Math 130 – Fall 2014 – Boyle – Exam 3 Solutions

1. (12 points)
(a) (10 pts) Let $S(x) = 8\ln(7x) - 3x$ define a "satisfaction function" for x in $[2, 5]$. Find the value of x at which S achieves a maximum.
Solution.
$S'(x) = 8/x - 3$ equals zero at $x = 8/3$. S achieves a maximum at $x = 8/3$.

(b) (2 pts) *Briefly* explain how you know the maximum is achieved at this x.
Solution.
$S'(x)$ is positive on $[2, 8/3)$ and is negative on $(8/3, 5]$, so S is increasing on $[2, 8/3)$ and decreasing on $(8/3, 5]$.

2. (13 points)

(a) (9 pts) Find the equation of the tangent line to the curve

$$xy^3 + \ln(y) = x^2 - 6$$

at the point $(x, y) = (3, 1)$.

Solution.

Use implicit differentiation: differentiate both sides with respect to x, then substitute $x = 3, y = 1$:

$$\begin{aligned}
(x)'y^3 + x(y^3)' + \frac{y'}{y} &= 2x \\
y^3 + x(3y^2y') + \frac{y'}{y} &= 2x \\
1^3 + 3(3(1^2)y') + \frac{y'}{1} &= 2(3) \\
1 + 10y' &= 6 \\
y' &= \frac{1}{2} \ .
\end{aligned}$$

An equation (in point-slope form) for the tangent line is $(y-1) = (1/2)(x-3)$.

(b) (4 pts) The top half of the circle $x^2 + y^2 = 25$ is the graph of the function $y = \sqrt{25 - x^2}$.

What is $\int_{-5}^{5} \sqrt{25 - x^2}\, dx$?

Solution.

This definite integral is half the area of a circle of radius 5, so the answer is $(1/2)\pi 5^2 = 25\pi/2$.

3. (10 points)
Find the area of the region bounded between the two curves $y = x^3$ and $y = x^2$.

Solution.
The curves intersect at $x = 0$ and $x = 1$. The bounded region lies over the interval $0 \le x \le 1$, where $x^2 \ge x^3$. So the area is

$$\begin{aligned} \int_0^1 x^2 - x^3\, dx &= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{x=0}^{1} \\ &= \left(\frac{1}{3} - \frac{1}{4}\right) - \left(0 - 0\right) \\ &= \frac{1}{12}\ . \end{aligned}$$

4. (13 points) Evaluate the following definite integrals.

(*a*) (6 pts) $\displaystyle\int_0^1 x^2\sqrt{x^3+1}\,dx$ (*b*) (7 pts) $\displaystyle\int_1^3 \frac{\ln x}{x}\,dx$

Solution.

For (a), we use a subtstitution $u = x^3 + 1$, with $\frac{du}{dx} = 3x^2$. Then $u(1) = 2$ and $u(0) = 1$, and

$$\begin{aligned}\int_0^1 x^2\sqrt{x^3+1}\,dx &= \frac{1}{3}\int_0^1 \sqrt{x^3+1}\;3x^2\,dx \\ &= \frac{1}{3}\int_0^1 \sqrt{u(x)}\;\frac{du}{dx}\,dx = \frac{1}{3}\int_{u=1}^2 \sqrt{u}\,du \\ &= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^2 = \frac{2}{9}\left[u^{3/2}\right]_{u=1}^2 \\ &= \frac{2}{9}\left(2^{3/2} - 1^{3/2}\right) = \frac{2}{9}\left(\sqrt{8} - 1\right)\ .\end{aligned}$$

For (b), we could use the substitution $u = \ln(x)$ and proceed as above, or just observe we have an antiderivative as follows:

$$\begin{aligned}\int_1^3 \frac{\ln x}{x}\,dx &= \left[\frac{1}{2}\big(\ln(x)\big)^2\right]_{x=1}^3 \\ &= \left(\frac{\big(\ln(3)\big)^2}{2} - \frac{\big(\ln(1)\big)^2}{2}\right) \\ &= \frac{\big(\ln(3)\big)^2}{2}\ .\end{aligned}$$

