ON THE UNIQUENESS OF SOLUTIONS TO THE GROSS-PITAEVSKII HIERARCHY

SERGIU KLAINERMAN AND MATEI MACHEDON

ABSTRACT. The purpose of this note is to give a new proof of uniqueness of the Gross- Pitaevskii hierarchy, first established in [1], in a different space, based on space-time estimates similar in spirit to those of [2].

1. INTRODUCTION

The Gross-Pitaevskii hierarchy refers to a sequence of functions $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k)$, $k = 1, 2, \cdots$, where $t \in \mathbb{R}$, $\mathbf{x}_k = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{3k}$, $\mathbf{x}'_k = (x'_1, x'_2, \cdots, x'_k) \in \mathbb{R}^{3k}$ which are symmetric, in the sense that

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \overline{\gamma^{(k)}(t, \mathbf{x}'_k, \mathbf{x}_k)}$$

and

$$\gamma^{(k)}(t, x_{\sigma(1)}, \cdots x_{\sigma(k)}, x'_{\sigma(1)}, \cdots x'_{\sigma(k)}) = \gamma^{(k)}(t, x_1, \cdots x_k, x'_1, \cdots x'_k) \quad (1)$$

for any permutation σ, and satisfy the Gross-Pitaevskii infinite linear hierarchy of equations,

$$\left(i\partial_t + \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k}\right) \gamma^{(k)} = \sum_{j=1}^{k} B_{j,\,k+1}(\gamma^{(k+1)}). \quad (2)$$

with prescribed initial conditions

$$\gamma^{(k)}(0, \mathbf{x}_k, \mathbf{x}'_k) = \gamma_0^{(k)}(\mathbf{x}_k, \mathbf{x}'_k).$$

Here $\Delta_{\mathbf{x}_k}$, $\Delta_{\mathbf{x}'_k}$ refer to the standard Laplace operators with respect to the variables $\mathbf{x}_k, \mathbf{x}'_k \in \mathbb{R}^{3k}$ and the operators $B_{j,\,k+1} = B^1_{j,\,k+1} - B^2_{j,\,k+1}$

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are defined according to,

$$B^1_{j,\,k+1}(\gamma^{(k+1)})(t,\mathbf{x}_k,\mathbf{x}'_k)$$
$$= \int\int \delta(x_j - x_{k+1})\delta(x_j - x'_{k+1})\gamma^{(k+1)}(t,\mathbf{x}_{k+1},\mathbf{x}'_{k+1})\,dx_{k+1}\,dx'_{k+1}$$
$$B^2_{j,\,k+1}(\gamma^{(k+1)})(t,\mathbf{x}_k,\mathbf{x}'_k)$$
$$= \int\int \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})\gamma^{(k+1)}(t,\mathbf{x}_{k+1},\mathbf{x}'_{k+1})\,dx_{k+1}\,dx'_{k+1}.$$

In other words $B^1_{j,\,k+1}$, resp. $B^2_{j,\,k+1}$, acts on $\gamma^{(k+1)}(t,\mathbf{x}_{k+1},\mathbf{x}'_{k+1})$ replacing both variables x_{k+1} and x'_{k+1} by x_j, resp x'_j. We shall also make use of the operators,

$$B^{k+1} = \sum_{1\le j\le k} B_{j,\,k+1}$$

One can easily verify that a particular solution to (2) is given by,

$$\gamma^{(k)}(t,\mathbf{x}_k,\mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t,x_j)\overline{\phi}(t,x'_j) \tag{3}$$

where each ϕ satisfies the non-linear Schrödinger equation in 3+1 dimensions

$$(i\partial_t + \Delta)\,\phi = \phi|\phi|^2, \quad \phi(0,x) = \phi(x) \tag{4}$$

In [1] L. Erdös, B. Schlein and H-T Yau provide a rigorous derivation of the cubic non-linear Schrödinger equation (4) from the quantum dynamics of many body systems. An important step in their program is to prove uniqueness to solutions of (2) corresponding to the special initial conditions

$$\gamma^{(k)}(0,\mathbf{x}_k,\mathbf{x}'_k) = \gamma_0^{(k)}(\mathbf{x}_k,\mathbf{x}'_k) = \prod_{j=1}^{k} \phi(x_j)\overline{\phi}(x'_j) \tag{5}$$

with $\phi \in H^1(\mathbb{R}^3)$. To state precisely the uniqueness result of [1], denote $S_j = (1-\Delta_{x_j})^{1/2}$, $S'_j = (1-\Delta_{x'_j})^{1/2}$ and $S^{(k)} = \prod_{j=1}^k S_j \cdot \prod_{j=1}^k S'_j$. If the operator given by the integral kernel $\gamma^{(k)}(\mathbf{x}_k,\mathbf{x}'_k)$ is positive (as an operator), then so is $S^{(k)}\gamma^{(k)}(\mathbf{x}_k,\mathbf{x}'_k)$, and the trace norm of $S^{(k)}\gamma^{(k)}$ is

$$\|\gamma^{(k)}\|_{\mathcal{H}_k} = \int \left(S^{(k)}\gamma^{(k)}(\mathbf{x}_k,\mathbf{x}'_k)\right)\big|_{\mathbf{x}'_k=\mathbf{x}_k}\,d\mathbf{x}_k.$$

The authors of [1] prove uniqueness of solutions to (2) in the set of symmetric, positive operators γ_k satisfying, for some $C > 0$,

$$\sup_{0\le t\le T} \|\gamma^{(k)}(t,\cdot,\cdot)\|_{\mathcal{H}_k} \le C^k \tag{6}$$

In that work, the equations (2) are obtained as a limit of the BBGKY hierarchy (see [1]), and it is proved that solutions to BBGKY with initial conditions (5) converge, in a weak sense, to a solution of (2) in the space (6).

The purpose of this note is to give a new proof of uniqueness of the Gross- Pitaevskii hierarchy (2), in a different space, motivated , in part, by space-time type estimates, similar in spirit to those of [2].

Our norms will be

$$\|R^{(k)}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \tag{7}$$

Here, $R_j = (-\Delta_{x_j})^{1/2}$, $R_j' = (-\Delta_{x_j'})^{1/2}$ and $R^{(k)} = \prod_{j=1}^k R_j \cdot \prod_{j=1}^k R_j'$. Notice that for a symmetric, smooth kernel γ, for which the associated linear operator is positive we have

$$\begin{aligned} & \|R^{(k)}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ \le\ & \|S^{(k)}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ \le\ & \|\gamma^{(k)}(t,\cdot,\cdot)\|_{\mathcal{H}_k} \end{aligned}$$

since $|S^{(k)}\gamma^{(k)}(\mathbf{x},\mathbf{x}')|^2 \le S^{(k)}\gamma^{(k)}(\mathbf{x},\mathbf{x})S^{(k)}\gamma^{(k)}(\mathbf{x}',\mathbf{x}')$. This is similar to the condition $a_{ii}a_{jj} - |a_{ij}|^2 \ge 0$ which is satisfied by all $n \times n$ positive semi-definite Hermitian matrices.

Our main result is the following:

Theorem 1.1 (Main Theorem)**.** *Consider solutions $\gamma^{(k)}(t,\mathbf{x}_k,\mathbf{x}_k')$ of the Gross-Pitaevskii hierarchy (2), with zero initial conditions, which verify the estimates,*

$$\int_0^T \|R^{(k)}B_{j,k+1}\gamma^{(k+1)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}\, dt \le C^k \tag{8}$$

for some $C > 0$ and all $1 \le j < k$. Then $\|R^{(k)}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} = 0$ for all k and all t.

Therefore, solutions to (2) verifying the initial conditions (5), are unique in the space-time norm (8). We plan to address the connection with solutions of BBGKY in a future paper. The following remark is however reassuring.

