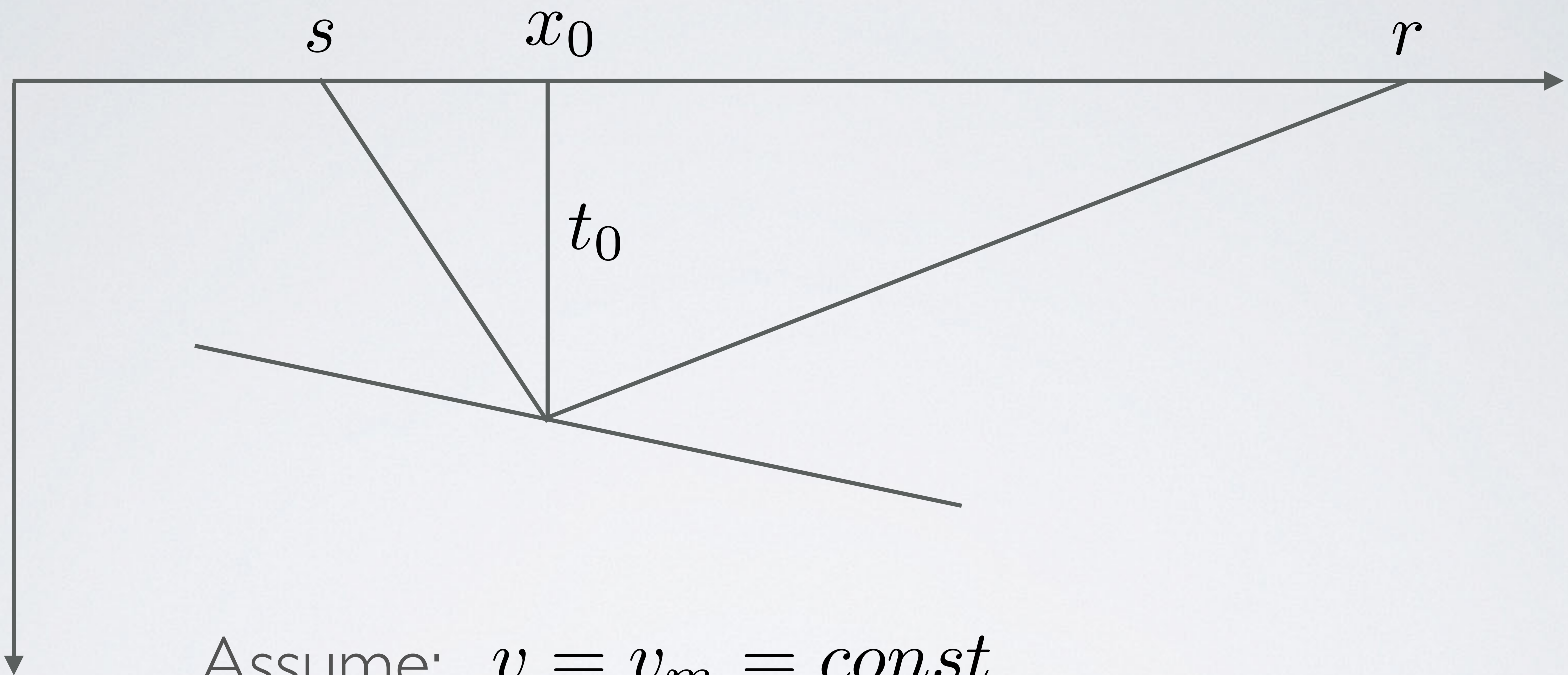
The high frequency approximation:

$$P(x,y,z) = A(x,y,z)e^{-i\omega(t-T(x,y,z))}$$

The eikonal equation:

$$|\nabla T|^2 = \frac{1}{v^2(x,y,z)}$$

# The travel time approximation

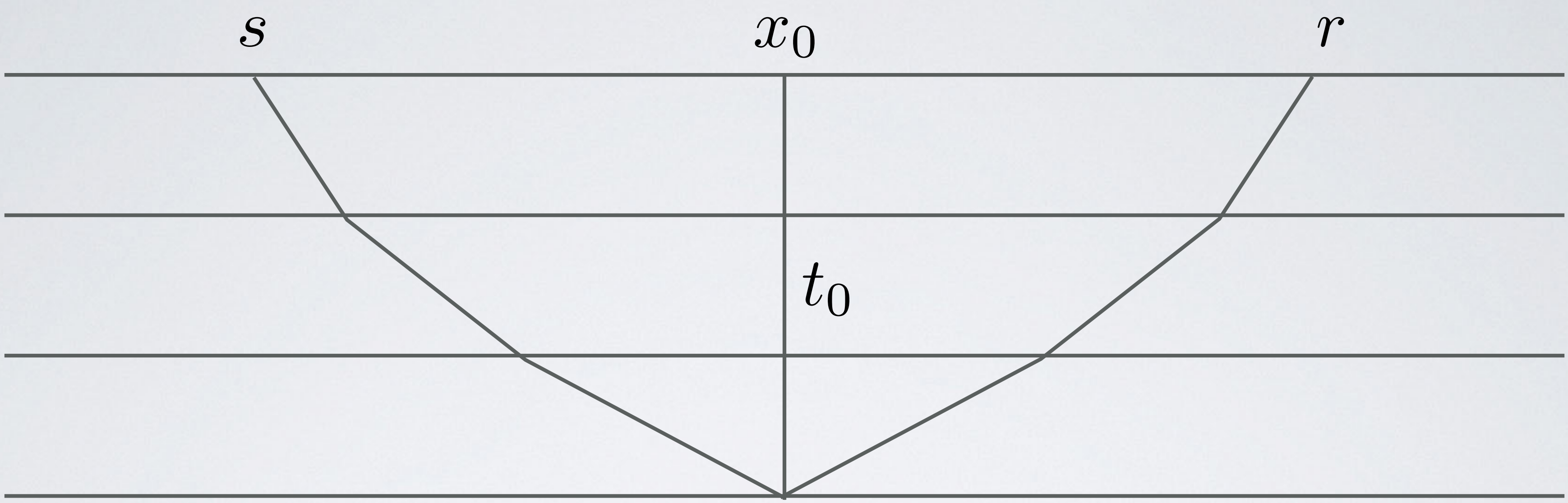



Assume: $v = v_m = const$

Using the Pythagorean theorem we get:

$$T(x_0, t_0; s, r) = \sqrt{t_0^2 + \frac{|x_0 - s|^2}{v_m^2}} + \sqrt{t_0^2 + \frac{|x_0 - r|^2}{v_m^2}}$$

# The Dix inversion

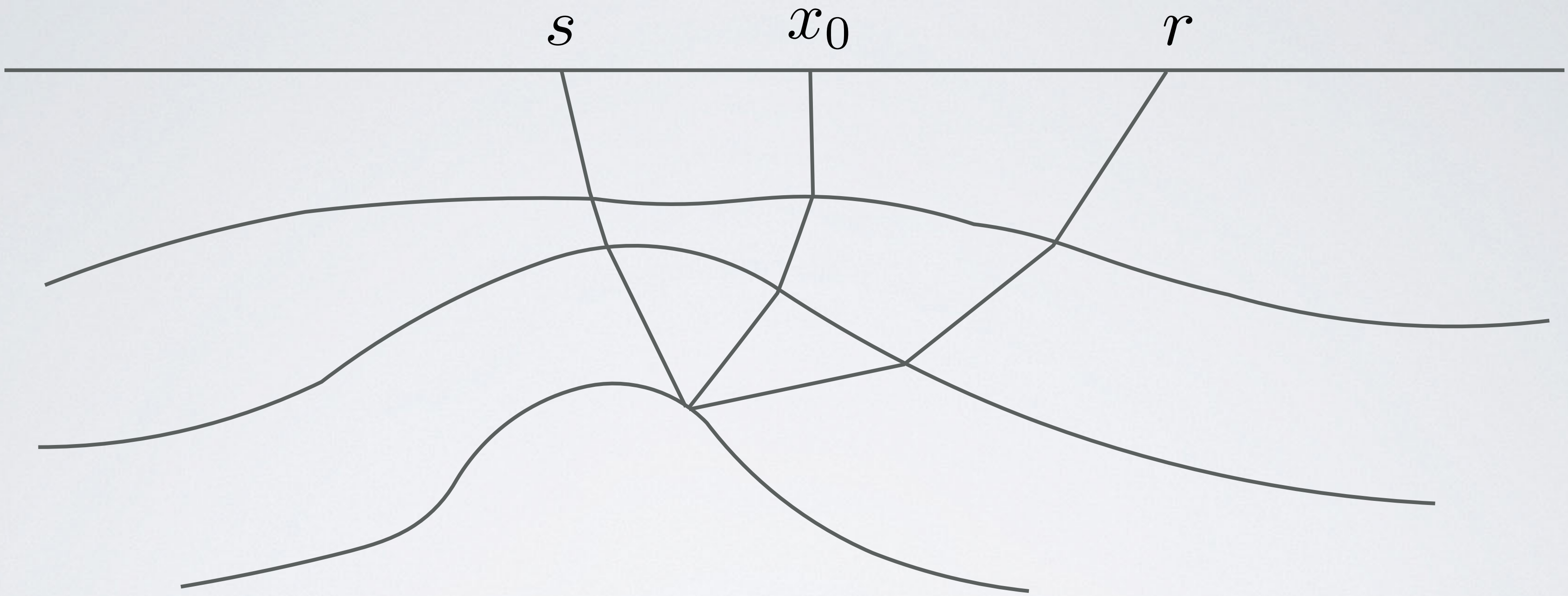



$$T(x_0, t_0; s, r) = \sqrt{t_0^2 + \frac{|x_0 - s|^2}{v_m^2(t_0)}} + \sqrt{t_0^2 + \frac{|x_0 - r|^2}{v_m^2(t_0)}}$$