5. (12 points)
Evaluate the following indefinite integrals.

$$(a)\ (6\text{ pts})\quad \int \frac{1}{(5x+2)^2}\,dx \qquad\qquad (b)\ (6\text{ pts})\quad \int \tan(x)\,dx$$

Solution.

$$\begin{aligned}\int \frac{1}{(5x+2)^2}\,dx &= \int (5x+2)^{-2}\,dx\\ &= -\frac{1}{5}(5x+2)^{-1} + C\\ &= -\frac{1}{5(5x+2)} + C\ .\end{aligned}$$

To integrate $\tan(x)$, use the substitution $u(x) = \cos(x)$, with $du/dx = -\sin(x)$. Then

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\\ &= -\int \frac{1}{\cos(x)}(-\sin(x))\,dx\\ &= -\int \frac{1}{u(x)}\frac{du}{dx}\,dx\\ &= \int \frac{1}{u}\,du = -\ln|u| + C\\ &= -\ln|\cos(x)| + C = \ln|\sec(x)| + C\ .\end{aligned}$$

6. (10 points)
At time t days, the rate at which a certain substance grows is $200\, e^{0.1t}$ milligrams per day.

What is the total accumulated growth of the substance over the time interval $0 \leq t \leq 3$?

Solution.

$$\begin{aligned}\int_0^3 200\, e^{0.1t}\, dt &= \left[\frac{200}{0.1} e^{0.1t}\right]_{x=0}^{3} \\ &= \left(2000e^{(0.1)3} - 2000e^{(0.1)(0)}\right) \\ &= 2000(e^{0.3} - 1)\ .\end{aligned}$$

The answer is $2000(e^{0.3} - 1)$ milligrams.

7. (15 points) A 13-foot ladder is placed against a building. The base of the ladder is slipping away from the building at a rate of 4 feet per minute.

Find the rate at which the top of the ladder is sliding down the building at the instant when the bottom of the latter is 5 feet from the base of the building.

Solution.

Let x be the distance from the bottom of the latter to the building and let y be the distance from the top of the ladder to the ground. Then $x^2+y^2 = 13^2$; we know dx/dt; we are looking for dy/dt. Differentiating with respect to t, we get

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ 2y\frac{dy}{dt} &= -2x\frac{dx}{dt} \\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \ . \end{aligned}$$

When $x = 5$, we have $y = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$. So, substituting $x = 5$ and $y = 12$ we get

$$\frac{dy}{dt} = -\frac{5}{12}(4) = -\frac{5}{3} \ .$$

Thus at $x = 5$ the ladder is sliding down the building at a rate of $5/3$ feet per minute.

(The "down" in "sliding down" expresses the negative sign in $dy/dt = -5/3$.)

8. (15 points) A fence must be built to enclose a rectangular area of 5,000 square feet. Fencing material costs \$2.50 per foot for the two sides facing north and south, and it costs \$3.20 per foot for the other two sides.

For the rectangle of area 5,000 square feet which is enclosed by the cheapest possible fence, what is the length of the side facing north?

Solution.

Let x be the length of the north-facing side and y the length of the east-facing side. The cost in dollars is $C = (2.50)(2x) + (3.20)(2y)$. The area is $xy = 5000$, so $y = 5000/x$. Then

$$\begin{aligned} C &= (2.50)(2x) + (3.20)(2(5000x^{-1})) \\ \frac{dC}{dx} &= 5 - (32,000)x^{-2} \ . \end{aligned}$$

This derivative is zero when

$$\begin{aligned} 5 &= (32,000)x^{-2} \\ 5 &= \frac{32,000}{x^2} \\ x^2 &= 6400 \\ x &= 80 \ . \end{aligned}$$

For that cheapest fence, the length of the side facing north is 80 feet.

(We can check $C(x)$ is minimum at $x = 80$ by, for example, noticing $C'(x) > 0$ for $x > 80$ and $C'(x) < 0$ for $x < 80$.)