Remark 1.2. The sequence $\gamma^{(k)}$, given by (3) with ϕ an arbitrary solution of (4) with H^1 data, verifies (8) for every $T > 0$ sufficiently small. Moreover, if the H^1 norm of the initial data is sufficiently small then (8) is verified for all values of $T > 0$.

Proof. Observe that $R^{(k)} B_{1,k+1} \gamma^{(k+1)}(t, x_1, \ldots, x_k; x'_1 \ldots, x'_k)$ can be written in the form,

$$\begin{aligned} R^{(k)} B_{1,k+1} \gamma^{(k+1)}(t, \cdot, \cdot) &= R_1 \left(|\phi(t, x_1)|^2 \phi(t, x_1) \right) R_2(\phi(t, x_2) \cdots R_k(\phi(t, x_k) \\ &\cdot \quad R'_1(\phi(t, x'_1) \cdots R'_k(\phi(t, x'_k) \end{aligned}$$

Therefore, in $[0, T] \times \mathbb{R}^{3k} \times \mathbb{R}^{3k}$ we derive

$$\begin{aligned} \|R^{(k)} B_{j,k+1} \gamma^{(k+1)}\|_{L^1_t L^2} &\leq \| |R_1(|\phi|^2 \phi)\|_{L^1_t L^2_x} \cdot \|R_2 \phi\|_{L^\infty_t L^2_x} \cdots \|R_k \phi\|_{L^\infty_t L^2_x} \\ &\cdot \quad \|R'_1 \phi\|_{L^\infty_t L^2_x} \|R'_2 \phi\|_{L^\infty_t L^2_x} \cdots \|R'_k \phi\|_{L^\infty_t L^2_x} \\ &\leq C \|\nabla(|\phi|^2 \phi)\|_{L^1_t L^2_x} \times \|\nabla \phi\|^{2k-1}_{L^\infty_t L^2_x} \end{aligned}$$

where the norm on the left is in $[0, T] \times \mathbb{R}^{3k} \times \mathbb{R}^{3k}$ and all norms on the right hand side are taken relative to the space-time domain $[0, T] \times \mathbb{R}^3$.

In view of the standard energy identity for the nonlinear equation (4) we have apriori bounds for $\sup_{t \in [0,T]} \|\nabla \phi(t)\|_{L^2(\mathbb{R}^3)}$. Therefore we only need to provide a uniform bound for the norm $\|\nabla(|\phi|^2 \phi)\|_{L^1_t L^2_x}$. We shall show below that this is possible for all values of $T > 0$ provided that the H^1 norm of $\phi(0)$ is sufficiently small. The case of arbitrary size for $\|\phi(0)\|_{H^1}$ and sufficiently small T is easier and can be proved in a similar manner.

We shall rely on the following Strichartz estimate (see [3]) for the linear, inhomogeneous, Schrödinger equation $i\partial_t \phi + \Delta \phi = f$ in $[0, T] \times \mathbb{R}^3$,

$$\|\phi\|_{L^2_t L^6_x} \leq C \big(\|f\|_{L^1_t L^2_x} + \|\phi\|_{L^\infty_t L^2_x} \big) \tag{9}$$

We start by using Hölder inequality, in $[0, T] \times \mathbb{R}^3$,

$$\|\nabla(|\phi|^2 \phi)\|_{L^1_t L^2_x} \leq C \|\nabla \phi\|_{L^2_t L^6_x} \|\phi^2\|_{L^2_t L^3_x} \leq C \|\nabla \phi\|_{L^2_t L^6_x} \|\phi\|^2_{L^4_t L^6_x}$$

Using (9) for $f = |\phi|^2\phi$ we derive,

$$\|\nabla\phi\|_{L^2_t L^6_x} \leq C\big(\|\nabla(|\phi|^2\phi)\|_{L^1_t L^2_x} + \|\nabla\phi\|_{L^\infty_t L^2_x}\big)$$

Denoting,

$$\begin{aligned} A(T) &= \| \, |\phi|^2\phi\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^3)} \\ B(T) &= \|\nabla(|\phi|^2\phi)\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^3)} \end{aligned}$$

we derive,

$$\begin{aligned} B(T) &\leq C\big(B(T) + \|\nabla\phi(0)\|_{L^2}\big)\|\phi\|^2_{L^4_t L^6_x} \\ &\leq C\big(B(T) + \|\nabla\phi(0)\|_{L^2}\big)\|\phi\|_{L^\infty_t L^6_x}\|\phi\|_{L^2_t L^6_x} \\ &\leq C\big(A(T) + \|\phi(0)\|_{L^2}\big)\big(B(T) + \|\nabla\phi(0)\|_{L^2}\big)\|\nabla\phi(0)\|_{L^2_x} \end{aligned}$$

On the other hand, using (9) again,

$$\begin{aligned} A(T) &\leq C\big(\|\phi^3\|_{L^1_t L^2_x} + \|\phi(0)\|_{L^2}\big) \\ &\leq C(A(T)^3 + \|\phi(0)\|_{L^2}\big) \end{aligned}$$

Observe this last inequality implies that, for sufficiently small $\|\phi(0)\|_{L^2}$, $A(T)$ remains uniformly bounded for all values of T. Thus, for all values of T, with another value of C,

$$B(T) \leq C\big(B(T) + \|\nabla\phi(0)\|_{L^2}\big)\|\nabla\phi(0)\|_{L^2_x}$$

from which we get a uniform bound for $B(T)$ provided that $\|\nabla\phi(0)\|_{L^2}$ is also sufficiently small.

□

The proof of Theorem (1.1) is based on two ingredients. One is expressing $\gamma^{(k)}$ in terms of the *future iterates* $\gamma^{(k+1)} \cdots, \gamma^{(k+n)}$ using Duhamel's formula. Since $B^{(k+1)} = \sum_{j=1}^k B_{j,\,k+1}$ is a sum of k terms, the iterated Duhamel's formula involves $k(k+1)\cdots(k+n-1)$ terms. These have to be grouped into much fewer $O(C^n)$ sets of terms. This part of our paper follows in the spirit of the Feynman path combinatorial arguments of [1]. The second ingredient is the main novelty of our work. We derive a space-time estimate, reminiscent of the bilinear estimates of [2].

Theorem 1.3. *Let $\gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$ verify the homogeneous equation,*

$$\left(i\partial_t + \Delta_{\pm}^{k+1}\right)\gamma^{(k+1)} = 0, \qquad \Delta_{\pm}^{(k+1)} = \Delta_{\mathbf{x}_{k+1}} - \Delta_{\mathbf{x}'_{k+1}} \tag{10}$$
$$\gamma^{(k+1)}(0, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) = \gamma_0^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}).$$

Then there exists a constant C, independent of j, k, such that

$$\begin{aligned} &\|R^{(k)}B_{j,k+1}(\gamma^{(k+1)})\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ &\leq C\|R^{(k+1)}\gamma_0^{(k+1)}\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})} \end{aligned} \tag{11}$$