The Dix formula:

$$v_{Dix}(t_0) = \sqrt{\frac{\partial}{\partial t_0}\left(t_0 v_m^2(t_0)\right)}$$

# The travel time approximation based on the Dix formula

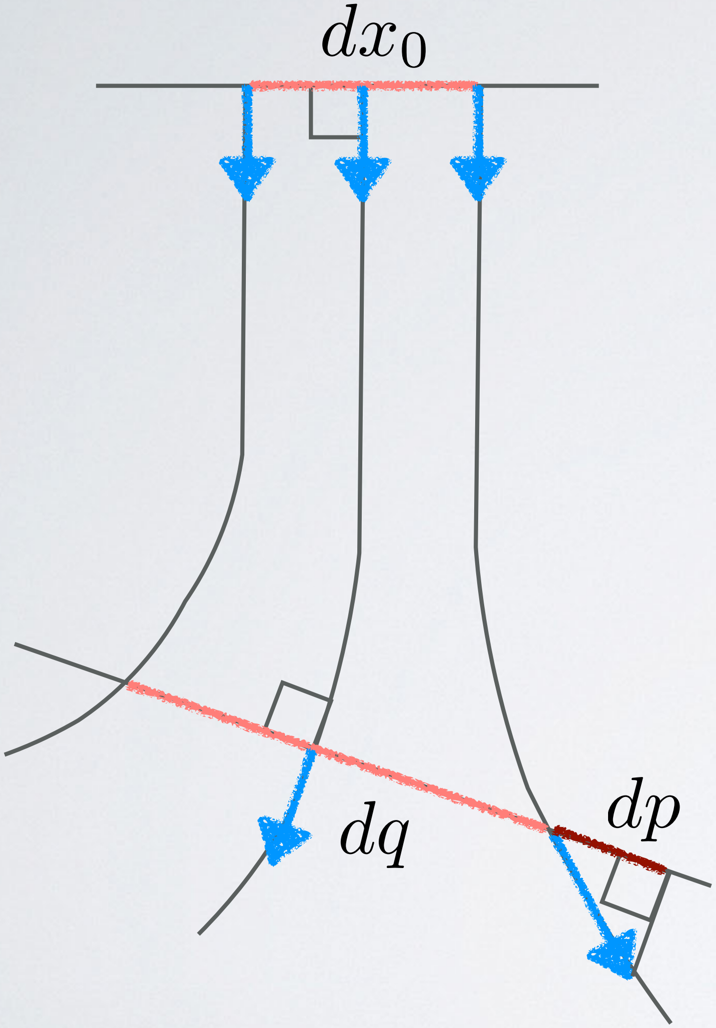



$$T(x_0, t_0; s, r) = \sqrt{t_0^2 + \frac{|x_0 - s|^2}{v_m^2(x_0, t_0)}} + \sqrt{t_0^2 + \frac{|x_0 - r|^2}{v_m^2(x_0, t_0)}}$$

$$v_{Dix}(x_0, t_0) = \sqrt{\frac{\partial}{\partial t_0}\left(t_0 v_m^2(x_0, t_0)\right)}$$

# Paraxial ray tracing

M. Popov




$$Q := \frac{dq}{dx_0}, \quad P := \frac{dp}{dx_0}$$

$|Q|$ = the geometrical spreading of the image rays

$$\frac{d}{dt_0}\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 & v^2 \\ -\frac{v_{qq}}{v} & 0 \end{bmatrix}\begin{bmatrix} Q \\ P \end{bmatrix}$$

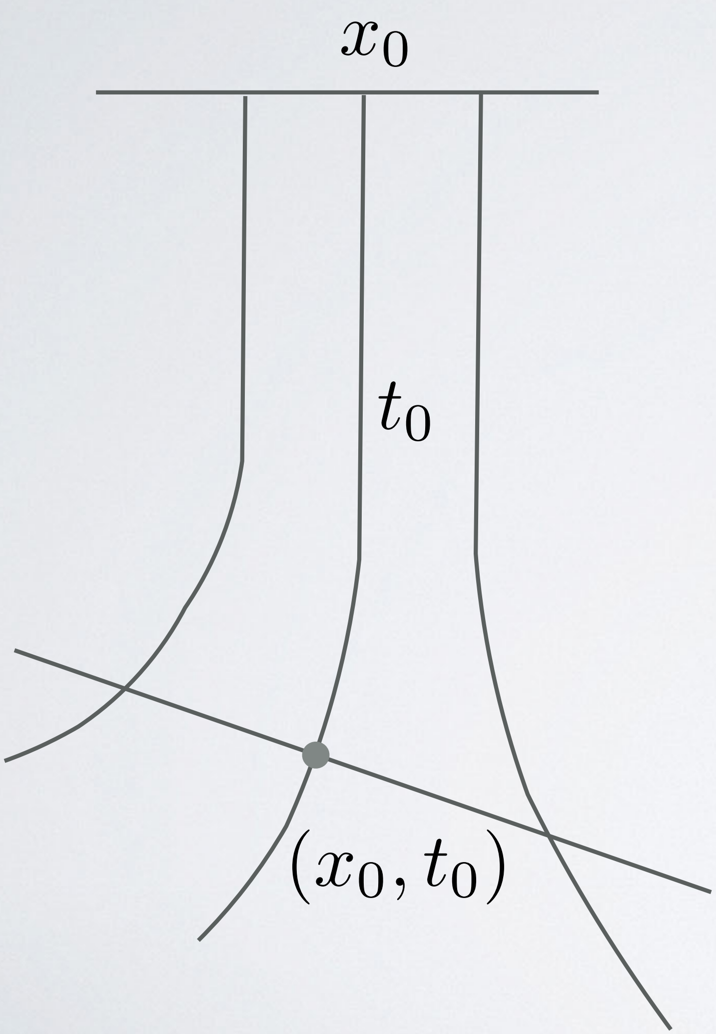
$$Q(x_0, 0) = 1, \quad P(x_0, 0) = 0$$

$$v = v(x_0, t_0)$$

$v_{qq}$ = the 2nd derivative along the line normal to the central ray

# The relationship between time migration velocities and seismic velocities

Cameron, Fomel, Sethian, 2006




2D:

$$v_{Dix}(x_0, t_0) = \frac{v(x_0, t_0)}{|Q(x_0, t_0)|}$$

where

$$v_{Dix}(x_0, t_0) = \sqrt{\frac{\partial}{\partial t_0}\left(t_0 v_m^2(x_0, t_0)\right)}$$

3D:

