MATH 240 – Spring 2013 – Exam 3 Solutions

There are 5 questions. Answer each on a separate sheet of paper. Use the back side if needed.

On each sheet, put your name, your section leader's name and your section meeting time.

When a question has a short final answer, put a BOX around that answer.

NOTATION: If A is a matrix, then A^T denotes the transpose of A.

1. (20 points) Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$. (Do not compute P^{-1}.)

SOLUTION.

Because A is triangular, its eigenvalues are the diagonal entries $1, 2, 0$. For each eigenvalue λ, solve $(A - \lambda I)x = 0$ to get an eigenvector for λ. We let the columns of P be the eigenvectors and D the diagonal matrix with their eigenvalues:

$$P = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. (21 points) Determine which of the following matrices are diagonalizable:

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 3 \\ 4 & 2 & 0 & 1 \\ 5 & 3 & 1 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Justify your answers.

SOLUTION.

- A is diagonalizable because it is symmetric.
- The only eigenvalue of the matrix B is 1; its algebraic multiplicity (multiplicity as a root of the characteristic polynomial) is four, but its geometric multiplicity (dimension of the eigenspace, i.e. the dimension of the null space of $B - I$) is less than four (it is one). Therefore B is not diagonalizable.
- The characteristic polynomial of C is $t^2 - 2t + 5$, and by the quadratic formula its roots are $(1/2)(2 \pm \sqrt{-16}) = 1 \pm 2i$, which are not real numbers. Therefore C is not diagonalizable.

3. (a) **(10 points)** Compute the area of the parallelogram P in $\mathbb{R}^3$ whose four corners are the points

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\3\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}.$$

SOLUTION.

The vectors in order have the form $0, u, v, u+v$. Let A be the 3×2 matrix with columns u, v. Then we can compute

$$A^T A = \begin{pmatrix}0 & -1 & 3\\2 & 1 & 1\end{pmatrix} \begin{pmatrix}0 & 2\\-1 & 1\\3 & 1\end{pmatrix} = \begin{pmatrix}10 & 2\\2 & 6\end{pmatrix}$$

$$\text{area}(P) = \sqrt{\det(A^T A)} = \sqrt{56} = 2\sqrt{14}\ .$$

(b) **(5 points)** Give an example of two matrices with the same characteristic polynomial which are not similar. No justification required.

SOLUTION.

For example: $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.

(The identity matrix is diagonalizable and the other matrix is not.)

4. (a) **(5 points)** Let W be the span of the vectors $\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 8 \end{pmatrix}$.

Write down a matrix A such that $Ax = 0$ if and only if x is in $W^{\perp}$. No justification necessary.

SOLUTION.

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 0 & 1 \\ 7 & 7 & 7 & 7 & 8 \end{pmatrix}$$

(b) **(5 points)** Let $P = \begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}$. As n goes to infinity, P^n approaches a matrix Q. What is Q?

SOLUTION.

As in the MATLAB homework, Q is the matrix who rows equal the stochastic left (row vector) eigenvector of P. For the eigenvalue q, first solve

$$0 = v(P - I) = (v_1, v_2) \begin{pmatrix} -.8 & .8 \\ .4 & -.4 \end{pmatrix} .$$

One solution is $v = (1, 2)$; divide by $1 + 2$ to get the stochastic eigenvector. Then

$$Q = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} .$$

(c) (15 points) For each statement below, write TRUE or FALSE. No justification necessary.

i. If W is a subspace of $\mathbb{R}^n$ and W is not the trivial space $\{0\}$, then W has an orthonormal basis.
TRUE. The Gram-Schmidt algorithm will produce one.

ii. If $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of a matrix A, then A is a symmetric matrix.
TRUE.

iii. The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.
TRUE.

iv. If A is a symmetric matrix, then A is an orthogonal matrix.
FALSE.

v. If x is the closest vector in the column space of A to b, then x is a least squares solution for the equation $Ax = b$.
FALSE.
Let $\hat{b}$ denote the closest vector in the column space of A to b. Then x is a least squares solution for the equation $Ax = b$ iff $Ax = \hat{b}$ (not $x = \hat{b}$).

5. In this problem A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

(a) **(5 points)** There is an equation involving A^T whose solutions are precisely the least squares solutions to $Ax = b$. What is that equation?

SOLUTION.
$A^T Ax = A^T b$.

(b) **(12 points)** Now consider the specific matrices

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} .$$

Let $A = QR$. Find a least squares solution to $Ax = b$.

SOLUTION.
Here $A = QR$ is indeed the QR factorization of A. Recall that the least squares solutions are the solutions of $Rx = Q^T b$ – this follows easily by plugging into the equation of part (a):

$$\begin{aligned} A^T Ax &= A^T b \\ (QR)^T (QR)x &= (QR)^T b \\ R^T Q^T QRx &= R^T Q^T b \\ Q^T QRx &= Q^T b \\ Rx &= Q^T b \,. \end{aligned}$$

Now substitute and solve:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1/2\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

(c) **(4 points)** Is the least squares solution in part (b) unique? Briefly justify your answer.

SOLUTION. The solution is unique; in general the least squares solution of $Ax = b$ is unique if and only if the columns of A are linearly independent, as they are here.