2. Proof of the estimate

Without loss of generality, we may take $j = 1$ in $B_{j,k+1}$. It also suffices to estimate the term in $B^1_{j,k+1}$, the term in $B^2_{j,k+1}$ can be treated in the same manner. Let $\gamma^{(k+1)}$ be as in (10). Then the Fourier transform of $\gamma^{(k+1)}$ with respect to the variables $(t, \mathbf{x}_k, \mathbf{x}'_k)$ is given by the formula,

$$\delta(\tau + |\underline{\xi}_k|^2 - |\underline{\xi}'_k|^2)\hat{\gamma}(\xi, \xi')$$

where τ corresponds to the time t and $\underline{\xi}_k = (\xi_1, \xi_2, \ldots, \xi_k)$, $\underline{\xi}'_k = (\xi'_1, \xi'_2, \ldots, \xi'_k)$ correspond to the space variables $\mathbf{x}_k = (x_1, x_2, \ldots, x_k)$ and $\mathbf{x}'_k = (x'_1, x'_2, \ldots, x'_k)$. We also write $\underline{\xi}_{k+1} = (\underline{\xi}_k, \xi_{k+1})$, $\underline{\xi}'_{k+1} = (\underline{\xi}'_k, \xi'_{k+1})$ and,

$$\begin{aligned} |\underline{\xi}_{k+1}|^2 &= |\xi_1|^2 + \ldots + |\xi_k|^2 + |\xi_{k+1}|^2 = |\underline{\xi}_k|^2 + |\xi_{k+1}|^2 \\ |\underline{\xi}'_{k+1}|^2 &= |\xi'_1|^2 + \ldots + |\xi'_k|^2 + |\xi'_{k+1}|^2 \end{aligned}$$

The Fourier transform of $B^1_{1,k+1}(\gamma^{(k+1)})$, with respect to the same variables $(t, \mathbf{x}_k, \mathbf{x}'_k)$, is given by,

$$\int\int \delta(\cdots)\hat{\gamma}(\xi_1 - \xi_{k+1} - \xi'_{k+1}, \xi_2, \cdots, \xi_{k+1}, \underline{\xi}'_{k+1})d\xi_{k+1}\, d\xi'_{k+1} \tag{12}$$

where,

$$\delta(\cdots) \;=\; \delta(\tau + |\xi_1 - \xi_{k+1} - \xi'_{k+1}|^2 + |\underline{\xi}_{k+1}|^2 - |\xi_1|^2 - |\underline{\xi}'_{k+1}|^2)$$

and γ denotes the initial condition $\gamma_0^{(k+1)}$. By Plancherel's theorem, estimate (11) is equivalent to the following estimate,

$$\|I_k[f]\|_{L^2(\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k)} \leq C\|\hat{f}\|_{L^2(\mathbb{R}^{k+1}\times\mathbb{R}^{k+1})}, \tag{13}$$

applied to $f = R^{(k+1)}\gamma$, where,

$$I_k[f](\tau, \underline{\xi}_k, \underline{\xi}'_k) = \int\int \delta(\dots) \frac{|\xi_1| \hat{f}(\xi_1 - \xi_{k+1} - \xi'_{k+1}, \xi_2, \cdots, \xi_{k+1}, \underline{\xi}'_{k+1})}{|\xi_1 - \xi_{k+1} - \xi'_{k+1}||\xi_{k+1}||\xi'_{k+1}|} d\xi_{k+1}\, d\xi'_{k+1}.$$

Applying the Cauchy-Schwarz inequality with measures, we easily check that,

$$\begin{aligned} |I_k[f]|^2 &\leq \int\int \delta(\dots) \frac{|\xi_1|^2}{|\xi_1 - \xi_{k+1} - \xi'_{k+1}|^2 |\xi_{k+1}|^2 |\xi'_{k+1}|^2} d\xi_{k+1}\, d\xi'_{k+1} \\ &\quad \cdot \int\int \delta(\dots) |\hat{f}(\xi_1 - \xi_{k+1} - \xi'_{k+1}, \xi_2, \cdots, \xi_{k+1}, \underline{\xi}'_{k+1})|^2 d\xi_{k+1}\, d\xi'_{k+1} \end{aligned}$$

If we can show that the supremum over τ, $\xi_1 \cdots \xi_k, \xi'_1 \cdots \xi'_k$ of the first integral above is bounded by a constant C^2, we infer that,

$$\begin{aligned} \|I_k[f]\|^2_{L^2} &\leq C^2 \int\int\int \delta(\dots) |\hat{f}(\xi_1 - \xi_{k+1} - \xi'_{k+1}, \xi_2, \cdots, \xi_{k+1}, \underline{\xi}'_{k+1})|^2 d\underline{\xi}_{k+1} d\underline{\xi}'_{k+1} d\tau \\ &\leq C^2 \|\hat{f}\|^2_{L^2(\mathbb{R}^{k+1} \times \mathbb{R}^{k+1})} \end{aligned}$$

Thus,

$$\|I_k[f]\|^2_{L^2(\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k)} \leq C^2 \|\hat{f}\|^2_{L^2(\mathbb{R}^{k+1} \times \mathbb{R}^{k+1})}$$

as desired. Thus we have reduced matters to the following,

Proposition 2.1. *There exists a constant C such that*

$$\begin{aligned} \int \quad & \delta(\tau + |\xi_1 - \xi_{k+1} - \xi'_{k+1}|^2 + |\xi_{k+1}|^2 - |\xi'_{k+1}|^2) \\ & \frac{|\xi_1|^2}{|\xi_1 - \xi_{k+1} - \xi'_{k+1}|^2 |\xi_{k+1}|^2 |\xi'_{k+1}|^2} d\xi_{k+1} d\xi'_{k+1} \leq C \end{aligned}$$

uniformly in τ, ξ_1.

The proof is based on the following lemmas

Lemma 2.2. *Let P be a 2 dimensional plane or sphere in $\mathbb{R}^3$ with the usual induced surface measure dS. Let $0 < a < 2$, $0 < b < 2$, $a + b > 2$. Let $\xi \in \mathbb{R}^3$. Then there exists C independent of ξ and P such that*

$$\int_P \frac{1}{|\xi - \eta|^a |\eta|^b} dS(\eta) \leq \frac{C}{|\xi|^{a+b-2}}$$

Proof. If $P = \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$ this is well known. The same proof works in our case, by breaking up the integral $I \le I_1 + I_2 + I_3$ over the overlapping regions:

Region 1: $\frac{|\xi|}{2} < |\eta| < 2|\xi|$. In this region $|\xi - \eta| < 3|\xi|$ and

$$\begin{aligned} I_1 \le & C\frac{1}{|\xi|^b}\int_{P\cap\{|\xi-\eta|<3|\xi|\}}\frac{1}{|\xi-\eta|^a}dS(\eta) \\ \le & C\frac{1}{|\xi|^b}\sum_{i=-\infty}^{1}\int_{P\cap\{3^{i-1}|\xi|<|\xi-\eta|<3^i|\xi|\}}\frac{1}{|\xi-\eta|^a}dS(\eta) \\ \le & C\frac{1}{|\xi|^b}\sum_{i=-\infty}^{1}\frac{1}{|3^i\xi|^a}(3^i|\xi|)^2 = \frac{C}{|\xi|^{a+b-2}} \end{aligned}$$

where we have used the obvious fact that the area of $P \cap \{3^{i-1}|\xi| < |\xi - \eta| < 3^i|\xi|\}$ is $\le C(3^i|\xi|)^2$.

Region 2: $\frac{|\xi|}{2} < |\xi - \eta| < 2|\xi|$. In this region $|\eta| < 3|\xi|$ and

$$\begin{aligned} I_2 \le & C\frac{1}{|\xi|^a}\int_{P\cap\{|\eta|<3|\xi|\}}\frac{1}{|\eta|^b}dS(\eta) \\ \le & \frac{C}{|\xi|^{a+b-2}} \end{aligned} \tag{14}$$

in complete analogy with region 1.