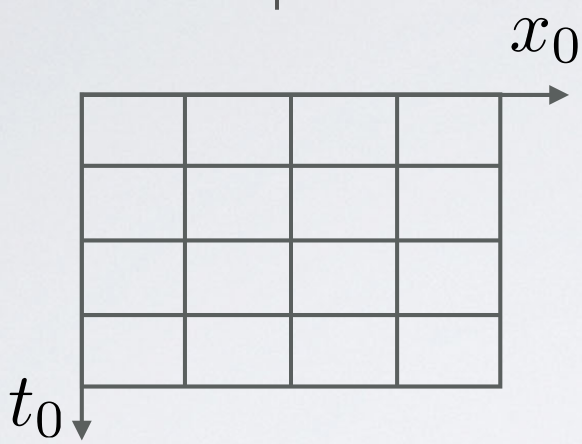
$$\frac{\partial}{\partial t_0}\left(t_0 v_m^2(x_0, t_0)\right) = v^2 \left[Q^T Q\right]^{-1}$$

where

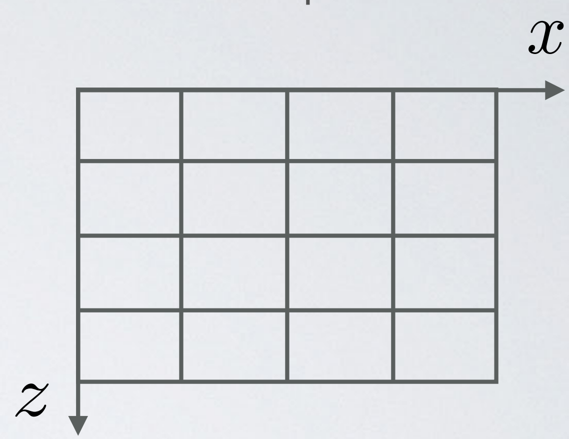
$$v = v(x_0, t_0), \quad Q = Q(x_0, t_0)$$

# The statement of the inverse problem

Input




Output




2D: $$v_{Dix}(x_0, t_0) = \frac{v(x_0, t_0)}{|Q(x_0, t_0)|} \quad \Rightarrow \quad v(x, z)$$

3D: $$f(x_0, t_0) := \frac{\partial}{\partial t_0}\left(t_0 v_m^2(x_0, t_0)\right)$$

$$\det f(x_0, t_0) = v^4 (\det Q)^{-2} \quad \Rightarrow \quad v(x, y, z)$$

# PDE for Q in 2D

Cameron, Fomel, Sethian, 2008

Notation: $f(x_0, t_0) := v_{Dix}(x_0, t_0)$

Look at $\begin{cases} \frac{\partial Q}{\partial t_0} = v^2 P \\ \frac{\partial P}{\partial t_0} = -\frac{v_{qq}}{v} Q \end{cases}$ and $v = fQ$

Since $Q = \dfrac{dq}{dx_0}$ we have: $\dfrac{d}{dq} = Q^{-1} \dfrac{d}{dx_0}$

$$\left(\frac{Q_{t_0}}{Q^2 f^2}\right)_{t_0} = -\frac{1}{fQ}\left(\frac{(fQ)_{x_0}}{Q}\right)_{x_0}$$

$$Q(x_0, 0) = 1, \quad Q_{t_0}(x_0, 0) = 0$$

# The reconstruction scheme

Dix velocities $f(x_0, t_0)$

A finite difference scheme

A Chebyshev spectral method

The geometrical spreading $Q(x_0, t_0)$

Seismic velocities in the time coordinates

$$v(x_0, t_0) = f(x_0, t_0) Q(x_0, t_0)$$

Time-to-depth conversion

Transition matrices from time- to depth coordinates

$$x_0(x, z), \quad t_0(x, z)$$

Seismic velocities in the depth coordinates

$$v(x, z)$$

# Cauchy problem for elliptic equation

Notation change: $t_0 \to t, \quad x_0 \to x$

Variable change: $y = -\dfrac{1}{Q}$

The PDE for Q becomes: $$\left(\frac{y_t}{f^2}\right)_t = \frac{y}{f}\left(\left(\frac{f}{y}\right)_x y\right)_x$$

The expanded form: $$\boxed{\frac{y_{tt}}{f^2}} - 2\frac{y_t f_t}{f^3} = y\frac{f_{xx}}{f} - y_x\frac{f_x}{f} \boxed{- y_{xx}} + \frac{y_x^2}{y}$$

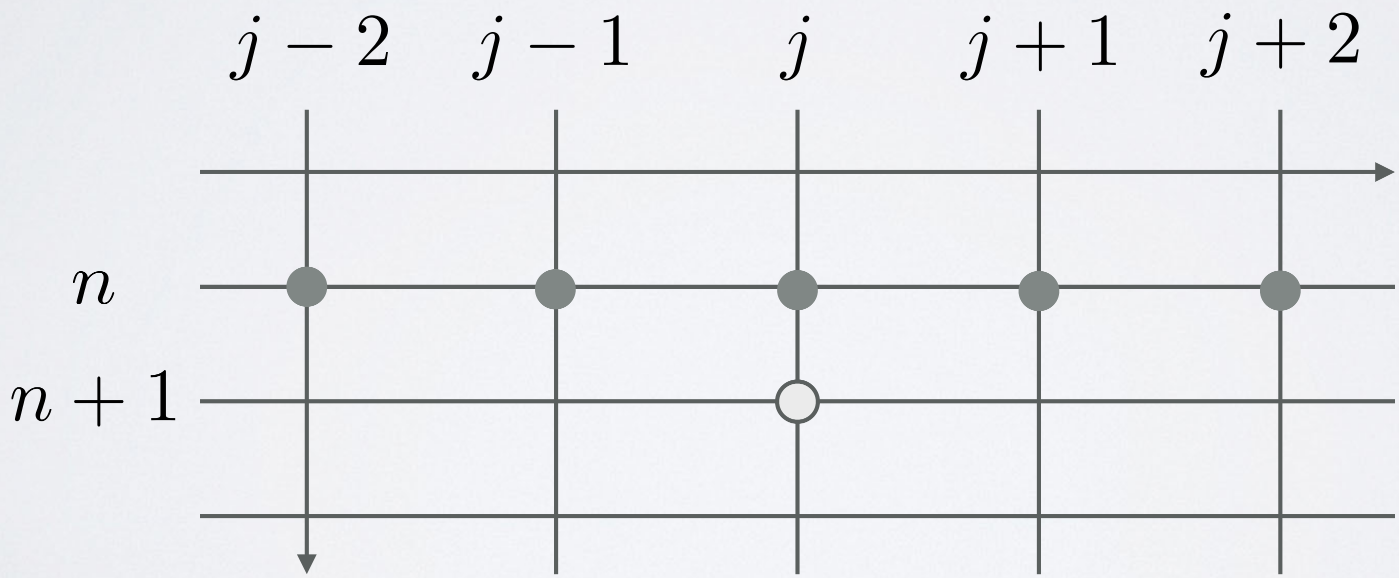
Initial conditions: $y(x,0) = -1, \quad y_t(x,0) = 0$

Boundary conditions: $y(0,t) = y(L,t) = -1$

# The finite difference scheme

$$P_j^{n+1} = \frac{P_{j+1}^n + P_{j-1}^n}{2} - \frac{\Delta t}{4\Delta x^2}\frac{1}{(fQ)_j^n}\left(\frac{(fQ)_{j+2}^n - (fQ)_j^n}{Q_{j+1}^n} - \frac{(fQ)_j^n - (fQ)_{j-2}^n}{Q_{j-1}^n}\right)$$

$$-\frac{1}{Q_j^{n+1}} = -\frac{1}{Q_j^n} + \frac{\Delta t}{2}\left((f_j^n)^2 P_j^n + (f_j^{n+1})^2 P_j^{n+1}\right)$$



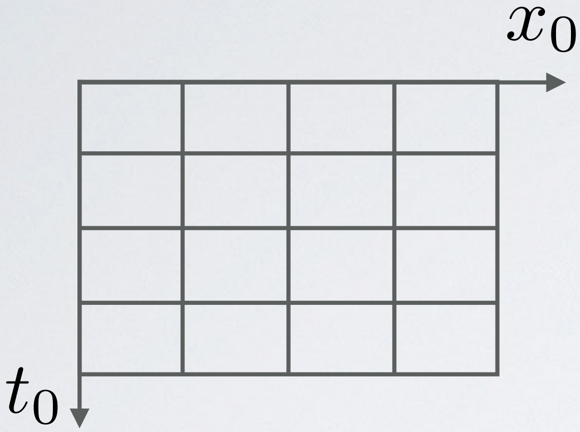
# The Chebyshev spectral method

$$\left(\frac{y_t}{f^2}\right)_t = \frac{y}{f}\left(\left(\frac{f}{y}\right)_x y\right)_x$$

- Choose the number of Chebyshev points $N$
- Choose the number of Chebyshev polynomials for function evaluation $N_{eval}$
- Interpolate the input data $f(x,t)$ at the Chebyshev points
- Solve the PDE to find $y$ at the Chebyshev points
- Compute $Q(x,t) = -1/y(x,y)$ on the regular grid

