MATH 240 – Spring 2013 – Exam 2 – Selected solutions

No proof is needed for TRUE-FALSE questions; just write clearly. You may assume given matrix expressions are well defined (i.e. the matrix sizes are compatible).

1. (a) Below are a matrix A and the matrix rref(A) produced by MATLAB.

$$A = \begin{pmatrix} 22 & -22 & 44 & 14 & 7 & 182 \\ 4 & -4 & 8 & 5 & 2 & 37 \\ 14 & -14 & 28 & 4 & 3 & 108 \\ 0 & 0 & 0 & 0 & 10 & 20 \\ 15 & -15 & 30 & 10 & 25 & 165 \end{pmatrix} , \qquad \mathrm{rref}(A) = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Write down a basis for each of the following (no justification required).

i. (4 points) Row(A), the row space of A.
SOLUTION: The first three rows of rref(A).
(The first three rows of A are NOT a basis.)

ii. (4 points) Col(A), the column space of A.
SOLUTION: Columns 1, 4 and 5 of A.

iii. (6 points) Nul(A), the null space of A.
SOLUTION: The following vectors form a basis:

$$\begin{pmatrix} -7 \\ 0 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix} , \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

(b) (6 points) For each of the following, answer TRUE or FALSE.

i. **FALSE** $\mathrm{rank}(A+B) \geq \mathrm{rank}(A)$ whenever A and B are 10×21 matrices.

ii. **TRUE** $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$ whenever A and B are 10×10 matrices.

2. (a) (15 points) Define

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad D = \{x \in \mathbb{R}^2 : \sqrt{(x_1 - \pi)^2 + (x_2 - 17)^2} \leq 3\} .$$

Compute the area of the set $E = \{Ax : x \text{ is in } D\}$.
SOLUTION:
$\mathrm{area}(E) = |\det(A)|\mathrm{area}(D) = |-7|(\pi 3^2) = 63\pi$.

(b) (5 points) Suppose A and B are 5×5 matrices with $\det(A) = 10$ and $\det(B) = 4$.
Compute the determunant of the matrix $M = -2\,A^3 B^{-1}$.
SOLUTION
$\det(M) = (-2)^5(\det(A))^3(1/\det(B)) = (-32)(10^3)(1/4) = -8000$.

3. (a) (14 points) Let $\mathbb{P}^1$ denote the vector space of polynomials of degree at most 1; then $\mathcal{B} = \{3+t, 5+5t\}$ is a basis of $\mathcal{B}$.
Find the coordinates vector $x = [-7+t]_{\mathcal{B}}$ of the polynomial $-7+t$.
SOLUTION. This vector is the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ **such that** $x_1(3+t)+x_2(5+5t) = -7+t$**. This** x **is the solution of**

$$\begin{pmatrix} 3 & 5 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

and this solution is $x = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ **.**

(b) (6 points) For each of the following, answer TRUE or FALSE.

i. **FALSE.** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

ii. **FALSE.** If $\mathcal{C}$ is a set of 14 vectors which span $\mathbb{R}^5$, then $\mathcal{C}$ contains every basis of $\mathbb{R}^5$.

4. (a) (8 points) Suppose that A and B are similar matrices. Prove that their characteristic polynomials are equal.
SOLUTION:
We have $U^{-1}AU = B$ for some invertible U. Therefore

$$\begin{aligned} \det(tI - B) &= \det(tI - U^{-1}BU) \\ &= \det(U^{-1}(tI - A)U) \\ &= \det(U^{-1}) \det(tI - A) \det(U) \\ &= \frac{1}{\det(U)} \det(tI - A) \det(U) \\ &= \det(tI - A) \ . \end{aligned}$$

(b) (12 points) Let V be the vector space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.
For each of the following, answer TRUE or FALSE.

i. **TRUE** The set of functions $\{\cos^2 t, \sin^2 t\}$ is a linearly independent subset of V.

ii. **TRUE** The map $T : V \to \mathbb{R}$ defined by the rule $T(f) = f'(1)$ is a linear transformation.

iii. **FALSE** The map $T : V \to \mathbb{R}$ defined by the rule $T(f) = f(1) - 1$ is a linear transformation.

iv. **TRUE** If S is a nonempty subset of a vector space V, then the set of all linear combnations of S is a subspace of V.

5. (a) (8 points) The determinant of an $n \times n$ matrix A is a polynomial function of the entries of A.

i. What is the degree of this polynomial, if $n = 5$?
Solution. The degree is 5.

ii. This polynomial is a sum of monomials; how many monomials are there in this sum, if $n = 5$?
Solution. The number of monomials here is $5! = 120$**.**

(b) (12 points) For each of the following, answer TRUE or FALSE.

i. **TRUE** If A is a 2×2 matrix with no eigenvalue, then $\det(A) > 0$.

ii. **TRUE** If 3 is an eigenvalue of a 4×4 matrix A, then 15 is an eigenvalue of $5A$.

iii. **FALSE** If A and B are 2×2 matrices with equal characteristic polynomials, then A and B are similar matrices.

iv. **TRUE** Two finite dimensional vector spaces are isomorphic vector spaces if they have the same dimension.