Region 3: $|\eta| > 2|\xi|$ or $|\xi - \eta| > 2|\xi|$. In this region, $|\eta| > |\xi|$ and $2|\xi - \eta| \ge |\xi - \eta| + |\xi| \ge |\eta|$, thus

$$\begin{aligned} I_3 \le & C\int_{P\cap\{|\eta|>|\xi|\}}\frac{1}{|\eta|^{a+b}}dS(\eta) \\ \le & C\sum_{i=1}^{\infty}\int_{P\cap\{2^{i-1}|\xi|<|\eta|<2^i|\xi|\}}\frac{1}{|\eta|^{a+b}}dS(\eta) \\ \le & C\sum_{i=1}^{\infty}\frac{1}{|2^i\xi|^{a+b}}(2^i|\xi|)^2 = \frac{C}{|\xi|^{a+b-2}} \end{aligned} \tag{15}$$

□

We also have

Lemma 2.3. *Let P, ξ as in Lemma* (2.2) *and let* $\epsilon = \frac{1}{10}$. *Then*

$$\int_P \frac{1}{|\xi-\eta|^{2-\epsilon}|\frac{\xi}{2}-\eta||\eta|^{2-\epsilon}} dS(\eta) \leq \frac{C}{|\xi|^{3-2\epsilon}}$$

Proof. Consider separately the regions $|\eta| > \frac{|\xi|}{2}$, and $|\xi-\eta| > \frac{|\xi|}{2}$, and apply the previous lemma. □

We are ready to prove the main estimate of Proposition (2.1)

Proof. Changing $k+1$ to 2, we have to show

$$I = \int \delta(\tau + |\xi_1-\xi_2-\xi_2'|^2 + |\xi_2|^2 - |\xi_2'|^2) \frac{|\xi_1|^2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2|^2|\xi_2'|^2} d\xi_2 d\xi_2' \leq C$$

The integral is symmetric in $\xi_1-\xi_2-\xi_2'$ and ξ_2 so we can integrate, without loss of generality, over $|\xi_1-\xi_2-\xi_2'| > |\xi_2|$.

Case 1. Consider the integral I_1 restricted to the region $|\xi_2'| > |\xi_2|$ and integrate ξ_2' first.

Notice

$$\begin{aligned}
&\delta(\tau + |\xi_1-\xi_2-\xi_2'|^2 + |\xi_2|^2 - |\xi_2'|^2) d\xi_2' \\
&= \delta(\tau + |\xi_1-\xi_2|^2 - 2(\xi_1-\xi_2)\cdot\xi_2' + |\xi_2|^2) d\xi_2' \\
&= \frac{dS(\xi_2')}{2|\xi_1-\xi_2|}
\end{aligned} \tag{16}$$

where dS is surface measure on a plane P in $\mathbb{R}^3$, i.e. the plane $\xi'\cdot\omega = \lambda$ with $\omega \in \mathbb{S}^2$ and $\lambda = \frac{\tau+|\xi_1-\xi_2|^2+|\xi_2|^2}{2|\xi_1-\xi_2|}$.

In this region

$$
\begin{aligned}
I_1 &\le |\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2}{|\xi_2|^2|\xi_1-\xi_2|} \int_P \frac{dS(\xi_2')}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2} \\
&\le |\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2}{|\xi_2|^{2+2\epsilon}|\xi_1-\xi_2|} \int_P \frac{dS(\xi_2')}{|\xi_1-\xi_2-\xi_2'|^{2-\epsilon}|\xi_2'|^{2-\epsilon}} \\
&\le C|\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2}{|\xi_2|^{2+2\epsilon}|\xi_1-\xi_2|^{3-2\epsilon}} \\
&\le C \qquad (17)
\end{aligned}
$$

Case 2. Consider the integral I_2 restricted to the region $|\xi_2'| < |\xi_2|$ and integrate ξ_2 first.

Notice

$$
\begin{aligned}
&\delta(\tau + |\xi_1-\xi_2-\xi_2'|^2 + |\xi_2|^2 - |\xi_2'|^2) d\xi_2 \\
&= \delta\left(\tau + |\frac{\xi_1-\xi_2'}{2} - (\xi_2 - \frac{\xi_1-\xi_2'}{2})|^2 + |(\xi_2 - \frac{\xi_1-\xi_2'}{2}) + \frac{\xi_1-\xi_2'}{2}|^2 - |\xi_2'|^2\right) d\xi_2 \\
&= \delta\left(\tau + \frac{|\xi_1-\xi_2'|^2}{2} + 2|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2 - |\xi_2'|^2\right) d\xi_2 \qquad (18) \\
&= \frac{dS(\xi_2)}{4|\xi_2 - \frac{\xi_1-\xi_2'}{2}|}
\end{aligned}
$$

where dS is surface measure on a sphere P, i.e. the sphere in ξ_2 centered at $\frac{1}{2}(\xi_1-\xi_2')$ and radius $\frac{1}{2}\left(|\xi_2'|^2 - \tau - \frac{|\xi_1-\xi_2'|^2}{2}\right)$.

Thus

$$
\begin{aligned}
I_2 &\le |\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2'}{|\xi_2'|^2} \int_P \frac{dS(\xi_2)}{|\xi_2|^2|\xi_2 - \frac{\xi_1-\xi_2'}{2}||\xi_1-\xi_2-\xi_2'|^2} \\
&\le |\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2'}{|\xi_2'|^{2+2\epsilon}} \int_P \frac{dS(\xi_2)}{|\xi_2|^{2-\epsilon}|\xi_2 - \frac{\xi_1-\xi_2'}{2}||\xi_1-\xi_2-\xi_2'|^{2-\epsilon}} \\
&\le |\xi_1|^2 \int_{\mathbb{R}^3} \frac{d\xi_2'}{|\xi_2'|^{2+2\epsilon}|\xi_1-\xi_2'|^{3-2\epsilon}} \le C
\end{aligned}
$$

□

3. Duhamel expansions and regrouping

This part of our note is based on a somewhat shorter version of the combinatorial ideas of [1]. We are grateful to Schlein and Yau for explaining their arguments to us.

Recalling the notation $\Delta_{\pm}^{(k)} = \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k}$ and $\Delta_{\pm,x_j} = \Delta_{x_j} - \Delta_{x'_j}$ we write,

$$\begin{aligned}
\gamma^{(1)}(t_1,\cdot) &= \int_0^{t_1} e^{i(t_1-t_2)\Delta_{\pm}^{(1)}} B_2(\gamma^2(t_2))\,dt_2 \\
&= \int_0^{t_1}\int_0^{t_2} e^{i(t_1-t_2)\Delta_{\pm}^{(1)}} B_2 e^{i(t_2-t_3)\Delta_{\pm}^{(2)}}(\gamma^3(t_3))\,dt_2 dt_3 \\
&= \cdots\cdots\cdots\cdots \\
&= \int_0^{t_1}\cdots\int_0^{t_n} J(\underline{t}_{n+1})\,dt_2\,\cdots dt_{n+1} \qquad (19)
\end{aligned}$$

where, $\underline{t}_{n+1} = (t_1,\ldots,t_{n+1})$ and

$$J(\underline{t}_{n+1}) = e^{i(t_1-t_2)\Delta_{\pm}^{(1)}} B_2 \cdots e^{i(t_n-t_{n+1})\Delta_{\pm}^{(n)}} B_{n+1}(\gamma^{(n+1)})(t_{n+1},\cdot)$$

Expressing $B^{(k+1)} = \sum_{j=1}^{k} B_{j,\,k+1}, \cdots$, the integrand $J(\underline{t}_{n+1}) = J(t_1,\ldots,t_{n+1})$ in (19) can be written as

$$J(\underline{t}_{n+1}) = \sum_{\mu\in M} J(\underline{t}_{n+1};\,\mu) \qquad (20)$$

where,

$$\begin{aligned}
J(\underline{t}_{n+1};\,\mu) &= e^{i(t_1-t_2)\Delta_{\pm}^{(1)}} B_{1,2} e^{i(t_2-t_3)\Delta_{\pm}^{(2)}} B_{\mu(3),3}\cdots \\
&\quad e^{i(t_n-t_{n+1})\Delta_{\pm}^{(n)}} B_{\mu(n+1),n+1}(\gamma^{(n+1)})(t_{n+1},\cdot)
\end{aligned}$$

Here we have denoted by M the set of maps $\mu : \{2,\cdots,n+1\} \to \{1,\cdots n\}$ satisfying $\mu(2) = 1$ and $\mu(j) < j$ for all j.