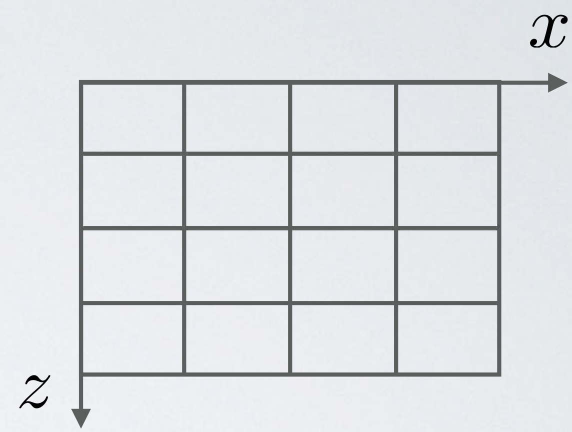
# Time-to-depth conversion

Input




$$v(x_0, t_0)$$

Output




$$v(x, z), \quad x_0(x, z), \quad t_0(x, z)$$

Motivation and a building block:
Fast Marching Method (Sethian, 1996)

$$|\nabla t_0|^2 = \frac{1}{v^2(x_0, t_0)}$$

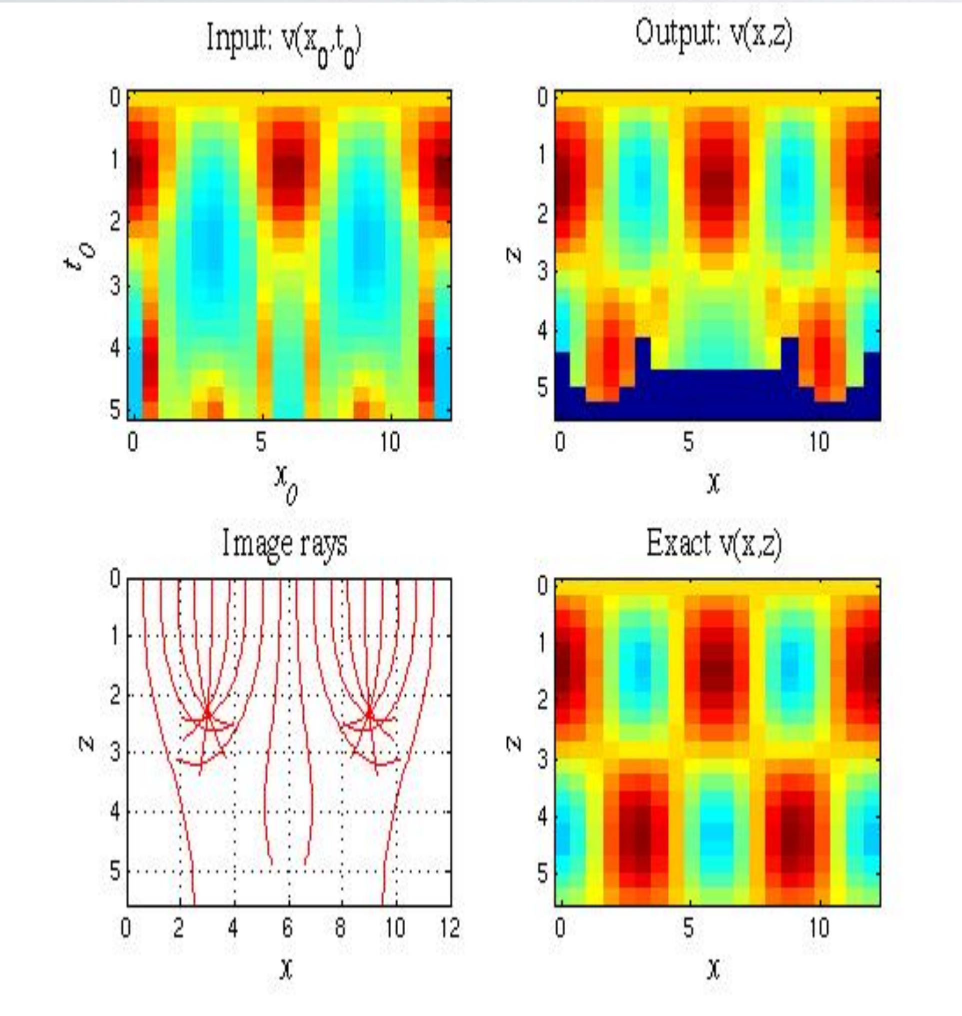
$$\nabla t_0 \cdot \nabla x_0 = 0$$

Eikonal equation with unknown RHS

Orthogonality relationship:
image rays are orthogonal
to equitime curves

# Movie: time-to-depth conversion

The input and the results




Computed points in coordinates

$(x_0, t_0)$ $(x, z)$

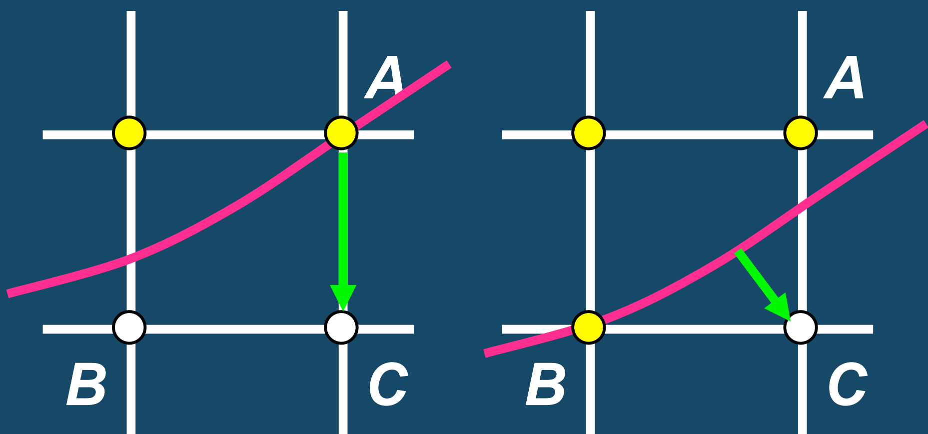


The points are computed
in the order of increase of $t_0$

# An issue

## Fast Marching Method

$$\left|\nabla t_0\right|^2 = \frac{1}{v^2(x,z)}$$



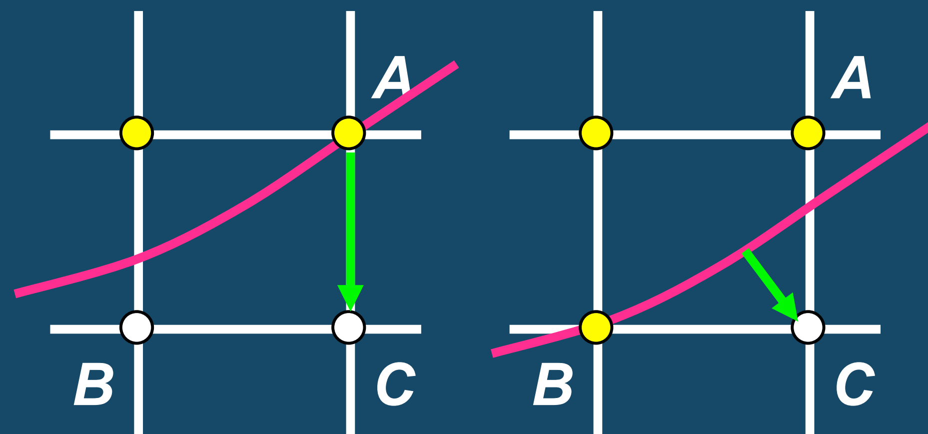

$T_1(C)$ $T_2(C)$

$T_1(C) \geq T_2(C)$

## Time-to-depth conversion

$$\left|\nabla t_0\right|^2 = \frac{1}{v^2(x_0(x,z),t_0(x,z))}$$

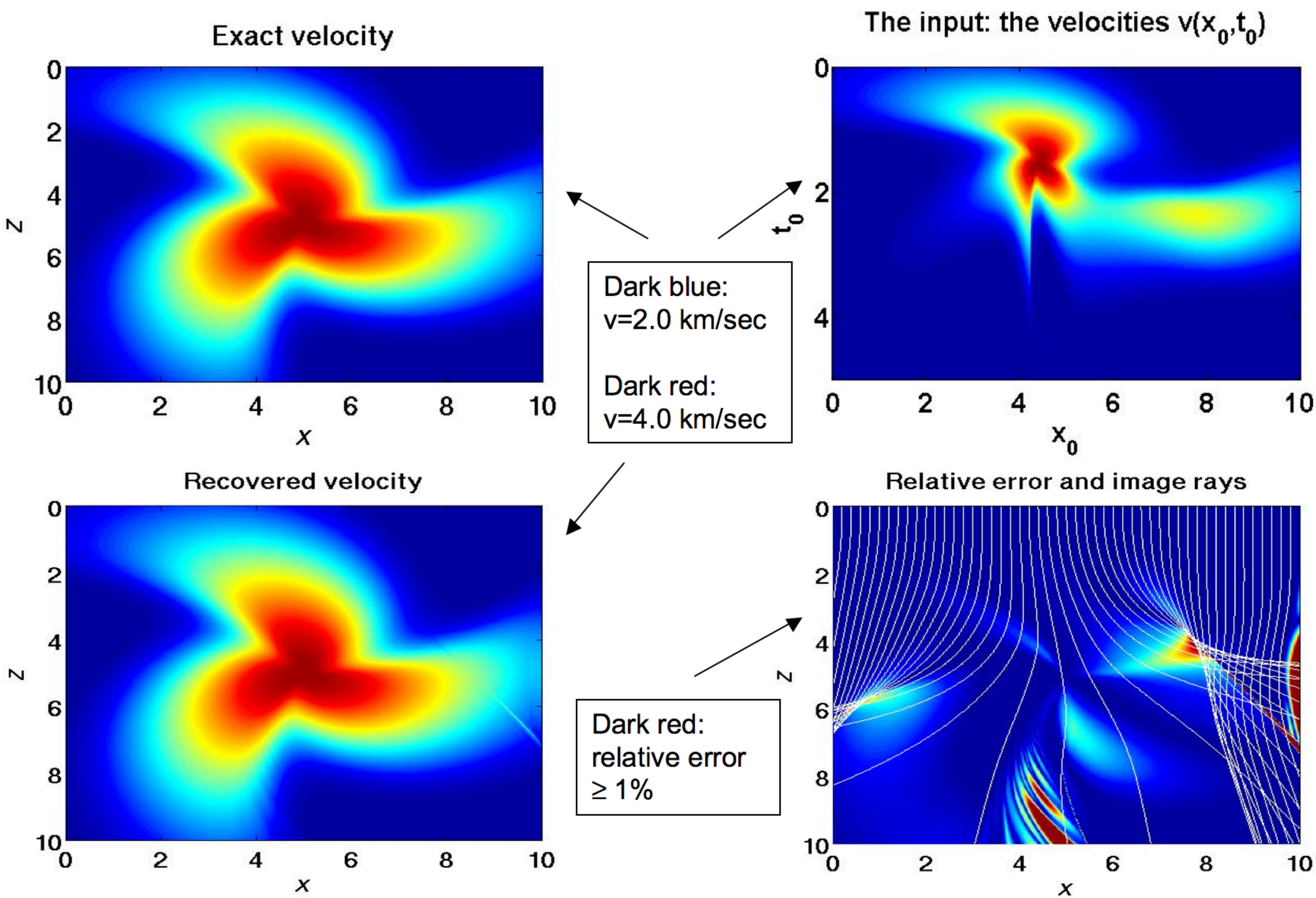
$$\nabla t_0 \cdot \nabla x_0 = 0$$



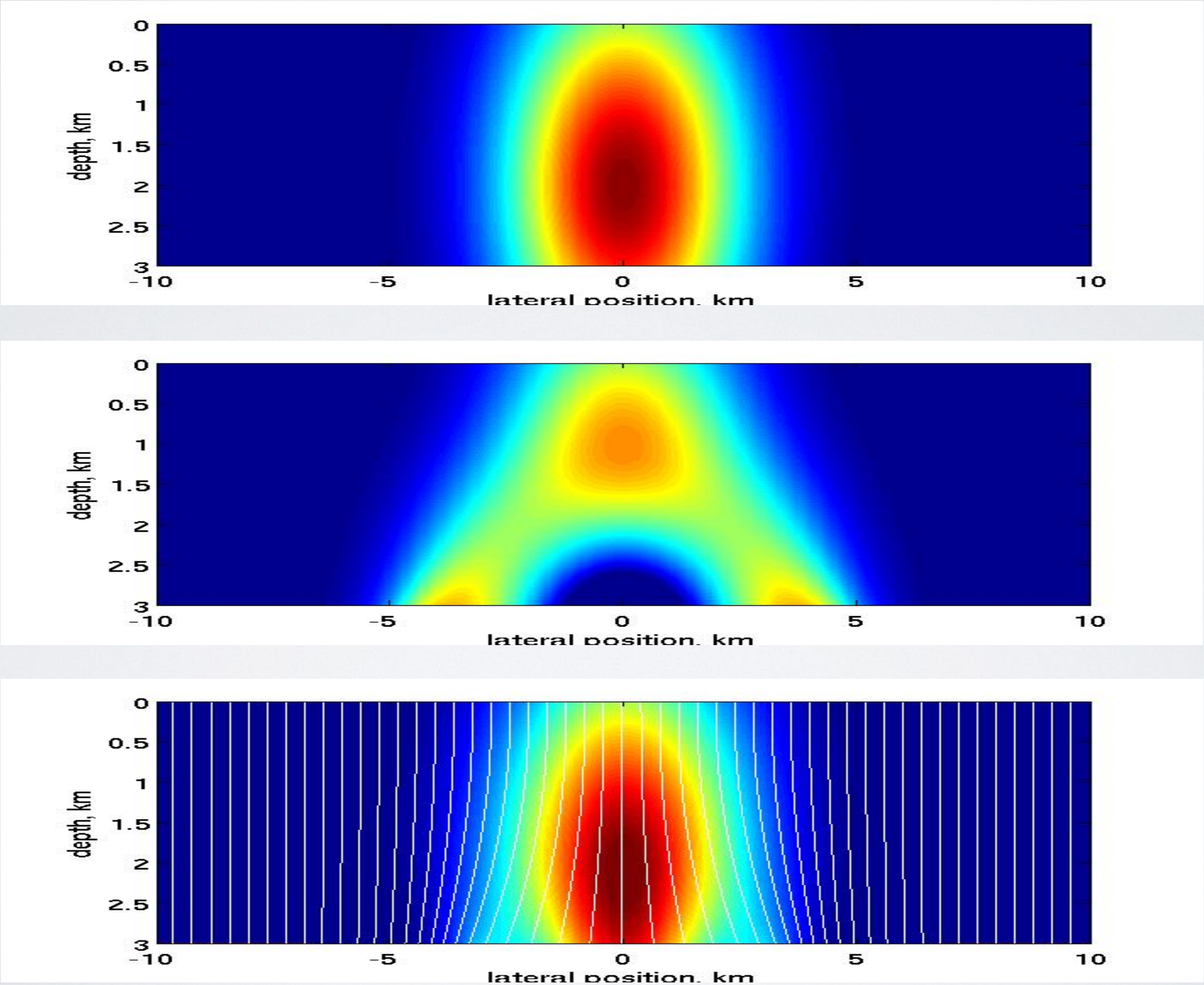

$T_1(C)$ $T_2(C)$

*Not necessarily* $T_1(C) \geq T_2(C)$

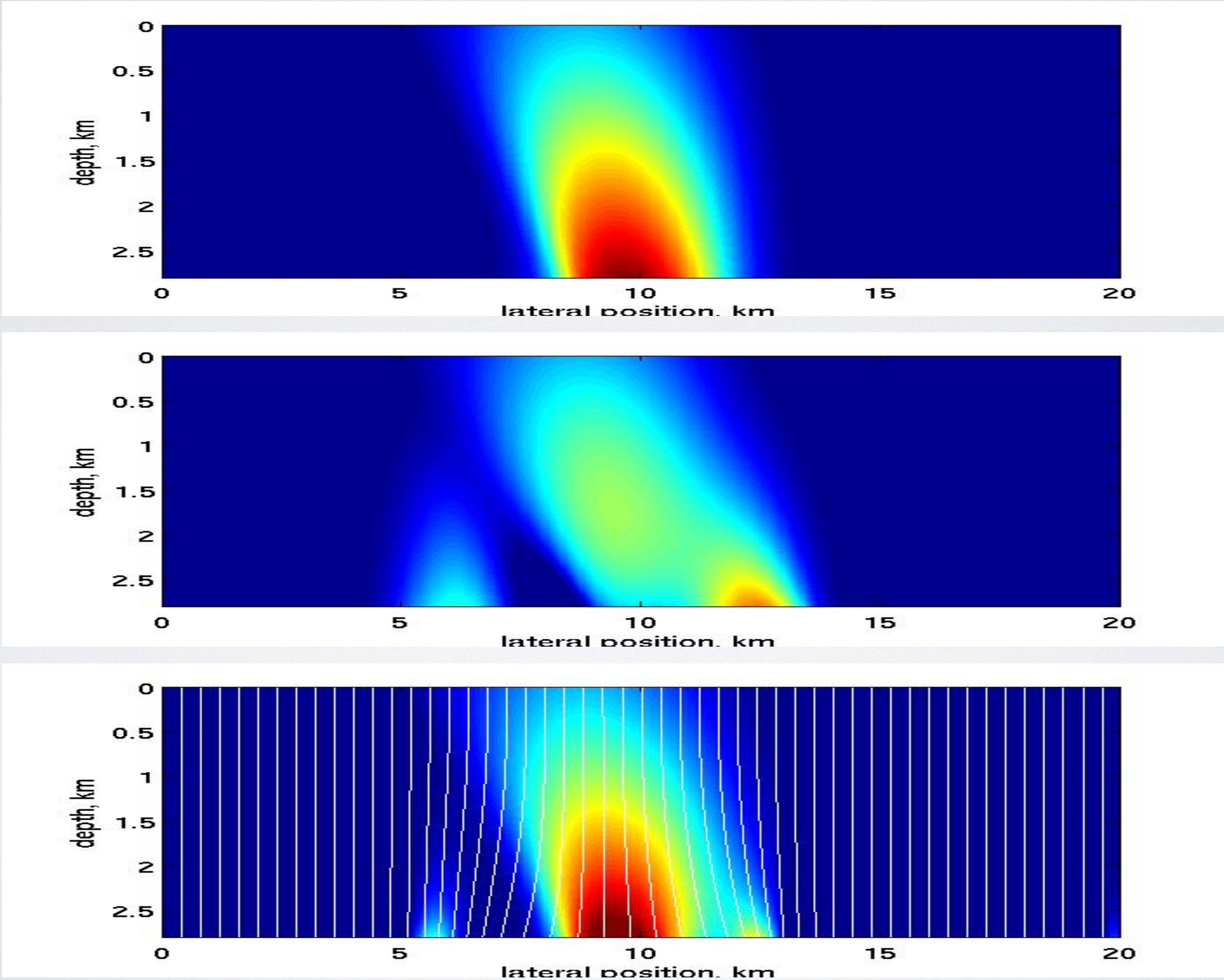
# Example 1. Time-to-depth conversion.



# 2D example: Gaussian anomaly



# 2D example: asymmetric anomaly



# Why does this work?

- Special input corresponding to a positive finite velocity
- Special initial conditions corresponding to the image rays
- Our finite difference scheme damps high harmonics.
- High harmonics are truncated in the Chebyshev spectral method.