Graphically, such a μ can be represented by selecting one $\mathbf{B}$ entry from each column of an $n\times n$ matrix

such as, for example, (if $\mu(2) = 1$, $\mu(3) = 2$, $\mu(4) = 1$, etc),

$$\begin{pmatrix}
\mathbf{B_{1,2}} & B_{1,3} & \mathbf{B_{1,4}} & \cdots & \mathbf{B_{1,n+1}} \\
0 & \mathbf{B_{2,3}} & B_{2,4} & \cdots & B_{2,n+1} \\
0 & 0 & B_{3,4} & \cdots & B_{3,n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & B_{n,n+1}
\end{pmatrix} \qquad (21)$$

To such a matrix one can associate a Feynman graph whose nodes are the selected entries, as in [1]. However, our exposition will be self-contained and will not rely explicitly on the Feynman graphs.

We will consider

$$\mathcal{I}(\mu,\sigma)=\int_{t_1\geq t_{\sigma(2)}\geq t_{\sigma(3)}\geq\cdots\geq t_{\sigma(n+1)}} J(\underline{t}_{n+1};\,\mu)dt_2\cdots dt_{n+1} \tag{22}$$

where σ is a permutation of $\{2,\cdots,n+1\}$. The integral $\mathcal{I}(\mu,\sigma)$ is represented by (μ,σ), or, graphically, by the matrix

$$\begin{pmatrix} t_{\sigma^{-1}(2)} & t_{\sigma^{-1}(3)} & t_{\sigma^{-1}(4)} & \cdots & t_{\sigma^{-1}(n+1)} & \\ \mathbf{B_{1,2}} & B_{1,3} & \mathbf{B_{1,4}} & \cdots & \mathbf{B_{1,n+1}} & \text{row 1} \\ 0 & \mathbf{B_{2,3}} & B_{2,4} & \cdots & B_{2,n+1} & \text{row 2} \\ 0 & 0 & B_{3,4} & \cdots & B_{3,n+1} & \text{row 3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & B_{n,n+1} & \text{row n} \\ \text{column 2} & \text{column 3} & \text{column 4} & \cdots & \text{column n+1} & \end{pmatrix} \tag{23}$$

Notice the columns of this matrix are labelled 2 to $n+1$, while the rows are labelled 1 through n.

We will define a set of "acceptable moves" on the set of such matrices. Imagine a board game where the names $B_{i,j}$ are carved in, and one entry $B_{\mu(j),j}$, $\mu(j)<j$, in each column is highlighted. If $\mu(j+1)<\mu(j)$, the player is allowed to exchange the highlighted entries in columns j and $j+1$ and, at the same time, exchange the highlighted entries in rows j and $j+1$. This changes μ to a new $\mu'=(j,j+1)\circ\mu\circ(j,j+1)$. Here $(j,j+1)$ denotes the permutation which reverses j and $j+1$. The rule for changing σ is $\sigma'^{-1}=\sigma^{-1}\circ(j,j+1)$. In other words, σ^{-1} changes according to column exchanges. Thus going from

$$\begin{pmatrix} t_2 & t_5 & t_4 & t_3 \\ \mathbf{B_{1,2}} & B_{1,3} & \mathbf{B_{1,4}} & B_{1,5} \\ 0 & \mathbf{B_{2,3}} & B_{2,4} & B_{2,5} \\ 0 & 0 & B_{3,4} & B_{3,5} \\ 0 & 0 & 0 & \mathbf{B_{4,5}} \end{pmatrix} \tag{24}$$

to

$$\begin{pmatrix} t_2 & t_4 & t_5 & t_3 \\ \mathbf{B_{1,2}} & \mathbf{B_{1,3}} & B_{1,4} & B_{1,5} \\ 0 & B_{2,3} & \mathbf{B_{2,4}} & B_{2,5} \\ 0 & 0 & B_{3,4} & \mathbf{B_{3,5}} \\ 0 & 0 & 0 & B_{4,5} \end{pmatrix} \tag{25}$$

is an acceptable move. In the language of [1] the partial order of the graphs is preserved by an acceptable move.

The relevance of this game to our situation is explained by the following,

Lemma 3.1. *Let* (μ, σ) *be transformed into* (μ', σ') *by an acceptable move. Then, for the corresponding integrals* (22), $\mathcal{I}(\mu, \sigma) = \mathcal{I}(\mu', \sigma')$

Proof. We will first explain the strategy of the proof by focusing on an explicit example. Consider the integrals $\mathcal{I}_1$ corresponding to (24) and $\mathcal{I}_2$ corresponding to (25):

$$\begin{aligned} \mathcal{I}_1 &= \int_{t_1 \geq t_2 \geq t_5 \geq t_4 \geq t_3} e^{i(t_1-t_2)\Delta_\pm^{(1)}} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{2,3} \\ &\quad e^{i(t_3-t_4)\Delta_\pm^{(3)}} B_{1,4} e^{i(t_4-t_5)\Delta_\pm^{(4)}} B_{4,5}(\gamma^{(5)}(t_5,\cdot) dt_2 \cdots dt_5 \end{aligned} \tag{26}$$

$$\begin{aligned} \mathcal{I}_2 &= \int_{t_1 \geq t_2 \geq t_5 \geq t_3 \geq t_4} e^{i(t_1-t_2)\Delta_\pm^{(1)}} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{1,3} \\ &\quad e^{i(t_3-t_4)\Delta_\pm^{(3)}} B_{2,4} e^{i(t_4-t_5)\Delta_\pm^{(4)}} B_{3,5}(\gamma^{(5)}(t_5,\cdot) dt_2 \cdots dt_5 \end{aligned} \tag{27}$$

We first observe the identity,

$$\begin{aligned} & e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{2,3} e^{i(t_3-t_4)\Delta_\pm^{(3)}} B_{1,4} e^{i(t_4-t_5)\Delta_\pm^{(4)}} \\ = \quad & e^{i(t_2-t_4)\Delta_\pm^{(2)}} B_{1,4} e^{-i(t_3-t_4)(\Delta_\pm^{(3)} - \Delta_{\pm,x_3} + \Delta_{\pm,x_4})} B_{2,3} e^{i(t_3-t_5)\Delta_\pm^{(4)}} \end{aligned}$$

In other words, (t_3, x_3, x_3') and (t_4, x_4, x_4') and the position of $B_{2,3}$ and $B_{1,4}$ have been exchanged. This is based on trivial commutations, and is proved below in general, see(37).