- Short enough interval of time on which we need to compute the solution, such that low harmonics do not grow significantly.

# Special input

**Claim 1** *Consider the following initial and boundary value problem for equation*

$$\frac{y_{tt}}{f^2} - 2\frac{y_t f_t}{f^3} = y\frac{f_{xx}}{f} - y_x\frac{f_x}{f} - y_{xx} + \frac{y_x^2}{y}, \tag{1}$$

$$y(x,0) = 0, \quad y_t(x,0) = 0 \tag{2}$$

$$y(a,t) = y(b,t) = -1. \tag{3}$$

*Suppose the function $f$ in Eq. (1) is analytic and satisfies the following conditions:*

1. *$f(x,t)$ is independent of $t$;*
2. *$f(x)$ is bounded: $0 < m \leq f(x) \leq M$;*
3. *$f_{xx} \leq 0$ on $(a,b)$ and $f_{xx}(a) = f_{xx}(b) = 0$.*
4. *$f$ and $ff_{xx}$ reach their absolute maximums at the same point $x_0 \in (a,b)$ and $f_{xx}(x_0) < 0$.*

*Then the solution to the problem (1), (2), (3) becomes zero or $-\infty$ in a finite time. This corresponds to $Q$ becoming infinitely large or zero respectively.*

# Special initial conditions

**Claim 1** *Suppose* $f(x,t) = 1$. *Consider the following initial and boundary value problem:*

$$y_{tt} = -y_{xx} + \frac{y_x^2}{y}, \quad a \leq x \leq b, \tag{1}$$

$$y(x,0) = \alpha(x), \quad y_t(x,0) = 0, \tag{2}$$

$$y(a,t) = y(b,t) = -1. \tag{3}$$

*Let* $\alpha(x)$ *be a smooth analytic function such that*

1. $-M \leq \alpha \leq -m < 0$, $\alpha(a) = \alpha(b) = -1$,
2. $\alpha$ *has an absolute maximum at a point* $x_0 \in (a,b)$ $\alpha_{xx}(x_0) < 0$;
3. $\alpha_{xx}(a) = \alpha_{xx}(b) = 0$.

*Then the solution to the problem (1)-(3) becomes zero or* $-\infty$ *in a finite time. This corresponds to* $Q$ *becoming infinitely large or zero respectively.*

# Damping high harmonics

- Write the modified equation for our finite difference scheme
- Set $f = 1, \quad y = -1$ for simplicity
- Consider a perturbed problem: $f = 1 + \delta f, \quad y = 1 + \delta y$
- Linearize the modified equation around $f = 1, \quad y = -1$ to obtain an equation for $\delta y$
- Plot the root diagram for the eigenroots of the linearized modified equation for the Fourier harmonics supported by the grid:

$$0 \leq k \leq \pi/\Delta x$$

# Analysis of the modified equation

Original equation

$$y = -\frac{1}{Q} \qquad \left(\frac{y_t}{f^2}\right)_t = \frac{y}{f}\left(\left(\frac{f}{y}\right)_x y\right)_x$$

Modified equation

$$\left(\frac{y_t}{f^2}\right)_t = \frac{y}{f}\left(\left(\frac{f}{y}\right)_x y\right)_x + \frac{\Delta x^2}{2\Delta t}\left(\frac{y_t}{f^2}\right)_{xx} - \frac{\Delta t}{2}\left(\frac{y_t}{f^2}\right)_{xx} - \Delta x^2 \frac{y}{f}\left(\frac{1}{3}v_{xxxx}y + \frac{1}{3}v_{xxx}y_x - \frac{1}{2}v_{xx}y_{xx} + \frac{1}{6}v_x y_{xxx}\right)$$

Modified equation linearized around $f = 1$ and $y = -1$

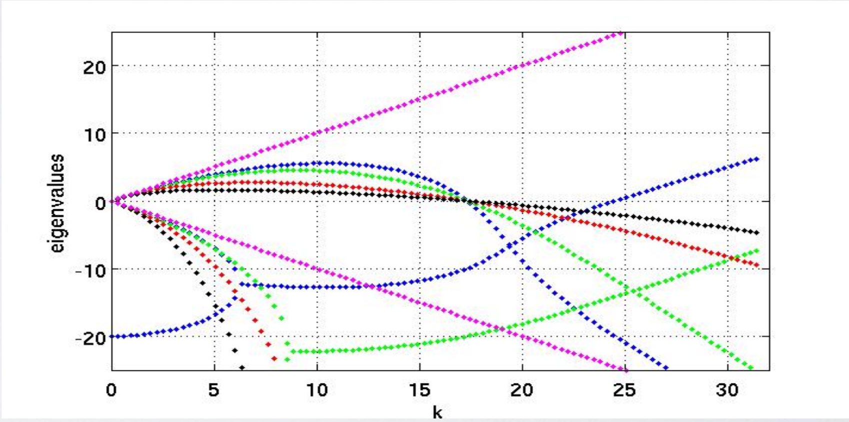
$$\delta y_{tt} + \delta y_{xx} - \frac{\Delta x^2}{2\Delta t}\delta y_{txx} + \frac{\Delta t^2}{2}\delta y_{ttt} + \frac{\Delta x^2}{3}\delta y_{xxxx} = F$$

Equation for the Fourier harmonics

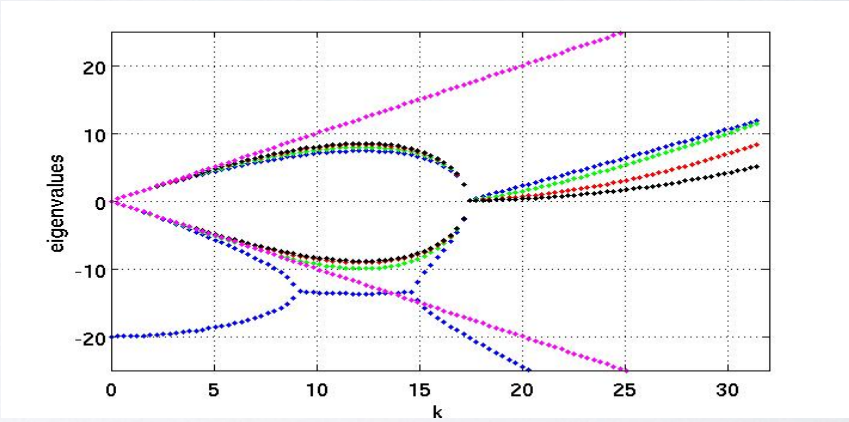
$$\frac{\Delta t^2}{2}a_{ttt} + a_{tt} + \frac{\Delta x^2}{2\Delta t}k^2 a_t + \left(\frac{\Delta x^2}{3}k^4 - k^2\right)a = \hat{F}$$