Recalling the definition (3) we abbreviate the integral kernel of $B_{j,k+1}$,

$$\delta_{j,k+1} = \delta(x_j - x_{k+1})\delta(x_j - x'_{k+1}) - \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1}).$$

We also denote,

$$\begin{aligned}\gamma_{3,4} &= \gamma^{(5)}(t_5, x_1, x_2, x_3, x_4, x_5; x_1', x_2', x_3', x_4', x_5')\\ \gamma_{4,3} &= \gamma^{(5)}(t_5, x_1, x_2, x_4, x_3, x_5; x_1', x_2', x_4', x_3', x_5').\end{aligned}$$

By the symmetry assumption (1), $\gamma_{3,4} = \gamma_{4,3}$. Thus the integral $\mathcal{I}_1$ of (26) equals

$$\begin{aligned}\mathcal{I}_1 &= \int_{t_1\ge t_2\ge t_5\ge t_4\ge t_3}\int_{\mathbf{R}^{24}} e^{i(t_1-t_2)\Delta_\pm^{(1)}}\delta_{1,2} \\ &\quad e^{i(t_2-t_4)\Delta_\pm^{(2)}}\delta_{1,4}e^{-i(t_3-t_4)(\Delta_\pm^{(3)}-\Delta_{\pm,x_3}+\Delta_{\pm,x_4})}\delta_{2,3}e^{i(t_3-t_5)\Delta_\pm^{(4)}}\\ &\quad \delta_{4,5}\gamma_{4,3}dt_2\cdots dt_5dx_2\cdots dx_5dx_2'\cdots dx_5'\end{aligned} \tag{28}$$

In the above integral we perform the change of variables which exchanges (t_3, x_3, x_3') with (t_4, x_4, x_4'). Thus, in particular, Δ_{x_3} becomes Δ_{x_4}, and $\Delta_\pm^{(3)} = \Delta_\pm^{(2)} + \Delta_{\pm,x_3}$ becomes $\Delta_\pm^{(2)} + \Delta_{\pm,x_4}$. Thus, $\Delta_\pm^{(3)} - \Delta_{\pm,x_3} + \Delta_{\pm,x_4}$ changes to,

$$\Delta_\pm^{(2)} + \Delta_{\pm,x_4} - \Delta_{\pm,x_4} + \Delta_{\pm,x_3} = \Delta_\pm^{(3)},$$

and $\Delta_\pm^{(4)}$ stays unchanged. The integral (28) becomes

$$\begin{aligned}\mathcal{I}_1 &= \int_{t_1\ge t_2\ge t_5\ge t_3\ge t_4}\int_{\mathbf{R}^{24}} e^{i(t_1-t_2)\Delta_\pm^{(1)}}\delta_{1,2}\\ &\quad e^{i(t_2-t_3)\Delta_\pm^{(2)}}\delta_{1,3}e^{i(t_3-t_4)\Delta_\pm^{(3)}}\delta_{2,4}e^{i(t_4-t_5)\Delta_\pm^{(4)}}\\ &\quad \delta_{3,5}\gamma_{3,4}dt_2\cdots dt_5dx_2\cdots dx_5dx_2'\cdots dx_5'\\ &= \int_{t_1\ge t_2\ge t_5\ge t_3\ge t_4} e^{i(t_1-t_2)\Delta_\pm^{(1)}}B_{1,2}e^{i(t_2-t_3)\Delta_\pm^{(2)}}B_{1,3}\\ &\quad e^{i(t_3-t_4)\Delta_\pm^{(3)}}B_{2,4}e^{i(t_4-t_5)\Delta_\pm^{(4)}}B_{3,5}(\gamma^{(5)}(t_5,\cdot)dt_2\cdots dt_5\\ &= \mathcal{I}_2\end{aligned}$$

Therefore, $\mathcal{I}_1 = \mathcal{I}_2$ as stated above.

Notice the domain of integration corresponds to $\sigma'(2) = 2$, $\sigma'(3) = 5$, $\sigma'(4) = 3$, $\sigma'(5) = 4$, that is, $\sigma' = (3,4)\circ\sigma$.

Now we proceed to the general case. Consider a typical term,

$$\mathcal{I}(\mu,\sigma) = \int_{t_1\geq\cdots t_{\sigma(j)}\geq t_{\sigma(j+1)}\geq\cdots t_{\sigma(n+1)}\geq 0} J(\underline{t}_{n+1};\,\mu)dt_2\dots dt_{n+1} \tag{29}$$

$$= \int_{t_1\geq\cdots t_{\sigma(j)}\geq t_{\sigma(j+1)}\geq\cdots t_{\sigma(n+1)}\geq 0} \cdots e^{i(t_{j-1}-t_j)\Delta_\pm^{(j-1)}} B_{l,j} e^{i(t_j-t_{j+1})\Delta_\pm^{(j)}} B_{i,j+1} e^{i(t_{j+1}-t_{j+2})\Delta_\pm^{(j+1)}} (\cdots)dt_2\dots dt_{n+1} \tag{30}$$

with associated matrix of the form,

$$\begin{pmatrix} \cdots & t_{\sigma^{-1}(j)} & t_{\sigma^{-1}(j+1)} & \cdots \\ \cdots & B_{i,j} & \mathbf{B_{i,j+1}} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \mathbf{B_{l,j}} & B_{l,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \text{row j} \\ \cdots & \cdots & \cdots & \text{row j+1} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{31}$$

where $\mu(j) = l$ and $\mu(j+1) = i$ and $i < l < j < j+1$. It is understood that rows j or j+1 may in fact not have highlighted entries, as in the previous example. We plan to show

$$\mathcal{I} = \mathcal{I}' \tag{32}$$

where

$$\mathcal{I}' = \int_{t_1\geq\cdots t_{\sigma'(j)}\geq t_{\sigma'(j+1)}\geq\cdots t_{\sigma'(n+1)}\geq 0} \cdots e^{i(t_{j-1}-t_j)\Delta_\pm^{(j-1)}} B_{i,j} e^{i(t_j-t_{j+1})\Delta_\pm^{(j)}} B_{l,j+1} e^{i(t_{j+1}-t_{j+2})\Delta_\pm^{(j+1)}} (\cdots)'dt_2\dots dt_{n+1} \tag{33}$$

The $\cdots$ at the beginning of (30) and (33) are the same. Any $B_{j,\alpha}$ in $(\cdots)$ in (30) become $B_{j+1,\alpha}$ in $(\cdots)'$ in (33). Similarly, any $B_{j+1,\alpha}$ in $(\cdots)$ in (30) become $B_{j,\alpha}$ in $(\cdots)'$ in (33), while the rest is unchanged.

Thus $\mathcal{I}'$ is represented by the matrix,

$$\begin{pmatrix} \cdots & t_{\sigma'^{-1}(j)} & t_{\sigma'^{-1}(j+1)} & \cdots \\ \cdots & \mathbf{B_{i,j}} & B_{i,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & B_{l,j} & \mathbf{B_{l,j+1}} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \text{(row j)'} \\ \cdots & \cdots & \cdots & \text{(row j+1)'} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{34}$$

where the highlighted entries of (row j)', respectively (row j+1)' in (34) have the positions of the highligted entries of row j+1 , respectively row j in (31).

To prove (32) denote, $\tilde{\Delta}_{\pm}^{(j)} = \Delta_{\pm}^{(j)} - \Delta_{\pm,x_j} + \Delta_{\pm,x_{j+1}}$. We consider the terms,

$$P = B_{l,j} e^{i(t_j - t_{j+1})\Delta_{\pm}^{(j)}} B_{i,j+1} \tag{35}$$

and

$$\tilde{P} = B_{i,j+1} e^{-i(t_j - t_{j+1})\tilde{\Delta}_{\pm}^{(j)}} B_{l,j} \tag{36}$$

We will show that,

$$e^{i(t_{j-1}-t_j)\Delta_{\pm}^{(j-1)}} P e^{i(t_{j+1}-t_{j+2})\Delta_{\pm}^{(j+1)}} = e^{i(t_{j-1}-t_{j+1})\Delta_{\pm}^{(j-1)}} \tilde{P} e^{i(t_j - t_{j+2})\Delta_{\pm}^{(j+1)}} \tag{37}$$

Indeed in (35) we can write $\Delta_{\pm}^{(j)} = \Delta_{\pm,x_i} + (\Delta_{\pm}^{(j)} - \Delta_{\pm,x_i})$. Therefore,

$$e^{i(t_j - t_{j+1})\Delta_{\pm}^{(j)}} = e^{i(t_j - t_{j+1})\Delta_{\pm,x_i}} \cdot e^{i(t_j - t_{j+1})(\Delta_{\pm}^{(j)} - \Delta_{\pm,x_i})}$$