# Root diagrams

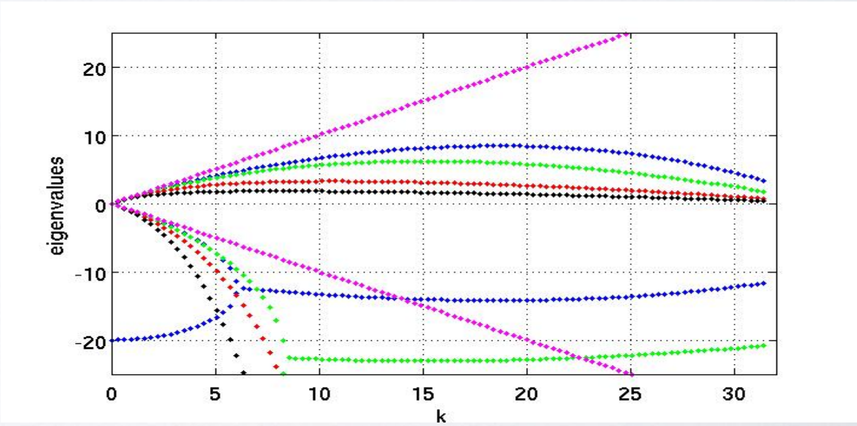
Our scheme:
"Lax-Friedrichs" averaging,
5 point stencil in space




Alternative scheme 1:
no "Lax-Friedrichs" averaging,
5 point stencil in space




Alternative scheme 2:
"Lax-Friedrichs" averaging,
3 point stencil in space



# PDE for Q in 3D

$$\left(\frac{1}{v^2}Q_t\right)_t = -\frac{1}{v}Q^{-T}\nabla\left[(\nabla v)^T Q^{-1}\right]Q$$

$$v = \sqrt[4]{\det f(\det Q)^2}$$

Input: $\sqrt{\det f}$

# 3D Example 1

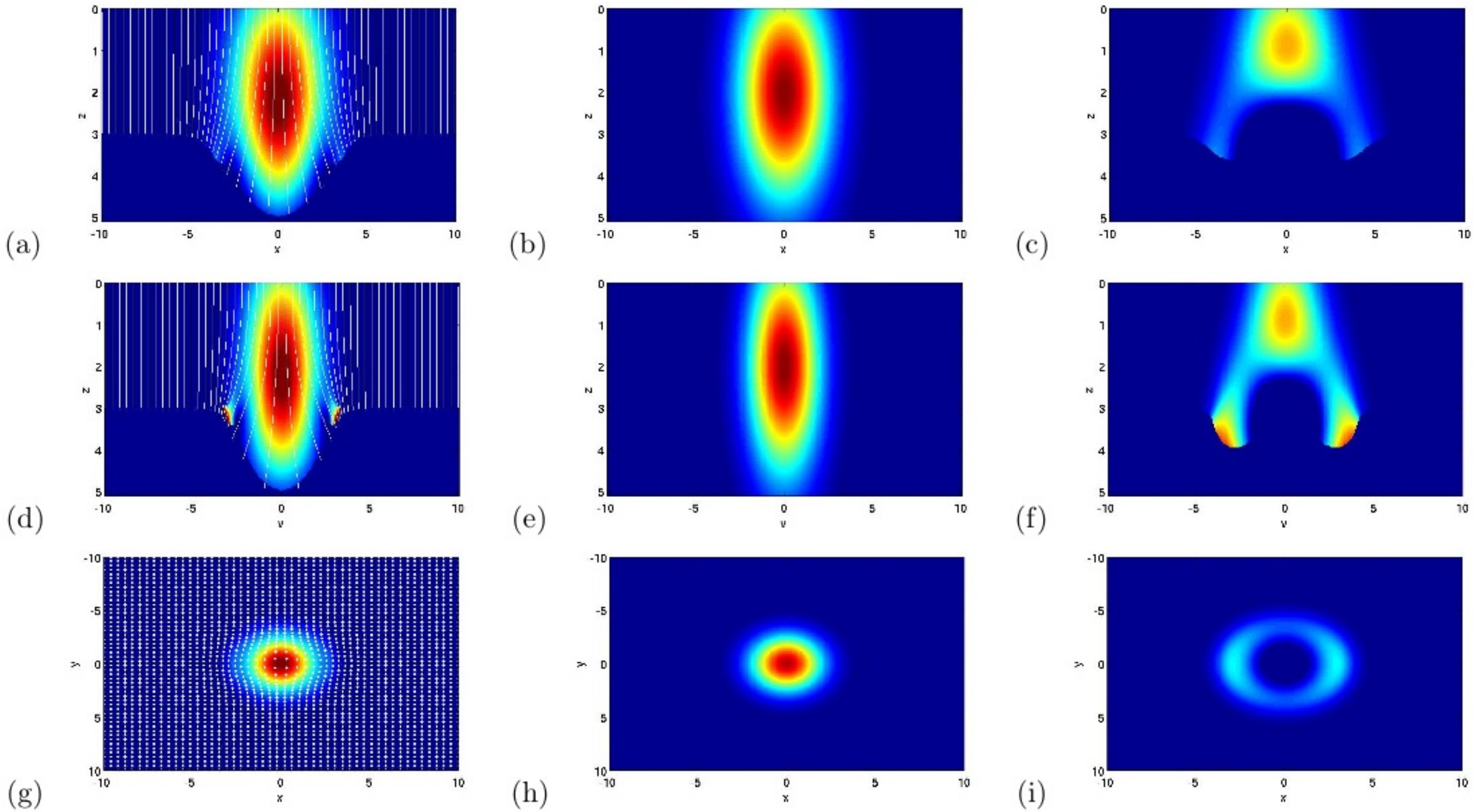



Figure 15: 3D example 1. The first row: the velocity on the vertical plane $y = 0$. The second row: the velocity on the vertical plane $x = 0$. The third row: the velocity on the horizontal plane $z = 2.55$ km. The first column ( (a),(d),(g)): the reconstructed velocity and the image rays; the second column ((b),(e),(h)): the exact velocity; the third column ((c),(f),(i)): the velocity estimate analogous to Dix inversion, converted to depth. Dark blue and dark red correspond to 2 km/s and 4 km/s respectively.