Observe that the first terms on the right can be commuted to the left of $B_{l,j}$, the second one to the right of $B_{i,j+1}$ in the expression for I. Thus,

$$P = e^{i(t_j - t_{j+1})(\Delta_{\pm,x_i})} B_{l,j} B_{i,j+1} e^{i(t_j - t_{j+1})(\Delta_{\pm}^{(j)} - \Delta_{\pm,x_i})}$$

and

$$\begin{aligned} & e^{i(t_{j-1}-t_j)\Delta_{\pm}^{(j-1)}} P e^{i(t_{j+1}-t_{j+2})\Delta_{\pm}^{(j+1)}} \\ = \; & e^{i(t_{j-1}-t_j)\Delta_{\pm}^{(j-1)}} e^{i(t_j - t_{j+1})(\Delta_{\pm,x_i})} B_{l,j} B_{i,j+1} \\ & e^{i(t_j - t_{j+1})(\Delta_{\pm}^{(j)} - \Delta_{\pm,x_i})} e^{i(t_{j+1}-t_{j+2})\Delta_{\pm}^{(j+1)}} \\ = \; & e^{i(t_{j-1}-t_j)\Delta_{\pm}^{(j-1)}} e^{i(t_j - t_{j+1})\Delta_{\pm,x_i}} B_{i,j+1} B_{l,j} \\ \cdot \; & e^{i(t_{j+1}-t_{j+2})(\Delta_{\pm,x_i} + \Delta_{\pm,x_{j+1}})} e^{i(t_j - t_{j+2})(\Delta_{\pm,x_1} \cdots + \hat{\Delta}_{\pm,x_i} + \cdots + \Delta_{\pm,x_j})} \end{aligned}$$

where a hat denotes a missing term.

Similarly, in view of the definition of $\tilde{\Delta}_{\pm}^{(j)}$, we can write,

$$\begin{aligned}\tilde{\Delta}_{\pm}^{(j)} &= \Delta_{\pm}^{(j)} - \Delta_{\pm,x_j} + \Delta_{\pm,x_{j+1}} \\ &= \Delta_{\pm}^{(j-1)} + \Delta_{\pm,x_{j+1}} \\ &= \Delta_{\pm}^{(j-1)} - \Delta_{\pm,x_i} + \Delta_{\pm,x_i} + \Delta_{\pm,x_{j+1}}\end{aligned}$$

Hence,

$$e^{-i(t_j-t_{j+1})\tilde{\Delta}_{\pm}^{(j)}} = e^{-i(t_j-t_{j+1})(\Delta_{\pm}^{(j-1)}-\Delta_{\pm,x_i})} \cdot e^{-i(t_j-t_{j+1})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{j+1}})}$$

and consequently,

$$\tilde{P} = e^{-i(t_j-t_{j+1})(\Delta_{\pm}^{(j-1)}-\Delta_{\pm,x_i})} B_{i,j+1} B_{l,j} e^{-i(t_j-t_{j+1})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{j+1}})}$$

Now,

$$\begin{aligned}& e^{i(t_{j-1}-t_{j+1})\Delta_{\pm}^{(j-1)}} \tilde{P} e^{i(t_j-t_{j+2})\Delta_{\pm}^{(j+1)}} \\ = \;& e^{i(t_{j-1}-t_{j+1})\Delta_{\pm}^{(j-1)}} e^{-i(t_j-t_{j+1})(\Delta_{\pm}^{(j-1)}-\Delta_{\pm,x_i})} B_{i,j+1} B_{l,j} \\ \cdot \;& e^{-i(t_j-t_{j+1})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{j+1}})} e^{i(t_j-t_{j+2})\Delta_{\pm}^{(j+1)}} \\ = \;& e^{i(t_{j-1}-t_j)\Delta_{\pm}^{(j-1)}} e^{i(t_j-t_{j+1})\Delta_{\pm,x_i}} B_{i,j+1} B_{l,j} \\ \cdot \;& e^{i(t_{j+1}-t_{j+2})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{j+1}})} e^{i(t_j-t_{j+2})(\Delta_{\pm,x_1}\cdots+\hat{\Delta}_{\pm,x_i}+\cdots+\Delta_{\pm,x_j})}\end{aligned}$$

and (37) is proved.

Now the argument proceeds as in the example. In the integral (29) use the symmetry (1) to exchange x_j, x_j' with x_{j+1}, x_{j+1}' in the arguments of $\gamma^{(n+1)}$ (only). Then use (37) in the integrand and also replace the $B's$ by their corresponding integral kernels δ. Then we make the change of variables which exchanges t_j, x_j, x_j' with $t_{j+1}, x_{j+1}, x_{j+1}'$ in the whole integral. To see the change in the domain of integration, say $\sigma(a) = j$ and $\sigma(b) = j+1$, and say $b < a$. Then the domain $t_1 \geq \cdots \sigma(b) \geq \cdots \geq \sigma(a) \cdots$ changes to $t_1 \geq \cdots \sigma(a) \geq \cdots \geq \sigma(b) \cdots$. In other words, $a = \sigma^{-1}(j)$ and $b = \sigma^{-1}(j+1)$ have been reversed. This proves (32). □

Next, we consider the subset $\{\mu_s\} \subset M$ of special, upper echelon, matrices in which each highlighted element of a higher row is to the left of each highlighted element of a lower row. Thus (25) is in upper echelon form, and (24) is not. According to our definition, the matrix

$$\begin{pmatrix} \mathbf{B_{1,2}} & \mathbf{B_{1,3}} & B_{1,4} & B_{1,5} \\ 0 & B_{2,3} & \mathbf{B_{2,4}} & B_{2,5} \\ 0 & 0 & B_{3,4} & B_{3,5} \\ 0 & 0 & 0 & \mathbf{B_{4,5}} \end{pmatrix} \tag{38}$$

is also in upper echelon form

Lemma 3.2. *For each element of M there is a finite set of acceptable moves which brings it to upper echelon form.*

Proof. The strategy is to start with the first row and do acceptable moves to bring all marked entries in the first row in consecutive order, $\mathbf{B}_{1,2}$ through $\mathbf{B}_{1,k}$. If there are any highlighted elements on the second row, bring them to $\mathbf{B}_{2,k+1}$, $\mathbf{B}_{2,l}$. This will not affect the marked entries of the first row. If no entries are highlighted on the second row, leave it blank and move to the third row. Continue to lower rows. In the end, the matrix is reduced to an upper echelon form. □

Lemma 3.3. *Let C_n be the number of $n \times n$ special, upper echelon matrices of the type discussed above. Then $C_n \leq 4^n$.*

Proof. The proof consists of 2 steps. First dis-assemble the original special matrix by "lifting" all marked entries to the first row. This partitions the first row into subsets $\{1, 2, \cdots k_1\}$, $\{k_1 + 1, \cdots, k_2\}$ etc. Let P_n be the number of such partitions. Look at the last subset of the partition. It can have 0 elements, in which case there is no last partition. This case contributes precisely one partitions to the total number P_n. If the last subset has k elements then the remaining $n-k$, can contribute exactly P_{n-k} partitions. Thus $P_n = 1 + P_1 + \cdots P_{n-1}$, and therefore $P_n \leq 2^n$ by induction. In the second step we will re-assemble the upper echelon matrix by lowering $\{1, 2, \cdots k_1\}$ to the first used row (we give up the requirement that only the upper triangle is used, thus maybe counting more matrices) $\{k_1 + 1, \cdots, k_2\}$ to the second used row etc. Now suppose that we have exactly i subsets in a given partition of the first row, which will be lowered in an order-preserving way to the available n rows. This can be done in exactly $\binom{n}{i}$ ways. Thus $C_n \leq P_n \sum_i \binom{n}{i} \leq 4^n$. This is in agreement with the combinatorial arguments of [1]. □

Theorem 3.4. *Let μ_s be a special, upper echelon matrix, and write $\mu \sim \mu_s$ if μ can be reduced to μ_s in finitely many acceptable moves.*

There exists D a subset of $[0,t_1]^n$ *such that*

$$\sum_{\mu\sim\mu_s}\int_0^{t_1}\cdots\int_0^{t_n} J(\underline{t}_{n+1};\,\mu)dt_2\cdots dt_{n+1} = \int_D J(\underline{t}_{n+1};\,\mu_s)dt_2\cdots dt_{n+1} \tag{39}$$

Proof. Start with the integral,

$$\mathcal{I}(\mu,\mathrm{id}) = \int_0^{t_1}\cdots\int_0^{t_n} J(\underline{t}_{n+1};\,\mu)dt_2\cdots dt_{n+1}$$

with its corresponding matrix,

$$\begin{pmatrix} t_2 & t_3 & t_4 & \cdots & t_{n+1} \\ \mathbf{B}_{\mu(\mathbf{2}),\mathbf{2}} & B_{1,3} & \mathbf{B}_{\mu(\mathbf{4}),\mathbf{4}} & \cdots & B_{1,n+1} \\ 0 & \mathbf{B}_{\mu(\mathbf{3}),\mathbf{3}} & B_{2,4} & \cdots & B_{2,n+1} \\ 0 & 0 & B_{3,4} & \cdots & B_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_{n,n+1} \end{pmatrix} \tag{40}$$

As in Lemma (3.2) perform finitely many acceptable moves on it, transforming the matrix determined by the pair (μ,id) to the special upper echelon form matrix corresponding to a pair (μ_s,σ),

$$\begin{pmatrix} t_{\sigma^{-1}(2)} & t_{\sigma^{-1}(3)} & t_{\sigma^{-1}(4)} & \cdots & t_{\sigma^{-1}(n+1)} \\ \mathbf{B}_{\mathbf{1},\mathbf{2}} & \mathbf{B}_{\mathbf{1},\mathbf{3}} & B_{1,4} & \cdots & B_{1,n+1} \\ 0 & B_{2,3} & \mathbf{B}_{\mathbf{2},\mathbf{4}} & \cdots & B_{2,n+1} \\ 0 & 0 & B_{3,4} & \cdots & B_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{41}$$

By Lemma (3.1), $\mathcal{I}(\mu,\mathrm{id}) = \mathcal{I}(\mu_s,\sigma)$. Now observe that if (μ_1, id) and (μ_2, id), with $\mu_1\neq\mu_2$ lead to the same echelon form μ_s the corresponding permutations σ_1 and σ_2 must be different. The lemma is thus proved with D the union of all $\{t_1\geq t_{\sigma(2)}\geq t_{\sigma(3)}\geq\cdots t_{\sigma(n_1)}\}$ for all permutations σ which occur in a given class of equivalence of a given μ_s. □

Proof of Main Theorem (1.1) We start by fixing t_1. Express

$$\gamma^{(1)}(t_1,\cdot) \;=\; \sum_{\mu}\int_0^{t_1}\cdots\int_0^{t_n} J(\underline{t}_{n+1},\mu) \tag{42}$$

where, we recall,

$$
\begin{aligned}
J(\underline{t}_{n+1},\mu) \quad &= \quad e^{i(t_1-t_2)\Delta_\pm^{(1)}} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu(3),3} \cdots \\
&\qquad e^{i(t_n-t_{n+1})\Delta_\pm^{(n)}} B_{\mu(n+1),n+1}(\gamma^{(n+1)})(t_{n+1},\cdot)
\end{aligned}
$$

Using Theorem (3.4) we can write $\gamma^{(1)}(t_1,\cdot)$ as a sum of at most 4^n terms of the form

$$
\int_D J(\underline{t}_{n+1},\mu_s) \tag{43}
$$

Let $C^n=[0,t_1]\times[0,t_1]\times\cdots\times[0,t_1]$ (product of n terms). Also, let $D_{t_2}=\{(t_3,\cdots,t_{n+1})|(t_2,t_3,\cdots,t_{n+1})\in D\}$. We have

$$
\begin{aligned}
&\|R^{(1)}\gamma^{(1)}(t_1,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&= \|R^{(1)}\int_D e^{i(t_1-t_2)\Delta_\pm^{(1)}} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\, dt_2\cdots dt_{n+1}\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&= \|\int_0^{t_1} e^{i(t_1-t_2)\Delta_\pm^{(1)}}\left(\int_{D_{t_2}} R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\, dt_3\cdots dt_{n+1}\right) dt_2\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\leq \int_0^{t_1}\|e^{i(t_1-t_2)\Delta_\pm^{(1)}}\int_{D_{t_2}} R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\, dt_3\cdots dt_{n+1}\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} dt_2 \\
&= \int_0^{t_1}\|\int_{D_{t_2}} R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\, dt_3\cdots dt_{n+1}\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} dt_2 \\
&\leq \int_0^{t_1}\left(\int_{D_{t_2}}\|R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\, dt_3\cdots dt_{n+1}\right) dt_2 \\
&\leq \int_{C^n}\|R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta_\pm^{(2)}} B_{\mu_s(3),3}\cdots\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} dt_2\, dt_3\cdots dt_{n+1}
\end{aligned}
\tag{44}
$$

Applying Cauchy-Schwarz in t and Theorem (1.3) n-1 times , we estimate

$$\int_{C^n} \|R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta^{(2)}_{\pm}} B_{\mu_s(3),3} \cdots \|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} dt_2\, dt_3 \cdots dt_{n+1}$$

$$\leq t_1^{\frac{1}{2}} \int_{C^{n-1}} \|R^{(1)} B_{1,2} e^{i(t_2-t_3)\Delta^{(2)}_{\pm}} \left(B_{\mu_s(3),3} \cdots\right) \|_{L^2((t_2\in[0,t_1])\times\mathbb{R}^3\times\mathbb{R}^3)} dt_3 \cdots dt_{n+1}$$

$$\leq C t_1^{\frac{1}{2}} \int_{C^{n-1}} \|R^{(2)} B_{\mu_s(3),3} e^{i(t_3-t_4)\Delta^{(3)}_{\pm}} B_{\mu_s(4),4} \cdots \|_{L^2(\mathbb{R}^6\times\mathbb{R}^6)} dt_3 \cdots dt_{n+1}$$

$$\cdots$$

$$\leq (C t_1^{\frac{1}{2}})^{n-1} \int_0^{t_1} \|R^{(n)} B_{\mu_s(n+1),n+1} \gamma^{(n+1)}(t_{n+1},\cdot)\|_{L^2(R^{3n}\times R^{3n})} dt_{n+1}$$

$$\leq C (C t_1^{\frac{1}{2}})^{n-1}$$

Consequently,

$$\|R^{(1)}\gamma^{(1)}(t_1,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq C(C t_1^{\frac{1}{2}})^{n-1} \tag{45}$$

If $Ct_1 < 1$ and we let $n \to \infty$ and infer that $\|R^{(1)}\gamma^{(1)}(t_1,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0$. The proof for all $\gamma^{(k)} = 0$ is similar. Clearly we can continue the argument to show that all $\gamma^{(k)}$ vanish for all $t \geq 0$ as desired.

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Department of Mathematics, Princeton University, Princeton, NJ, 08540

Department of Mathematics, University of Maryland, College Park, MD, 20742