# 3D example 2

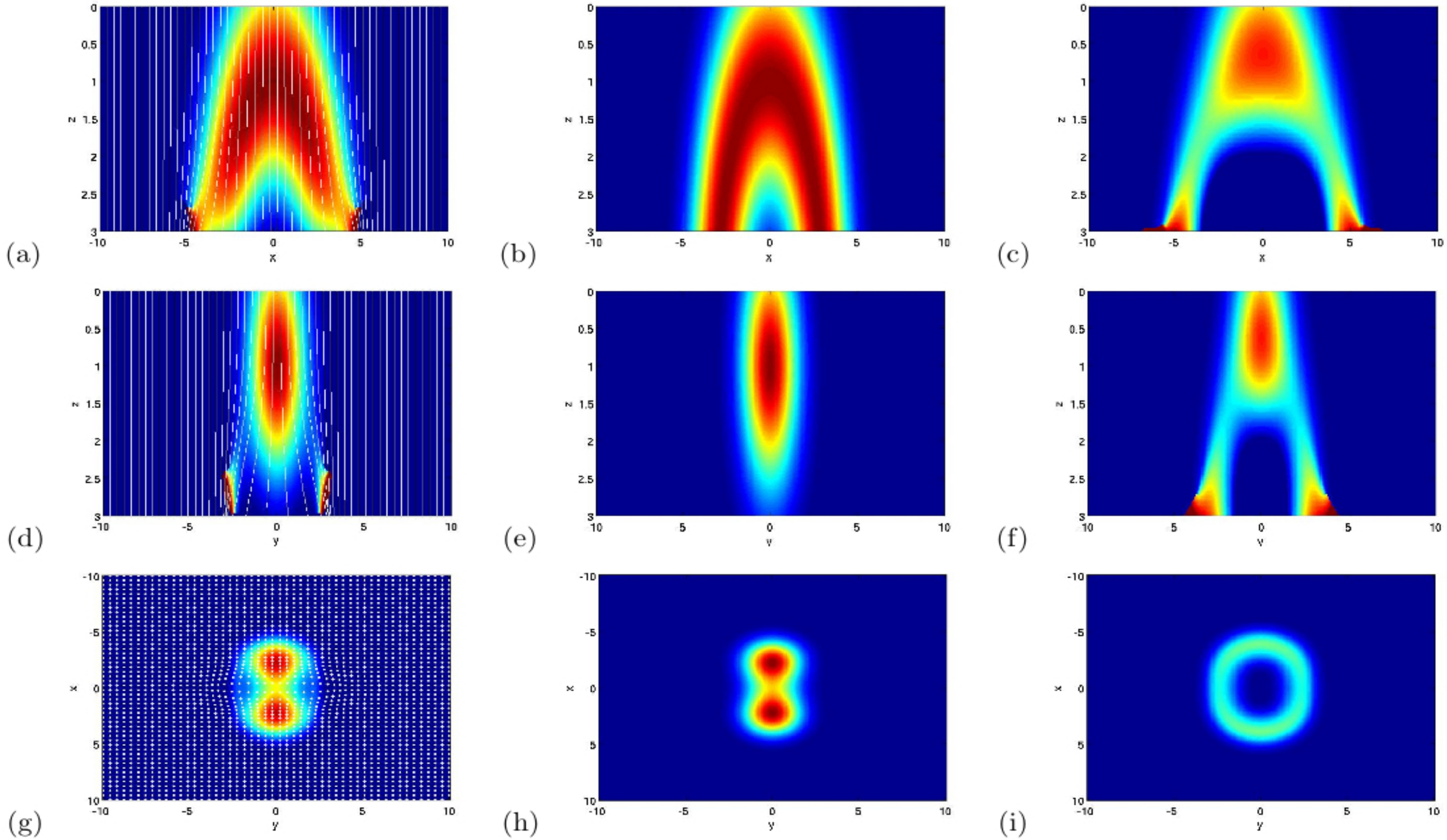
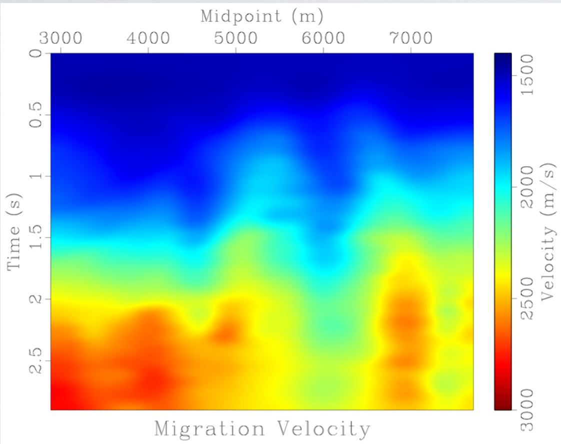
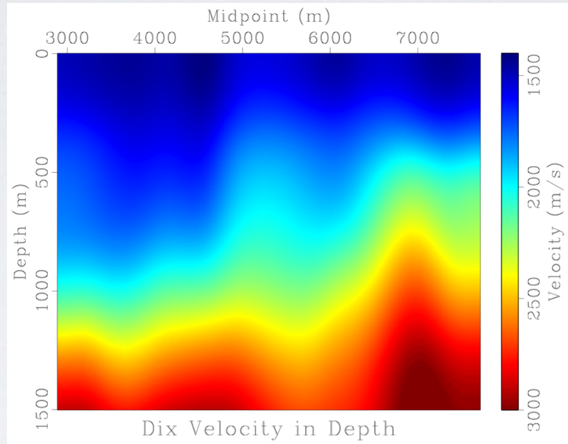
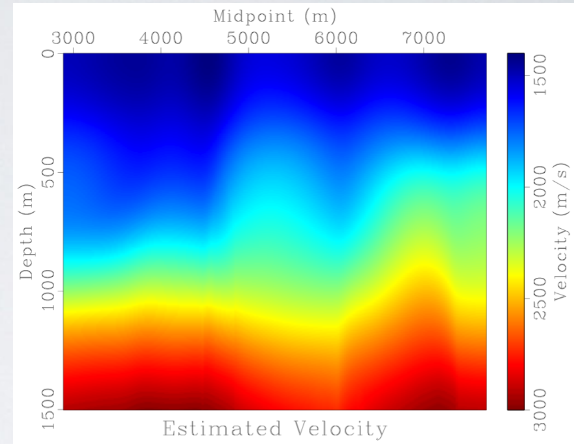


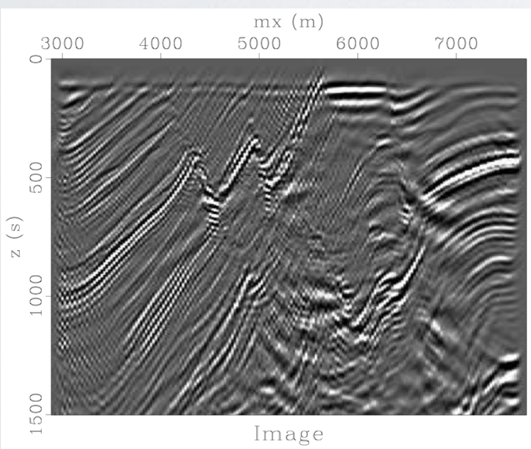

Figure 16: 3D example 2. The first row: the velocity on the vertical plane $y = 0$. The second row: the velocity on the vertical plane $x = 0$. The third row: the velocity on the horizontal plane $z = 2.0$ km. The first column ( (a),(d),(g)): the reconstructed velocity and the image rays; the second column ((b),(e),(h)): the exact velocity; the third column ((c),(f),(i)): the velocity estimate analogous to Dix inversion, converted to depth. Dark blue and dark red correspond to 2 km/s and 4 km/s respectively.

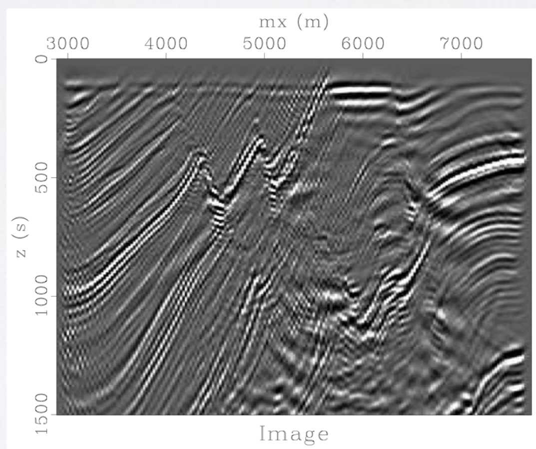
# Marmousi Example

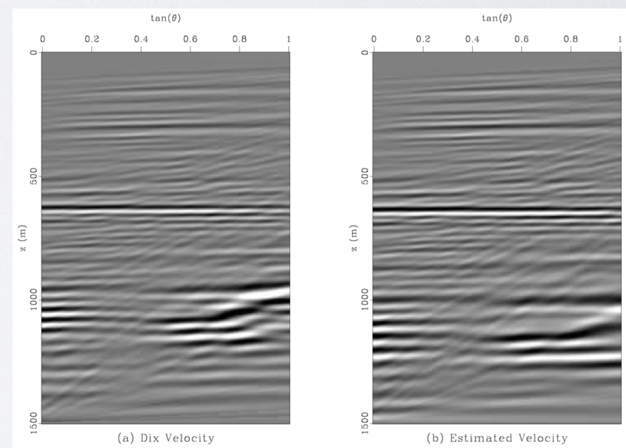



Prestack depth-migrated image with Dix velocities

Prestack depth-migrated image with our velocities

Angle domain common-image point gather at 4000 m using: Left: Dix, velocity, Right: our velocity

# Conclusions

- Relationships between $v_m(x_0,t_0)$ and $v(x,z)$ in 2D and 3D
- PDE's connecting $v(x_0,t_0)$ and $v_m(x_0,t_0)$ in 2D and 3D
- Difficulties of solving them arise from
  1. Sensetivity (dependence not only on the data but also on their derivatives)
  2. Ill-Posedness (Cauchy problems for elliptic PDE's)
- Finite difference ("Lax-Friedrichs") and spectral ("Chebyshev") numerical methods allow to solve these PDE's on a short interval of time due to
  1. Special input
  2. Special initial conditions
  3. Damping of high harmonics
- Efficient Dijkstra-like solver to compute $v(x,z)$, $x_0(x,z)$, $t_0(x,z)$ from $v(x_0,t_0)$